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Research Article

Asymptotic Dichotomy in a Class of Third-Order Nonlinear Differential Equations with Impulses

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Solutions of quite a few higher-order delay functional differential equations oscillate or converge to zero. In this paper, we obtain several such dichotomous criteria for a class of third-order nonlinear differential equation with impulses.

1. Introduction

It has been observed that the solutions of quite a few higher-order delay functional differential equations oscillate or converge to zero (see, e.g., the recent paper [1] in which a third order nonlinear delay differential equation with damping is considered). Such a dichotomy may yield useful information in real problems (see, e.g., [2] in which implications of this dichotomy are applied to the deflection of an elastic beam). Thus it is of interest to see whether similar dichotomies occur in different types of functional differential equations.

One such type consists of impulsive differential equations which are important in simulation of processes with jump conditions (see, e.g., [3–22]). But papers devoted to the study of asymptotic behaviors of third-order equations with impulses are quite rare. For this reason, we study here the third-order nonlinear differential equation with impulses of the form

$$\begin{gathered} \left(r(t)x''(t)\right)' + f(t,x) = 0, \quad t \ge t_0,\ t \ne t_k, \\ x^{(i)}\left(t_k^+\right) = g_k^{[i]}\left(x^{(i)}(t_k)\right), \quad i = 0,1,2;\ k = 1,2,\dots, \\ x^{(i)}\left(t_0^+\right) = x_0^{[i]}, \quad i = 0,1,2, \end{gathered} \tag{1.1}$$

where $x^{(0)}(t) = x(t)$, $0 \le t_0 < t_1 < \cdots < t_k < \cdots$ such that $\lim_{k\to\infty} t_k = +\infty$,

$$x^{(i)}(t_k^+) = \lim_{t\to t_k^+} x^{(i)}(t), \qquad x^{(i)}(t_k^-) = \lim_{t\to t_k^-} x^{(i)}(t) \tag{1.2}$$

for $i = 0,1,2$. Here $g_k^{[i]}$, $i = 0,1,2$ and $k = 1,2,\ldots$, are real functions and $x_0^{[i]}$, $i = 0,1,2$, are real numbers.

By a solution of (1.1), we mean a real function $x = x(t)$ defined on $[t_0,+\infty)$ such that

(i) $x^{(i)}(t_0^+) = x_0^{[i]}$ for $i = 0,1,2$;

(ii) $x^{(i)}(t), i = 0,1,2$, and $(r(t)x''(t))'$are continuous on $[t_0,+\infty) \setminus \{t_k\}$; for $i = 0,1,2$, $x^{(i)}(t_k^+)$ and $x^{(i)}(t_k^-)$ exist, $x^{(i)}(t_k^-) = x^{(i)}(t_k)$ and $x^{(i)}(t_k^+) = g_k^{[i]}(x^{(i)}(t_k))$ for any t_k;

(iii) $x(t)$ satisfies $(r(t)x''(t))' + f(t,x) = 0$ at each point $t \in [t_0,+\infty) \setminus \{t_k\}$.

A solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is said to be oscillatory.

We will establish dichotomous criteria that guarantee solutions of (1.1) that are either oscillatory or zero convergent based on combinations of the following conditions.

(A) $r(t)$ is positive and continuous on $[t_0,\infty)$, $f(t,x)$ is continuous on $[t_0,\infty) \times R$, $xf(t,x) > 0$ for $x \neq 0$, and $f(t,x)/\varphi(x) \geqslant p(t)$, where $p(t)$ is positive and continuous on $[t_0,\infty)$, and φ is differentiable in R such that $\varphi'(x) \ge 0$ for $x \in R$.

(B) For each $k = 1,2,\ldots$, $g_k^{[i]}(x)$ is continuous in R and there exist positive numbers $a_k^{[i]}, b_k^{[i]}$ such that $a_k^{[i]} \le g_k^{[i]}(x)/x \le b_k^{[i]}$ for $x \neq 0$ and $i = 0,1,2$.

(C) One has

$$\begin{gathered}\int_{t_0}^{\infty} \prod_{t_0<t_k<s} \left(\frac{a_k^{[1]}}{b_k^{[0]}}\right) ds = +\infty, \\ \int_{t_0}^{\infty} \frac{1}{r(s)} \prod_{t_0<t_k<s} \left(\frac{a_k^{[2]}}{b_k^{[1]}}\right) ds = +\infty.\end{gathered} \tag{1.3}$$

In the next section, we state four theorems to ensure that every solution of (1.1) either oscillates or tends to zero. Examples will also be given. Then in Section 3, we prove several preparatory lemmas. In the final section, proofs of our main theorems will be given.

