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*Research Article*

# Extended Extragradient Methods for Generalized Variational Inequalities

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We suggest a modified extragradient method for solving the generalized variational inequalities in a Banach space. We prove some strong convergence results under some mild conditions on parameters. Some special cases are also discussed.

## 1. Introduction

The well-known variational inequality problem is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.1}$$

where $C$ is a nonempty closed convex subset of a real Hilbert space $H$ and $A : C \to H$ is a nonlinear operator. This problem has been researched extensively due to its applications in industry, finance, economics, optimization, medical sciences, and pure and applied sciences; see, for instance, [1–19] and the reference contained therein. For solving the above variational inequality, Korpelevičh [20] introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= P_C(x_n - \lambda A x_n), \\ x_{n+1} &= P_C(x_n - \lambda A y_n) \end{aligned} \tag{1.2}$$

for every $n = 0, 1, 2, \ldots$, where $P_C$ is the metric projection from $R^n$ onto $C$ and $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.2) converge to the same point $z \in \mathrm{VI}(C, A)$. Since some methods related to extragradient methods have been considered in Hilbert spaces by many authors, please see, for example, [3, 5, 7, 14].

This naturally brings us to the following questions.

*Question 1.* Could we extend variational inequality from Hilbert spaces to Banach spaces?

*Question 2.* Could we extend the extragradient methods from Hilbert spaces to Banach spaces?

For solving Question 1, very recently, Aoyama et al. [21] first considered the following generalized variational inequality problem in a Banach space.

*Problem 1. Let $X$ be a smooth Banach space and $C$ a nonempty closed convex subset of $X$. Let $A$ be an accretive operator of $C$ into $X$. Find a point $x^* \in C$ such that*

$$\langle Ax^*, J(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{1.3}$$

This problem is connected with the fixed point problem for nonlinear mapping, the problem of finding a zero point of an accretive operator, and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, please consult [22]. In order to find a solution of Problem 1, Aoyama et al. [21] introduced the following iterative scheme for an accretive operator $A$ in a Banach space $X$:

$$\begin{gathered} x_1 = x \in C, \\ y_n = Q_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) y_n, \end{gathered} \tag{1.4}$$

for every $n = 1, 2, \ldots$, where $Q_C$ is a sunny nonexpansive retraction from $X$ onto $C$. Then, they proved a weak convergence theorem in a Banach space which is generalized simultaneously by theorems of [4, 23] as follows.

**Theorem 1.1.** *Let $X$ be a uniformly convex and 2-uniformly smooth Banach space, and let $C$ be a nonempty closed convex subset of $X$. Let $Q_C$ be a sunny nonexpansive retraction from $X$ onto $C$, let $\alpha > 0$, and let $A$ be an $\alpha$-inverse-strongly accretive operator of $C$ into $X$ with $S(C, A) \neq \emptyset$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen so that $\lambda_n \in [a, \alpha/K^2]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some $b, c$ with $0 < b < c < 1$, then $\{x_n\}$ defined by (1.4) converges weakly to some element $z$ of $S(C, A) := \{x^* \in C : \langle Ax^*, J(x - x^*) \rangle \geq 0,$ for all $x \in C\}$, where $K$ is the 2-uniformly smoothness constant of $X$.*

In this paper, motivated by the ideas in the literature, we first introduce a new iterative method in a Banach space as follows.

For fixed $u \in C$ and arbitrarily given $x_0 \in C$, define a sequence $\{x_n\}$ iteratively by

$$\begin{gathered} y_n = Q_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - \lambda_n A y_n), \end{gathered} \tag{1.5}$$

for every $n = 1, 2, \ldots$, where $Q_C$ is a sunny nonexpansive retraction from $X$ onto $C$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence of real numbers. We prove some strong convergence results under some mild conditions on parameters.

## 2. Preliminaries

Let $X$ be a real Banach space, and let $X^*$ denote the dual of $X$. Let $C$ be a nonempty closed convex subset of $X$. A mapping $A$ of $C$ into $X$ is said to be accretive if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq 0, \tag{2.1}$$

for all $x, y \in C$, where $J$ is called the duality mapping. A mapping $A$ of $C$ into $X$ is said to be $\alpha$-strongly accretive if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2, \tag{2.2}$$

for all $x, y \in C$. A mapping $A$ of $C$ into $X$ is said to be $\alpha$-inverse-strongly accretive if, for $\alpha > 0$,

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|Ax-Ay\|^2, \tag{2.3}$$

for all $x, y \in C$.

