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Research Article

A Basic Inequality for the Tanaka-Webster Connection

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For submanifolds tangent to the structure vector field in Sasakian space forms, we establish a Chen's basic inequality between the main intrinsic invariants of the submanifold (namely, its pseudosectional curvature and pseudosectional curvature on one side) and the main extrinsic invariant (namely, squared pseudomean curvature on the other side) with respect to the Tanaka-Webster connection. Moreover, involving the pseudo-Ricci curvature and the squared pseudo-mean curvature, we obtain a basic inequality for submanifolds of a Sasakian space form tangent to the structure vector field in terms of the Tanaka-Webster connection.

1. Introduction

One of the basic interests in the submanifold theory is to establish simple relationship between intrinsic invariants and extrinsic invariants of a submanifold. Gauss-Bonnet Theorem, Isoperimetric inequality, and Chern-Lashof Theorem are those such kind of study.

Chen [1] established a nice basic inequality-related intrinsic quantities and extrinsic ones of submanifolds in a space form with arbitrary codimension. Moreover, he studied the basic inequalities of submanifolds of complex space forms and characterize submanifolds when the equality holds.

In this paper, we introduce pseudosectional curvatures and pseudo-Ricci curvature for the Tanaka-Webster connection in a Sasakian space form. After then, we study basic inequalities for submanifolds of a Sasakian space form of a constant pseudosectional curvature and a pseudo-Ricci curvature in terms of the Tanaka-Webster connection.

2. Preliminaries

Let $\widetilde{M}$ be an odd-dimensional Riemannian manifold with a Riemannian metric $\tilde{g}$ satisfying

$$\begin{gathered}\eta(\xi) = 1, \qquad \varphi^2 = -I + \eta \otimes \xi, \qquad \eta(X) = \tilde{g}(X, \xi), \\ \tilde{g}(\varphi X, \varphi Y) = \tilde{g}(X, Y) - \eta(X)\eta(Y).\end{gathered} \tag{2.1}$$

Then $(\varphi, \xi, \eta, \tilde{g})$ is called the almost contact metric structure on $\widetilde{M}$. Let Φ denote the fundamental 2 form in $\widetilde{M}$ given by $\Phi(X, Y) = \tilde{g}(X, \varphi Y)$ for all $X, Y \in T\widetilde{M}$, the set of vector fields of $\widetilde{M}$. If $\Phi = d\eta$, then $\widetilde{M}$ is said to be a contact metric manifold. Moreover, if ξ is a Killing vector field with respect to $\tilde{g}$, and the contact metric structure is called a $\mathcal{K}$-contact structure. Recall that a contact metric manifold is $\mathcal{K}$-contact if and only if

$$\tilde{\nabla}_X \xi = -\varphi X \tag{2.2}$$

for any $X \in T\widetilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection of $\widetilde{M}$. The structure of $\widetilde{M}$ is said to be normal if $[\varphi, \varphi] + 2d\eta \otimes \xi - 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is a normal contact metric manifold. In fact, an almost contact metric structure is Sasakian if and only if

$$\left(\tilde{\nabla}_X \varphi\right) Y = \overline{g}(X, Y)\xi - \eta(Y)X \tag{2.3}$$

for all vector fields X and Y. Every Sasakian manifold is a $\mathcal{K}$-contact manifold.

Given a Sasakian manifold $\widetilde{M}$, a plane section π in $T_p\widetilde{M}$ is called a φ-section if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\widetilde{K}(\pi)$ of a φ-section π is called φ-sectional curvature. If a Sasakian manifold $\widetilde{M}$ has constant φ-sectional curvature c, $\widetilde{M}$ is called a Sasakian space form, denoted by $\widetilde{M}(c)$. (For more details, see [2]).

Now let M be a submanifold immersed in $(\widetilde{M}, \varphi, \xi, \eta, g)$. We also denote by g the induced metric on M. Let TM be the Lie algebra of vector fields in M and $T^\perp M$ the set of all vector fields normal to M. We denote by h the second fundamental form of M and by A_v the Weingarten endomorphism associated with any $v \in T^\perp M$. We put $h^r_{ij} = \overline{g}(h(e_i, e_j), e_r)$ for any orthonormal vector $e_i, e_j \in TM$ and $e_r \in T^\perp M$. The mean curvature vector field H is defined by $H = (1/\dim M)\mathrm{trace}(h)$. M is said to be totally geodesic if the second fundamental form vanishes identically.