2. Main Results

The main results of the paper are as follows.

Theorem 2.1. *Assume that the conditions (A)–(C) hold. Suppose further that there exists a positive integer k_0 such that for $k \geq k_0$, $a_k^{[0]} \geq 1$,*

$$\sum_{k=1}^{+\infty}\left(b_k^{[0]}-1\right) < +\infty, \tag{2.1}$$

$$\int_{t_0}^{+\infty}\prod_{t_0<t_k<s}\left(\frac{1}{b_k^{[2]}}\right)p(s)ds = +\infty. \tag{2.2}$$

Then every solution of (1.1) *either oscillates or tends to zero.*

Theorem 2.2. *Assume that the conditions (A)–(C) hold. Suppose further that there exists a positive integer k_0 such that for $k \geq k_0$, $b_k^{[0]} \leq 1$, $a_k^{[1]} \geq 1$,*

$$\prod_{t_0\leq t_k<+\infty} a_k^{[0]} \geq \sigma > 0, \tag{2.3}$$

$$\int_{t_0}^{+\infty}\frac{1}{r(s)}\left(\int_{s}^{+\infty}\prod_{s<t_k<u}\frac{1}{b_k^{[2]}}p(u)du\right)ds = +\infty. \tag{2.4}$$

Then every solution of (1.1) *either oscillates or tends to zero.*

Theorem 2.3. *Assume that the conditions (A)–(C) hold and that $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. Suppose further that there exists a positive integer k_0 such that for*

$$k \geq k_0, b_k^{[0]} \leq 1, \qquad b_k^{[2]} \leq 1, \qquad b_k^{[2]} \leq \varphi\left(a_k^{[0]}\right), \tag{2.5}$$

$$\int_{t_0}^{+\infty} p(s)ds = +\infty. \tag{2.6}$$

Then every solution of (1.1) *either oscillates or tends to zero.*

Theorem 2.4. *Assume that the conditions (A)–(C) hold and that $\varphi(ab) \geq \varphi(a)\varphi(b)$ for any $ab > 0$. Suppose further that $b_k^{[2]} \leq a_k^{[0]}$, $\{\prod_{k=1}^{n} b_k^{[0]}\}$ is bounded, that*

$$\begin{gathered}\sum_{k=1}^{+\infty}\max\left\{\left|a_k^{[0]}-1\right|,\left|b_k^{[0]}-1\right|\right\} < +\infty,\\ \sum_{k=1}^{+\infty}\left|b_k^{[2]}-1\right| < +\infty,\end{gathered} \tag{2.7}$$

$$\int_{t_0}^{+\infty} p(s)ds = +\infty. \tag{2.8}$$

Then every solution of (1.1) *either oscillates or tends to zero.*

Before giving proofs, we first illustrate our theorems by several examples.

Example 2.5. Consider the equation

$$
\begin{gathered}
(t x''(t))' + e^{t} x(t) = 0, \quad t \geq \frac{1}{2},\ t \neq k, \\
x^{(i)}(k^{+}) = \left(1 + \frac{1}{k^{2}}\right) x^{(i)}(k), \quad i = 0,1,2;\ k = 1,2,\ldots, \\
x\left(\frac{1}{2}\right) = x_{0}^{[0]}, \qquad x'\left(\frac{1}{2}\right) = x_{0}^{[1]}, \qquad x''\left(\frac{1}{2}\right) = x_{0}^{[2]},
\end{gathered}
\tag{2.9}
$$

where $a_{k}^{[i]} = b_{k}^{[i]} = (1 + (1/k^{2})) \geq 1$ for $i = 0,1,2$; $p(t) = e^{t}$, $r(t) = t$, $t_{k} = k$, $\varphi(x) = x$. It is not difficult to see that conditions (A)–(C) are satisfied. Furthermore,

$$
\begin{gathered}
\sum_{k=1}^{+\infty} \left(b_{k}^{[0]} - 1\right) = \sum_{k=1}^{+\infty} \frac{1}{k^{2}} < +\infty, \\
\int_{t_{0}}^{+\infty} \prod_{t_{0} < t_{k} < s} \frac{1}{b_{k}^{[2]}} p(s) ds = \int_{1/2}^{+\infty} \prod_{1/2 < t_{k} < s} \frac{k^{2}}{k^{2} + 1} e^{s} ds = +\infty.
\end{gathered}
\tag{2.10}
$$

Thus by Theorem 2.1, every solution of (2.9) either oscillates or tends to zero.