*Remark 2.1.* (1) Evidently, the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping.

(2) If $A$ is an $\alpha$-strongly accretive and $L$-Lipschitz continuous mapping of $C$ into $X$, then

$$\langle Ax - Ay, j(x-y)\rangle \geq \alpha\|x-y\|^2 \geq \frac{\alpha}{L^2}\|Ax-Ay\|^2, \quad \forall x, y \in C, \tag{2.4}$$

from which it follows that $A$ must be $(\alpha/L^2)$-inverse-strongly accretive mapping.

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space $X$ is said to be uniformly convex if, for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x-y\| \geq \epsilon \text{ implies } \left\|\frac{x+y}{2}\right\| \leq 1-\delta. \tag{2.5}$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space $X$ is said to be smooth if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.6}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (2.6) is attained uniformly for $x, y \in U$. The norm of $X$ is said to be Frechet differentiable if, for each $x \in U$, the limit (2.6) is attained uniformly for $y \in$ U. And we define a function $\rho : [0,\infty) \to [0,\infty)$ called the modulus of smoothness of $X$ as follows:

$$\rho(\tau) = \sup\left\{ \frac{1}{2}\left(\|x+y\| + \|x-y\|\right) - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \right\}. \tag{2.7}$$

It is known that $X$ is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let $q$ be a fixed real number with $1 < q \leq 2$. Then a Banach space $X$ is said to be $q$-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$.

*Remark 2.2.* Takahashi et al. [24] remind us of the following fact: no Banach space is $q$-uniformly smooth for $q > 2$. So, in this paper, we study a strong convergence theorem in a 2-uniformly smooth Banach space.

We need the following lemmas for the proof of our main results.

**Lemma 2.3** (see [25]). *Let $q$ be a given real number with $1 < q \leq 2$, and let $X$ be a $q$-uniformly smooth Banach space. Then,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + 2\|Ky\|^q, \tag{2.8}$$

*for all $x, y \in X$, where $K$ is the $q$-uniformly smoothness constant of $X$ and $J_q$ is the generalized duality mapping from $X$ into $2^{X^*}$ defined by*

$$J_q(x) = \left\{ f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1} \right\}, \tag{2.9}$$

*for all $x \in X$.*

Let $D$ be a subset of $C$, and let $Q$ be a mapping of $C$ into $D$. Then, $Q$ is said to be sunny if

$$Q(Qx + t(x - Qx)) = Qx, \tag{2.10}$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q$ of $C$ into itself is called a retraction if $Q^2 = Q$. If a mapping $Q$ of $C$ into itself is a retraction, then $Qz = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of $Q$. A subset $D$ of $C$ is called a sunny nonexpansive retract of $C$ if there exists a sunny nonexpansive retraction from $C$ onto $D$. We know the following lemma concerning sunny nonexpansive retraction.

**Lemma 2.4** (see [26]). *Let $C$ be a closed convex subset of a smooth Banach space $X$, $D$ a nonempty subset of $C$, and $Q$ a retraction from $C$ onto $D$. Then, $Q$ is sunny and nonexpansive if and only if*

$$\langle u - Qu, j(y - Qu)\rangle \leq 0, \tag{2.11}$$

*for all $u \in C$ and $y \in D$.*

*Remark 2.5.* (1) It is well known that, if $X$ is a Hilbert space, then a sunny nonexpansive retraction $Q_C$ is coincident with the metric projection from $X$ onto $C$.

(2) Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $X$, and let $T$ be a nonexpansive mapping of $C$ into itself with $F(T) \neq \emptyset$. Then, the set $F(T)$ is a sunny nonexpansive retract of $C$.

The following lemma is characterized by the set of solution Problem AIT by using sunny nonexpansive retractions.