From now on, we assume that the dimension of M is $n + 1$, and that of the ambient manifold $\widetilde{M}$ is $2m + 1$ $(m \geq 2)$. We also assume that the structure vector field ξ is tangent to M. Hence, if we denote by D the orthogonal distribution to ξ in TM, we have the orthogonal direct decomposition of TM by $TM = D \oplus \mathrm{span}\{\xi\}$. For any $X \in TM$, we write $\varphi X = TX + NX$, where TX (NX, resp.) is the tangential (normal, resp.) component of φX. If φM is a $\mathcal{K}$-contact manifold, (2.2) gives

$$h(X, \xi) = -NX, \tag{2.4}$$

for any X in TM. Given a local orthonormal frame $\{e_1, \ldots, e_n\}$ of D, we can define the squared norms of T and N by

$$\|T\|^2 = \sum_{i,j=1}^{n} g(e_i, Te_j)^2, \qquad \|N\|^2 = \sum_{i,j=1}^{n} g(e_i, Ne_j)^2, \tag{2.5}$$

resepectively. It is easy to show that both $\|T\|^2$ and $\|N\|^2$ are independent of the choice of the orthonormal frames. The submanifold M is said to be invariant if N is identically zero, that is, $\varphi X \in TM$ for any $X \in TM$. On the other hand, M is said to be an anti-invariant submanifold if T is identically zero, that is, $\varphi X \in T^{\perp}M$ for any $X \in TM$.

3. The Tanaka-Webster Connection for Sasakian Space Form

The Tanaka-Webster connection [3, 4] is the canonical affine connection defined on a nondegenerate pseudo-Hermitian CR-manifold. Tanno [5] defined the Tanaka-Webster connection for contact metric manifolds by the canonical connection which coincides with the Tanaka-Webester connection if the associated CR-structure is integrable. We define the Tanaka-Webster connection for submanifolds of Sasakian manifolds by the naturally extended affine connection of Tanno's Tanaka-Webster connection. Now we recall the Tanaka-Webster connection $\widehat{\nabla}$ for contact metric manifolds

$$\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi, \tag{3.1}$$

for all vector fields $X, Y \in T\widetilde{M}$. Together with (2.1), $\widehat{\nabla}$ is written by

$$\widehat{\nabla}_X Y = \widetilde{\nabla}_X Y + \eta(X)\varphi Y + \eta(Y)\varphi X - \widetilde{g}(Y, \varphi(X))\xi. \tag{3.2}$$

Also, by using (2.1) and (2.3), we can see that

$$\widehat{\nabla}\eta = 0, \qquad \widehat{\nabla}\xi = 0, \qquad \widehat{\nabla}\varphi = 0, \qquad \widehat{\nabla}\widetilde{g} = 0. \tag{3.3}$$

We define the Tanaka-Webster curvature tensor of $\widetilde{R}$ (in terms of $\widetilde{\nabla}$) by

$$\widehat{R}(X, Y)Z = \widehat{\nabla}_X \widehat{\nabla}_Y Z - \widehat{\nabla}_Y \widehat{\nabla}_Y Z - \widehat{\nabla}_{[X,Y]} Z, \tag{3.4}$$

for all vector fields X, Y, and Z in $\widetilde{M}$.

Let $\widetilde{M}(c)$ be a Sasakian space form of constant sectional curvature c and M a submanifold of $\widetilde{M}(c)$. Then, we have the following Gauss' equation:

$$\begin{aligned} \widehat{R}(X, Y)Z = {} & \frac{c+3}{4}\big[\{g(Y, Z) - \eta(Y)\eta(Z)\}X - \{g(X, Z) - \eta(X)\eta(Z)\}Y \\ & + \{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}\xi + 2g(X, \varphi Y)\varphi Z\big] \\ & + \frac{c+7}{4}\{g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y\} \end{aligned} \tag{3.5}$$

for any tangent vector fields X, Y, Z tangent to M.

Let us define the connection $\overset{\circ}{\nabla}$ on M induced from the Tanaka-Webster connection $\widehat{\nabla}$ on $\widetilde{M}$ given by

$$\overset{\circ}{\nabla}_X Y = \overset{\circ}{\nabla}_X Y + \widehat{h}(X,Y), \tag{3.6}$$

for any $X, Y \in \Gamma(TM)$, where $\widehat{h}$ is called the lightlike second fundamental form of M with respect to the induced connection $\overset{\circ}{\nabla}$. In the view of (3.2) and (3.6),

$$\overset{\circ}{\nabla}_X Y + \widehat{h}(X,Y) = \nabla_X Y + h(X,Y) + \eta(X)\varphi Y + \eta(Y)\varphi X - \overline{g}(Y,\varphi X)\xi. \tag{3.7}$$

From (3.7), we obtain

$$\overset{\circ}{\nabla}_X Y = \nabla_X Y + \eta(X)TY + \eta(Y)TX - \overline{g}(Y,\varphi X)\xi, \tag{3.8}$$

$$\widehat{h}(X,Y) = h(X,Y) + \eta(X)NY + \eta(Y)NX, \tag{3.9}$$

where $\varphi X = TX + NX$.

From (3.3), (3.8), and (3.9) it is easy to verify the following:

$$\overset{\circ}{\nabla}\eta = 0, \qquad \overset{\circ}{\nabla}\xi = 0, \qquad \overset{\circ}{\nabla}\varphi = 0, \qquad \overset{\circ}{\nabla} g = 0. \tag{3.10}$$

Moreover, for the induced connection ∇, we have the following

$$\nabla_X \xi = -TX, \qquad h(X,\xi) = -NX. \tag{3.11}$$

From the definition of $\widehat{R}$, together with (3.5), we have

$$\begin{aligned} g\left(\overset{\circ}{R}(X,Y)Z,W\right) = {} & \frac{c+3}{4}\left[\{g(Y,Z) - \eta(Y)\eta(Z)\}g(X,W) - \{g(X,Z) - \eta(X)\eta(Z)\}g(Y,W)\right. \\ & \left. + \{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}g(\xi,W) + 2g(X,\varphi Y)g(\varphi Z,W)\right] \\ & + \frac{c+7}{4}\{g(Z,\varphi Y)g(\varphi X,W) - g(Z,\varphi X)g(\varphi Y,W)\} \\ & + \overline{g}\left(\widehat{h}(X,W),\widehat{h}(Y,Z)\right) - \overline{g}\left(\widehat{h}(X,Z),\widehat{h}(Y,W)\right), \end{aligned} \tag{3.12}$$

for any $X, Y, Z, W \in TM$.

For an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of the tangent space T_pM, $p \in M$, the pseudoscalar curvature $\widehat{\tau}$ at p is defined by

$$\widehat{\tau} = \sum_{i<j} \widehat{K}(e_i \wedge e_j), \tag{3.13}$$

where $\widehat{K}(e_i \wedge e_j)$ denotes the pseudosectional curvature of M associated with the plane section spanned by e_i and e_j for the Tanaka-Webster connection $\widehat{\nabla}$. In particular, if we put $e_{n+1} = \xi_p$, then (3.13) implies that

$$2\widehat{\tau} = \sum_{i \neq j} \widehat{K}(e_i \wedge e_j) + 2\sum_{i=1}^{n} \widehat{K}(e_i \wedge \xi). \tag{3.14}$$

Moreover, from (3.9), we have

$$\begin{aligned} \widehat{h}^r_{ij} &= h_{ij}, \quad i, j \in \{1, \ldots, n\}, \\ \widehat{h}^r_{in+1} &= 0, \quad j \in \{1, \ldots, n+1\}. \end{aligned} \tag{3.15}$$

The pseudomean curvature vector field H is defined by $\widehat{H} = (1/\dim M)\mathrm{trace}(\widehat{h})$. M is said to be totally pseudogeodesic if the second fundamental $\widehat{h}$ form vanishes identically. From (2.5), (3.12) and (3.14), we obtain the following relationship between the pseudoscalar curvature and the pseudomean curvature of M,

$$2\widehat{\tau} = (n+1)^2 \left\| \widehat{H} \right\|^2 - \left\| \widehat{h} \right\|^2 + n(n-1)\frac{c+3}{4} + \frac{3c+13}{4}\|T\|^2. \tag{3.16}$$

We now recall the Chen's lemma.

Lemma 3.1 (see [6]). *Let $a_1, \ldots, a_n, c$ be $n+1$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} a_i \right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + c \right). \tag{3.17}$$

Then, $2a_1a_2 \geq c$, with the equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_n$.