**Lemma 2.6** (see [21]). *Let $C$ be a nonempty closed convex subset of a smooth Banach space $X$. Let $Q_C$ be a sunny nonexpansive retraction from $X$ onto $C$, and let $A$ be an accretive operator of $C$ into $X$. Then, for all $\lambda > 0$,*

$$S(C, A) = F(Q_C(I - \lambda A)), \tag{2.12}$$

*where $S(C, A) = \{x^* \in C : \langle Ax^*, J(x - x^*)\rangle \geq 0, \text{ for all } x \in C\}$.*

**Lemma 2.7** (see [27]). *Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $X$, and let $T$ be nonexpansive mapping of $C$ into itself. If $\{x_n\}$ is a sequence of $C$ such that $x_n \to x$ weakly and $x_n - Tx_n \to 0$ strongly, then $x$ is a fixed point of $T$.*

**Lemma 2.8** (see [28]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space $X$, and let $\{\alpha_n\}$ be a sequence in $[0,1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1. \tag{2.13}$$

*Suppose that*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n, \quad n \geq 0,$$
$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{2.14}$$

*Then,* $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

**Lemma 2.9** (see [26]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \quad n \geq 0, \tag{2.15}$$

*where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $R$ such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

*Then,* $\lim_{n\to\infty} a_n = 0$.

## 3. Main Results

In this section, we obtain a strong convergence theorem for finding a solution of Problem AIT for an $\alpha$-strongly accretive and $L$-Lipschitz continuous mapping in a uniformly convex and 2-uniformly smooth Banach space. First, we assume that $\alpha > 0$ is a constant, $L > 0$ a Lipschitz constant of $A$, and $K > 0$ the 2-uniformly smoothness constant of $X$ appearing in the following.

In order to obtain our main result, we need the following lemma concerning $(\alpha/L^2)$-inverse-strongly accretive mapping.

**Lemma 3.1.** *Let $X$ be a uniformly convex and 2-uniformly smooth Banach space, and let $C$ be a nonempty closed convex subset of $X$. Let $Q_C$ be a sunny nonexpansive retraction from $X$ onto $C$, and let $A$ be an $(\alpha/L^2)$-inverse-strongly accretive mapping of $C$ into $X$ with $S(C,A) \neq \emptyset$. For given $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by (1.5), where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are three sequences in $(0,1)$ and $\{\lambda_n\}$ is a real number sequence in $[a,b]$ for some $a,b$ with $0 < a < b < \alpha/K^2L^2$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

*Then we have* $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ *and* $\lim_{n\to\infty}\|Ay_n - Ax_n\| = 0$.

*Proof.* First, we observe that $I-\lambda_n A$ is nonexpansive. Indeed, for all $x, y \in C$, from Lemma 2.3, we have

$$\begin{aligned}
&\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 \\
&\quad = \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\
&\quad \leq \|x-y\|^2 - 2\lambda_n\langle Ax - Ay, j(x-y)\rangle + 2K^2\lambda_n^2\|Ax - Ay\|^2 \\
&\quad \leq \|x-y\|^2 - 2\lambda_n\frac{\alpha}{L^2}\|Ax - Ay\|^2 + 2K^2\lambda_n^2\|Ax - Ay\|^2 \\
&\quad = \|x-y\|^2 + 2\lambda_n\left(K^2\lambda_n - \frac{\alpha}{L^2}\right)\|Ax - Ay\|^2.
\end{aligned} \tag{3.1}$$

If $0 < a < \lambda_n < b < \alpha/(K^2L^2)$, then $I - \lambda_n A$ is a nonexpansive mapping.

Letting $p \in S(C,A)$, it follows from Lemma 2.6 that $p = Q_C(p - \lambda_n Ap)$. Setting $z_n = Q_C(y_n - \lambda_n Ay_n)$, from (3.1), we have

$$\begin{aligned}
\|z_n - p\| &= \|Q_C(y_n - \lambda_n Ay_n) - Q_C(p - \lambda_n Ap)\| \\
&\leq \|(y_n - \lambda_n Ay_n) - (p - \lambda_n Ap)\| \\
&\leq \|y_n - p\| \\
&= \|Q_C(x_n - \lambda_n Ax_n) - Q_C(p - \lambda_n Ap)\| \\
&\leq \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.2}$$

By (1.5) and (3.2), we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n u+\beta_n x_n+\gamma_n z_n-p\|\\ &\le \alpha_n\|u-p\|+\beta_n\|x_n-p\|+\gamma_n\|z_n-p\|\\ &= \alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\|\\ &\le \max\{\|u-p\|,\|x_0-p\|\}.\end{aligned} \tag{3.3}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{z_n\}$, $\{Ax_n\}$, and $\{Ay_n\}$ are also bounded. We observe that