Let $p \in M$ and let π be a plane section of T_pM which is generated by orthonormal vectors X and Y. We can define a function $\alpha(\pi)$ of tangent space T_pM into $[0, 1]$ by

$$\alpha(\pi) = g(TX, Y)^2, \tag{3.18}$$

which is well defined.

Now, we prove the following.

Theorem 3.2. *Let M be an $(n+1)$-dimensional $(n \geq 2)$ submanifold isometrically immersed in a m-dimensional Sasakian space form $\widetilde{M}(c)$ such that the structure vector field ξ is tangent to M in terms*

of the Tanaka-Wester connection $\widehat{\nabla}$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have the following:

$$\begin{aligned}\widehat{\tau} - \widehat{K}(\pi) \leq{}& \frac{(n+1)^2(n-1)}{2n}\left\|\widehat{H}\right\|^2 + \frac{1}{8}(n+1)(n+2)(c+3) \\ &+ \frac{3c+13}{8}\|T\|^2 - \frac{3c+13}{4}\alpha(\pi).\end{aligned} \tag{3.19}$$

Equality in (3.19) *holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that* (a) $\pi = \mathrm{Span}\{e_1, e_2\}$ *and* (b) *the shape operators $A_r = A_{e_r}$, $r = n+2, \ldots, m$, take the following forms:*

$$\begin{aligned}A &= \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \\ A &= \begin{pmatrix} h_{11}^r & h_{12}^r & 0 \\ h_{12}^r & -h_{11}^r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \quad r = n+3, \ldots, m.\end{aligned} \tag{3.20}$$

Proof. Let M^{n+1} be a submanifold of $\widetilde{M}(c)$. We introduce

$$\widehat{\rho} = 2\widehat{\tau} - \frac{(n+1)^2(n-1)}{2n}\left\|\widehat{H}\right\|^2 - (n+1)(n+2)\frac{c+3}{4} - \frac{3c+13}{4}\|T\|^2. \tag{3.21}$$

Then, from (3.16) and (3.21), we get

$$(n+1)\left\|\widehat{H}\right\|^2 = n\left\|\widehat{h}\right\|^2 + n\left(\widehat{\rho} - \frac{2(c+3)}{4}\right). \tag{3.22}$$

Let p be a point of M and let $\pi \subset T_pM$ be a plane section at p. We choose an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ for T_pM and $\{e_{n+2}, \ldots, e_m\}$ for $T_p^{\perp}M$ such that $e_{n+1} = \xi$, $\pi = \mathrm{Span}\{e_1, e_2\}$, and the pseudomean curvature vector $\widehat{H}$ is parallel to e_{n+2}. Then, from (3.22), we get

$$\left(\sum_{i=1}^{n+1}\widehat{h}_{ii}^{n+2}\right)^2 = n\left(\sum_{i=1}^{n+1}\left(\widehat{h}_{ii}^{n+2}\right)^2 + \sum_{i\neq j}\left(\widehat{h}_{ij}^{n+2}\right)^2 + \sum_{r=n+3}^{m}\sum_{i,j}\left(\widehat{h}_{ij}^{r}\right)^2 + \widehat{\rho} - \frac{2(c+3)}{4}\right) \tag{3.23}$$

and so, by applying Lemma 3.1, we obtain

$$2\widehat{h}_{11}^{n+2}\widehat{h}_{22}^{n+2} \geq \sum_{i\neq j}\left(\widehat{h}_{ij}^{n+2}\right)^2 + \sum_{r=n+3}^{m}\sum_{i,j}\left(\widehat{h}_{ij}^{r}\right)^2 + \widehat{\rho} - \frac{2(c+3)}{4}. \tag{3.24}$$

On the other hand, from (3.12), we have

$$\widehat{K}(\pi) = \widehat{h}_{11}^{n+2}\widehat{h}_{22}^{n+2} - \left(\widehat{h}_{12}^{n+2}\right)^2 + \sum_{r=n+3}^{m}\left(\widehat{h}_{11}^{r}\widehat{h}_{22}^{r} - \left(\widehat{h}_{12}^{r}\right)^2\right)\frac{c+3}{4} + \frac{3c+13}{4}g^2(e_1,\varphi e_2). \tag{3.25}$$