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|Q_C(y_{n+1}-\lambda_{n+1}Ay_{n+1})-Q_C(y_n-\lambda_n Ay_n)\|\\ &\le \|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_n Ay_n)\|\\ &= \|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_{n+1}Ay_n)+(\lambda_n-\lambda_{n+1})Ay_n\|\\ &\le \|(y_{n+1}-\lambda_{n+1}Ay_{n+1})-(y_n-\lambda_{n+1}Ay_n)\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\|\\ &\le \|y_{n+1}-y_n\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\|\\ &= \|Q_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})-Q_C(x_n-\lambda_n Ax_n)\|+|\lambda_n-\lambda_{n+1}|\|Ay_n\|\\ &\le \|x_{n+1}-x_n\|+|\lambda_n-\lambda_{n+1}|(\|Ax_n\|+\|Ay_n\|).\end{aligned} \tag{3.4}$$

Setting $x_{n+1}=\beta_n x_n+(1-\beta_n)w_n$ for all $n\ge 0$ we obtain

$$\begin{aligned}w_{n+1}-w_n &= \frac{\alpha_{n+1}u+\gamma_{n+1}z_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n u+\gamma_n z_n}{1-\beta_n}\\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right)u+\frac{\gamma_{n+1}}{1-\beta_{n+1}}(z_{n+1}-z_n)+\left(\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right)z_n.\end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\begin{aligned}\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\| &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|\|u\|+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\|\\ &\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1}-\lambda_n|(\|Ax_n\|+\|Ay_n\|)\\ &\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|z_n\|-\|x_{n+1}-x_n\|\\ &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n}{1-\beta_n}\right|(\|u\|+\|z_n\|)\\ &\quad+\frac{\gamma_{n+1}}{1-\beta_{n+1}}|\lambda_{n+1}-\lambda_n|(\|Ax_n\|+\|Ay_n\|);\end{aligned} \tag{3.6}$$

this together with (ii) and (iv) implies that

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\leq 0. \tag{3.7}$$

Hence, by Lemma 2.8, we obtain $\|w_n-x_n\|\to 0$ as $n\to\infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.8}$$

From (1.5), we can write $x_{n+1}-x_n=\alpha_n(u-x_n)+\gamma_n(z_n-x_n)$ and note that $0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1$ and $\lim_{n\to\infty}\alpha_n=0$. It follows from (3.8) that

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.9}$$

For $p\in S(C,A)$, from (3.1) and (3.2), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2&=\|\alpha_n u+\beta_n x_n+\gamma_n z_n-p\|^2\\
&\leq\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|z_n-p\|^2\\
&\leq\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2\\
&\leq\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2+\gamma_n\left\{\|(x_n-\lambda_nAx_n)-(p-\lambda_nAp)\|^2\right\}\\
&\leq\alpha_n\|u-p\|^2+\beta_n\|x_n-p\|^2\\
&\quad+\gamma_n\left\{\|x_n-p\|^2+2\lambda_n\left(K^2\lambda_n-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2\right\}\\
&\leq\alpha_n\|u-p\|^2+\|x_n-p\|^2+2\gamma_n a\left(K^2b-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2.
\end{aligned} \tag{3.10}$$

Therefore, we have

$$\begin{aligned}
0&\leq-2\gamma_n a\left(K^2b-\frac{\alpha}{L^2}\right)\|Ax_n-Ap\|^2\\
&\leq\alpha_n\|u-p\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&=\alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)(\|x_n-p\|-\|x_{n+1}-p\|)\\
&\leq\alpha_n\|u-p\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|.
\end{aligned} \tag{3.11}$$

Since $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, from (3.11), we obtain

$$\|Ax_n-Ap\|\longrightarrow 0\quad\text{as } n\longrightarrow\infty. \tag{3.12}$$

From the definition of $z_n$ and (3.1), we also have

$$\begin{aligned}\|z_n - p\|^2 &= \|Q_C(y_n - \lambda_n A y_n) - Q_C(p - \lambda_n Ap)\|^2 \\ &\le \|(y_n - \lambda_n A y_n) - (p - \lambda_n Ap)\|^2 \\ &\le \|y_n - p\|^2 + 2\lambda_n\left(K^2\lambda_n - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2.\end{aligned} \tag{3.13}$$