Then, from (3.24) and (3.25), we get

$$\widehat{K}(\pi) = \frac{\rho}{2} + \frac{3c+13}{4}g^2(e_1,\varphi e_2) + \sum_{r=n+2}^{m}\sum_{j>2}\left(\left(\widehat{h}_{1j}^{r}\right)^2 + \left(\widehat{h}_{2j}^{r}\right)^2\right) + \frac{1}{2}\sum_{i\neq j>2}\left(\widehat{h}_{ij}^{n+2}\right)^2 \tag{3.26}$$

$$\begin{aligned} &+ \frac{1}{2}\sum_{r=n+3}^{m}\sum_{i,j>2}\left(\widehat{h}_{ij}^{r}\right)^2 + \frac{1}{2}\sum_{r=n+3}^{m}\left(\widehat{h}_{11}^{r} + \widehat{h}_{22}^{r}\right)^2 \\ &\geq \frac{\rho}{2} + \frac{3c+13}{4}\alpha(\pi). \end{aligned} \tag{3.27}$$

Combining (3.21) and (3.27), the inequality (3.19) yields. If the equality in (3.19) holds, then the inequalities given by (3.24) and (3.27) become equalities. In this case, we have

$$\begin{gathered} \widehat{h}_{1j}^{n+2} = \widehat{h}_{2j}^{n+2} = \widehat{h}_{ij}^{n+2} = 0, \quad i \neq j > 2, \\ \widehat{h}_{1j}^{r} = \widehat{h}_{2j}^{r} = \widehat{h}ij^{r} = 0, \quad r \in \{n+3,\ldots,m\};\ i,j \in \{3,\ldots,n+1\}, \\ \widehat{h}_{11}^{n+3} + \widehat{h}_{22}^{n+3} = \cdots = \widehat{h}_{11}^{m} + \widehat{h}_{22}^{m} = 0. \end{gathered} \tag{3.28}$$

Moreover, choosing e_1 and e_2 such that $h_{12}^{n+2} = 0$, from (3.11), we also have the following

$$\widehat{h}_{11}^{n+2} + \widehat{h}_{22}^{n+2} = \widehat{h}_{33}^{n+2} = \cdots = \widehat{h}_{n+1n+1}^{n+2} = 0. \tag{3.29}$$

Thus, with respect to the chosen orthonormal basis $\{e_1,\ldots,e_m\}$, the shape operators of M take the forms. □

We now define a well-defined function $\widehat{\delta}_M$ on M by using $(\inf \widehat{K})(p) = \inf\{\widehat{K}(\pi) \mid \pi \text{ is a plane section} \subset T_pM\}$ in the following manner:

$$\widehat{\delta}_M = \widehat{\tau} - \inf \widehat{K}. \tag{3.30}$$

If $c = -13/3$, then we obtain directly from (3.19) the following result.

Corollary 3.3. *Let M be an $(n+1)$-dimensional $(n \geq 2)$ submanifold isometrically immersed in a m-dimensional Sasakian space form $\widetilde{M}(-13/3)$ such that the structure vector field ξ is tangent to M in terms of the Tanaka-Wester connection $\widehat{\nabla}$. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have the following:*

$$\widehat{\delta}_M \leq \frac{(n+1)^2(n-1)}{2n}\left\|\widehat{H}\right\|^2 - \frac{1}{6}(n+1)(n+2). \tag{3.31}$$

The equality in (3.31) *holds if and only if M is a anti-invariant submanifold with* $\operatorname{rank}(T) = 2$.

Proof. In order to estimate $\widehat{\delta}_M$, we minimize $\|T\|^2 - 2\alpha(\pi)$ in (3.19). For an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of T_pM with $\pi = \mathrm{Span}\{e_1, e_2\}$, we write

$$\|T\|^2 - 2\alpha(\pi) = \sum_{i,j=3}^{n+1} g^2(e_i, \varphi e_j) + 2\sum_{j=3}^{n+1}\left\{ g^2(e_1, \varphi e_j) + g^2(e_2, \varphi e_j)\right\}. \tag{3.32}$$

Thus, we see that the minimum value of $\|T\|^2 - 2\alpha(\pi)$ is zero, provided that $\pi = \mathrm{Span}\{e_1, e_2\}$ is orthogonal to ξ, and $\mathrm{span}\{\varphi e_j \mid j = 3, \ldots, n\}$ is orthogonal to T_pM. Thus we have (3.31) with equality case holding if and only if M is anti-invariant such that $\mathrm{rank}(T) = 2$. □