From the above results and assumptions, we note that $\|y_n - p\| \le \|x_n - p\|$, $0 < a < b < \alpha/(K^2L^2)$, $\|x_n - p\|$, $\|z_n - p\|$ are bounded, and $\|x_n - z_n\| \to 0$ as $n \to \infty$. Therefore, from (3.13), we have

$$\begin{aligned}0 &\le -2a\left(K^2 b - \frac{\alpha}{L^2}\right)\|Ay_n - Ap\|^2 \\ &\le \|y_n - p\|^2 - \|z_n - p\|^2 \\ &\le \|x_n - p\|^2 - \|z_n - p\|^2 \\ &= (\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|) \\ &\le (\|x_n - p\| + \|z_n - p\|)\|x_n - z_n\| \longrightarrow 0,\end{aligned} \tag{3.14}$$

which implies that

$$\|Ay_n - Ap\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

It follows from (3.12) and (3.15) that

$$\|Ay_n - Ax_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

This completes the proof. □

Now we state and study our main result.

**Theorem 3.2.** *Let $X$ be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping, and let $C$ be a nonempty closed convex subset of $X$. Let $Q_C$ be a sunny nonexpansive retraction from $X$ onto $C$, and let $A$ be an $\alpha$-strongly accretive and $L$-Lipschitz continuous mapping of $C$ into $X$ with $S(C,A) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{\lambda_n\}$ a real number sequence in $[a,b]$ for some $a,b$ with $0 < a < b < \alpha/(K^2L^2)$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \ge 0$;

(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

*Then* $\{x_n\}$ *defined by* (1.5) *converges strongly to* $Q'u$, *where* $Q'$ *is a sunny nonexpansive retraction of* $C$ *onto* $S(C,A)$.

*Proof.* From Remark 2.1(2), we have that $A$ is an $(\alpha/L^2)$-inverse-strongly accretive mapping. Then, from Lemma 3.1, we have

$$\lim_{n\to\infty}\|Ay_n - Ax_n\| = 0. \tag{3.17}$$

On the other hand, we note that

$$\|Ay_n - Ax_n\| \geq \alpha\|y_n - x_n\|, \tag{3.18}$$

which implies that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0, \tag{3.19}$$

that is,

$$\lim_{n\to\infty}\|Q_C(x_n - \lambda_n Ax_n) - x_n\| = 0. \tag{3.20}$$

Next, we show that

$$\limsup_{n\to\infty}\langle u - Q'u, j(x_n - Q'u)\rangle \leq 0. \tag{3.21}$$

To show (3.21), since $\{x_n\}$ is bounded, we can choose a sequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to $z$ such that

$$\limsup_{n\to\infty}\langle u - Q'u, j(x_n - Q'u)\rangle = \limsup_{i\to\infty}\langle u - Q'u, j(x_{n_i} - Q'u)\rangle. \tag{3.22}$$

We first prove $z \in S(C,A)$. Since $\lambda_n$ is in $[a,b]$, it follows that $\{\lambda_{n_i}\}$ is bounded, and so there exists a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ which converges to $\lambda_0 \in [a,b]$. We may assume, without loss of generality, that $\lambda_{n_i} \to \lambda_0$ as $i \to \infty$. Since $Q_C$ is nonexpansive, it follows that

$$\begin{aligned}
\|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - x_{n_i}\| &\leq \|Q_C(x_{n_i} - \lambda_0 Ax_{n_i}) - Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i})\| \\
&\quad + \|Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i}) - x_{n_i}\| \\
&\leq \|(x_{n_i} - \lambda_0 Ax_{n_i}) - (x_{n_i} - \lambda_{n_i} Ax_{n_i})\| \\
&\quad + \|Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i}) - x_{n_i}\| \\
&\leq |\lambda_{n_i} - \lambda_0|\|Ax_{n_i}\| + \|Q_C(x_{n_i} - \lambda_{n_i} Ax_{n_i}) - x_{n_i}\|,
\end{aligned} \tag{3.23}$$

which implies that (noting that (3.20))

$$\lim_{i\to\infty}\|Q_C(I-\lambda_0 A)x_{n_i}-x_{n_i}\|=0. \tag{3.24}$$

By Lemma 2.7 and (3.24), we have $z\in F(Q_C(I-\lambda_0 A))$, and it follows from Lemma 2.6 that $z\in S(C,A)$.