4. A Pseudo-Ricci Curvature for Sasakian Space Form

We denote the set of unit vectors in T_pM by T_p^1M by

$$T_p^1M = \{X \in T_pM \mid g(X, X) = 1\}. \tag{4.1}$$

Let $\{e_1, \ldots, e_k\}$, $2 \le k \le n$, be an orthonormal basis of a k-place section Π_k of T_pM. If $k = n$, then $\Pi_k = T_pM$, and if $k = 2$, then Π_2 is a plane section of T_pM. For a fixed $i \in \{1, \ldots, k\}$, a k-pseudo-Ricci curvature of Π_k at e_i, denoted by $\widehat{\mathrm{Ric}}_{\Pi_k}(e_i)$, is defined by [7]

$$\widehat{\mathrm{Ric}}_{\Pi_k}(e_i) = \sum_{j\neq i}^{k} \widehat{K}_{ij}, \tag{4.2}$$

where $\widehat{K}_{ij}$ is the pseudosectional curvature in terms of the Tanaka-Webster connection $\widehat{\nabla}$ of the plane section spanned by e_i and e_j. We note that an n-pseudo-Ricci curvature $\mathrm{Ric}_{T_pM}(e_i)$ is the usual pseudo-Ricci curvature of e_i, denoted by $\widehat{\mathrm{Ric}}(e_i)$. Thus, for any orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ for T_pM and for a fixed $i \in \{1, \ldots, n+1\}$, we have the following:

$$\widehat{\mathrm{Ric}}_{T_pM}(e_i) = \widehat{\mathrm{Ric}}(e_i) = \sum_{j\neq i}^{n+1} \widehat{K}_{ij}. \tag{4.3}$$

The pseudoscalar curvature $\widehat{\tau}(\Pi_k)$ of the k-plane section Π_k is given by

$$\widehat{\tau}(\Pi_k) = \sum_{1\le i<j\le n+1} \widehat{K}_{ij}. \tag{4.4}$$

The relative null spae of M at p is defined by [8]

$$\mathcal{N}_p = \left\{X \in T_pM \mid \widehat{h}(X, Y) = 0, \forall Y \in T_pM\right\}. \tag{4.5}$$

Theorem 4.1. *Let $\widetilde{M}(c)$ be a m-dimensional Sasakian space form and M an $n+1$-dimensional submanifold tangent to ξ with respect to the Tanaka-Webster connection $\widehat{\nabla}$. Then,*

(i) *for each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$4\widehat{\mathrm{Ric}}(X) \leq (n+1)^2\left\|\widehat{H}\right\|^2 + (n-1)(c+3) + (3c+13)\|TX\|^2, \tag{4.6}$$

(ii) *if $\widehat{H}(p) = 0$, then a unit tanget vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (4.6) if and only of $X \in \mathcal{N}_p$.*

(iii) *the equality case of (4.6) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally pseudogeodesic point in terms of the Tanaka-Webster connection.*

Proof. (i) Let $X \in T_pM$ be a unit tangent vector at p, orthogonal to ξ. We choose an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ for T_pM and $\{e_{n+2}, \ldots, e_m\}$ for $T_p^{\perp}M$ such that $e_1 = X$ and $e_{n+1} = \xi$. Then, from (3.16), we have

$$(n+1)^2\left\|\widehat{H}\right\|^2 = 2\widehat{\tau} + \left\|\widehat{h}\right\|^2 - n(n-1)\frac{c+3}{4} - \frac{3c+13}{4}\|T\|^2. \tag{4.7}$$

From (4.7), we get

$$\begin{aligned}(n+1)^2\left\|\widehat{H}\right\|^2 &= 2\widehat{\tau} + \sum_{r=n+2}^{m}\left[\left(\widehat{h}_{11}^r\right)^2 + \left(\widehat{h}_{22}^r + \cdots + \widehat{h}_{n+1n+1}^r\right)^2 + 2\sum_{i<j}\left(\widehat{h}_{ij}^r\right)^2\right] \\ &\quad - 2\sum_{r=n+2}^{m}\sum_{2\leq i<j\leq n}\widehat{h}_{ii}^r\widehat{h}_{jj}^r - n(n-1)\frac{c+3}{4} - \frac{3c+13}{4}\|T\|^2 \\ &= 2\widehat{\tau} + \frac{1}{2}\sum_{r=n+2}^{m}\left[\left(\widehat{h}_{11}^r + \widehat{h}_{22}^r + \cdots + \widehat{h}_{n+1n+1}^r\right)^2 + \left(\widehat{h}_{11}^r - \widehat{h}_{22}^r - \cdots - \widehat{h}_{n+1n+1}^r\right)^2\right] \\ &\quad + 2\sum_{n+2}^{m}\sum_{i<j}\left(\widehat{h}_{ij}^r\right)^2 - 2\sum_{r=n+2}^{m}\sum_{2\leq i<j\leq n}\widehat{h}_{ii}^r\widehat{h}_{jj}^r - n(n-1)\frac{c+3}{4} - \frac{3c+13}{4}\|T\|^2.\end{aligned} \tag{4.8}$$

From (3.12), we have

$$\widehat{K}_{ij} = \sum_{r=n+2}^{m}\left[\widehat{h}_{ii}^r\widehat{h}_{jj}^r - \left(\widehat{h}_{ij}^r\right)^2\right] + \frac{c+3}{4} + \frac{3c+13}{4}g^2(e_i, Te_j), \tag{4.9}$$

and consequently

$$\sum_{2\leq i<j\leq n+1}\widehat{K}_{ij} = \sum_{r=n+2}^{m}\left[\widehat{h}_{ii}^r\widehat{h}_{jj}^r - \left(\widehat{h}_{ij}^r\right)^2\right] + \frac{(n-1)(n-2)(c+3)}{8} + \frac{3c+13}{8}\left\{\|T\|^2 - 2\|Te_1\|^2\right\}. \tag{4.10}$$

Substituting (4.10) into (4.8), one gets

$$(n+1)^2\left\|\widehat{H}\right\|^2 \geq 2\hat{\tau} + \frac{(n+1)^2}{2}\left\|\widehat{H}\right\|^2 + 2\sum_{r=n+2}^{m}\sum_{j=2}\left(\hat{h}_{1j}^r\right)^2 - 2\sum_{2\leq i<j\leq n+1}\hat{K}_{ij} - \frac{(n-1)(c+3)}{2} - \frac{3c+13}{2}\|Te_1\|^2. \tag{4.11}$$

Therefore,

$$\frac{(n+1)^2}{2}\left\|\widehat{H}\right\|^2 \geq 2\widehat{\mathrm{Ric}}(X) - \frac{(n-1)(c+3)}{2} - \frac{3c+13}{2}\|TX\|^2, \tag{4.12}$$

which is equivalent to (4.6)

(ii) Assume that $\widehat{H}(p) = 0$. Equality holds in (4.6) if and only if

$$\begin{aligned} \hat{h}_{12}^r = \cdots = \hat{h}_{1n+1}^r = 0, \\ \hat{h}_{11}^r = \hat{h}_{22}^r + \cdots + \hat{h}_{n+1n+1}^r, \quad r \in \{n+2, \ldots, m\}. \end{aligned} \tag{4.13}$$

Then, $\hat{h}_{1j}^r = 0$ for each $j \in \{1, \ldots, n+1\}$, $r \in \{n+2, \ldots, m\}$, that is, $X \in \mathcal{N}_p$.

(iii) The equality case of (4.6) holds for all unit tangent vectors at p if and only if

$$\begin{aligned} \hat{h}_{ij}^r = 0, \quad i \neq j, \ r \in \{n+2, \ldots, m\}, \\ \hat{h}_{11}^r + \cdots + \hat{h}_{n+1n+1}^r - 2\hat{h}_{ii}^r = 0, \quad i \in \{1, \ldots, n+1\}, \ r \in \{n+2, \ldots, m\}. \end{aligned} \tag{4.14}$$

Since $\hat{h}(e_i, e_{n+1} = \xi) = 0$ from (3.10), p is a totally pseudogeodesic point, and, hence, $\varphi(T_pM) \subset T_pM$. The converse is trivial. □

Corollary 4.2. *Let M be an $n+1$-dimensional invariant submanifold of a Sasakian space form $\widetilde{M}(c)$. Then,*

(i) *for each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$4\widehat{\mathrm{Ric}}(X) \leq (n-1)(c+3) + (3c+13). \tag{4.15}$$

(ii) *A unit tanget vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of* (4.6) *if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of* (4.6) *holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally pseudogeodesic point in terms of the Tanaka-Webster connection.*

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