Now, from (3.22) and Lemma 2.4, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u-Q'u,j(x_n-Q'u)\rangle&=\limsup_{i\to\infty}\langle u-Q'u,j(x_{n_i}-Q'u)\rangle\\&=\langle u-Q'u,j(z-Q'u)\rangle\le 0.\end{aligned} \tag{3.25}$$

Finally, from (1.5) and (3.2), we have

$$\begin{aligned}\|x_{n+1}-z\|^2&=\langle\alpha_n u+\beta_n x_n+\gamma_n z_n-z,j(x_{n+1}-z)\rangle\\&=\alpha_n\langle u-z,j(x_{n+1}-z)\rangle+\beta_n\langle x_n-z,j(x_{n+1}-z)\rangle\\&\quad+\gamma_n\langle z_n-z,j(x_{n+1}-z)\rangle\\&\le\frac{1}{2}\beta_n\left(\|x_n-z\|^2+\|x_{n+1}-z\|^2\right)+\alpha_n\langle u-z,j(x_{n+1}-z)\rangle\\&\quad+\frac{1}{2}\gamma_n\left(\|z_n-z\|^2+\|x_{n+1}-z\|^2\right)\\&\le\frac{1}{2}(1-\alpha_n)\left(\|x_n-z\|^2+\|x_{n+1}-z\|^2\right)\\&\quad+\alpha_n\langle u-z,j(x_{n+1}-z)\rangle,\end{aligned} \tag{3.26}$$

which implies that

$$\|x_{n+1}-z\|^2\le(1-\alpha_n)\|x_n-z\|^2+2\alpha_n\langle u-z,j(x_{n+1}-z)\rangle. \tag{3.27}$$

Finally, by Lemma 2.9 and (3.27), we conclude that $x_n$ converges strongly to $Q'u$. This completes the proof. □

*Remark 3.3.* From (3.1), we know that $Q(I-\lambda_n A)$ is nonexpansive. If $S(C,A)\neq\emptyset$, it follows that there exists a sunny nonexpansive retraction $Q'$ of $C$ onto $F(Q(I-\lambda_n A))=S(C,A)$.

## 4. Application

In this section, we prove a strong convergence theorem in a uniformly convex and 2-uniformly smooth Banach space by using Theorem 3.2. We study the problem of finding a fixed point of a strictly pseudocontractive mapping.

A mapping $T$ of $C$ into itself is said to be strictly pseudocontractive if there exists $0 \le \sigma < 1$ such that for all $x, y \in C$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \sigma \|x-y\|^2. \tag{4.1}$$

This inequality can be written in the following form

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge (1-\sigma)\|x-y\|^2. \tag{4.2}$$

Now we give an application concerning a strictly pseudocontractive mapping.

**Theorem 4.1.** *Let $X$ be a uniformly convex and 2-uniformly smooth Banach space with weakly sequentially continuous duality mapping, and let $C$ be a nonempty closed convex subset and a sunny nonexpansive retract of $X$. Let $T$ be a strictly pseudocontractive and $L$-Lipschitz continuous mapping of $C$ into itself with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences in $(0,1)$ and $\{\lambda_n\}$ a real number sequence in $[a,b]$ for some $a, b$ with $0 < a < b < 1-\sigma/K^2(L+1)^2$ satisfying the following conditions:*

- (i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all* $n \ge 0$;
- (ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
- (iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
- (iv) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$.

*For fixed $u \in C$ and arbitrarily given $x_0 \in C$, define a sequence $\{x_n\}$ iteratively by*

$$\begin{aligned} y_n &= (1-\lambda_n)x_n + \lambda_n T x_n, \\ z_n &= (1-\lambda_n)y_n + \lambda_n T y_n, \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n z_n, \end{aligned} \tag{4.3}$$

*for every $n = 1, 2, \ldots$. Then $\{x_n\}$ converges strongly to a fixed point of $T$.*

*Proof.* Putting $A = I - T$, we have from (4.2) that $A$ is $(1-\sigma)$-strongly accretive. At the same time, since $T$ is $L$-Lipschitz continuous, then we have

$$\|Ax - Ay\| = \|(I-T)x - (I-T)y\| \le (L+1)\|x-y\|, \tag{4.4}$$

for all $x, y \in C$, that is, $A$ is $(L+1)$-Lipschitz continuous mapping. It follows from Remark 2.1 (2) that $A$ is $(1-\sigma)/(L+1)^2$-inverse-strongly accretive mapping. It is easy to show that $S(C, A) = S(C, I-T) = F(T) \neq \emptyset$. Therefore, using Theorem 3.2, we can obtain the desired conclusion. This completes the proof. □

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