Example 2.6. Consider the equation

$$
\begin{gathered}
\left(\sqrt{t}(2 - \sin t) g(t) x''(t)\right)' + t^{-3/2} x^{3}(t) = 0, \quad t \geq \frac{1}{2},\ t \neq k, \\
x(k^{+}) = \frac{k}{k+1} x(k), \quad x^{(i)}(k^{+}) = x^{(i)}(k), \quad i = 1,2;\ k = 1,2,\ldots, \\
x\left(\frac{1}{2}\right) = x_{0}^{[0]}, \qquad x'\left(\frac{1}{2}\right) = x_{0}^{[1]}, \qquad x''\left(\frac{1}{2}\right) = x_{0}^{[2]},
\end{gathered}
\tag{2.11}
$$

where $a_{k}^{[0]} = b_{k}^{[0]} = k/(k+1)$, $a_{k}^{[i]} = b_{k}^{[i]} = 1$ for $i = 1,2$; $p(t) = t^{-3/2}$, $t_{k} = k$, $\varphi(x) = x^{3}$, and

$$
r(t) = \sqrt{t}(2 - \sin t) g(t), \quad \text{here } g(t) = \left| t - k - \frac{1}{2} \right| + 1,\ t \in [k, k+1),\ k = 1,2,\ldots. \tag{2.12}
$$

Here, we do not assume that $r(t)$ is bounded, monotonic, or differential. It is not difficult to see that conditions (A)–(C) are satisfied. Furthermore,

$$\begin{aligned}
\int_{t_0}^{+\infty}\frac{1}{r(s)}\left(\int_s^{+\infty}\prod_{s<t_k<u}\frac{1}{b_k^{(2)}}p(u)du\right)ds &= \int_{1/2}^{+\infty}\frac{1}{\sqrt{s}(2-\sin s)g(s)}\left(\int_s^{+\infty}u^{-3/2}du\right)ds \\
&\geq \int_{1/2}^{+\infty}\frac{1}{3\sqrt{s}g(s)}\left(\int_s^{+\infty}u^{-3/2}du\right)ds \\
&\geq \int_{1/2}^{+\infty}\frac{2}{9\sqrt{s}}\left(\int_s^{+\infty}u^{-3/2}du\right)ds \\
&= \int_{1/2}^{+\infty}\frac{4}{9s}ds = +\infty.
\end{aligned} \tag{2.13}$$

Thus by Theorem 2.2, every solution of (2.11) either oscillates or tends to zero.

Example 2.7. Consider the equation

$$\begin{gathered}
\left(e^{-2t}x''(t)\right)' + e^{-2t}x(t) = 0, \quad t \geq \frac{1}{2},\ t \neq k, \\
x(k^+) = x(k), \quad x'(k^+) = x'(k), \quad x''(k^+) = \frac{k}{k+1}x''(k), \quad k = 1,2,\ldots, \\
x\left(\frac{1}{2}\right) = x_0^{[0]}, \qquad x'\left(\frac{1}{2}\right) = x_0^{[1]}, \qquad x''\left(\frac{1}{2}\right) = x_0^{[2]},
\end{gathered} \tag{2.14}$$

where $a_k^{[i]} = b_k^{[i]} = 1$ for $i = 0,1$, $a_k^{[2]} = b_k^{[2]} = k/(k+1)$; $p(t) = e^{-2t}$, $r(t) = e^{-2t}$, $t_k = k$; $\varphi(x) = x$. It is not difficult to see that conditions (A)–(C) are satisfied. Furthermore,

$$\begin{aligned}
\int_{t_0}^{+\infty}\frac{1}{r(s)}\left(\int_s^{+\infty}\prod_{s<t_k<u}\frac{1}{b_k^{(2)}}p(u)du\right)ds &= \int_{1/2}^{+\infty}e^{2s}\left(\int_s^{+\infty}\prod_{s<t_k<u}\frac{k+1}{k}e^{-2u}du\right)ds \\
&\geq \int_{1/2}^{+\infty}e^{2s}\left(\int_s^{+\infty}e^{-2u}du\right)ds \\
&= \int_{1/2}^{+\infty}\frac{1}{2}ds = +\infty.
\end{aligned} \tag{2.15}$$

Thus, by Theorem 2.2, every solution of (2.14) either oscillates or tends to zero.

Note that the ordinary differential equation

$$\left(e^{-2t}x''(t)\right)' + e^{-2t}x(t) = 0 \tag{2.16}$$

has a nonnegative solution $x(t) = e^t \to +\infty$ as $t \to +\infty$. This example shows that impulses play an important role in oscillatory and asymptotic behaviors of equations under perturbing impulses.

3. Preparatory Lemmas

To prove our theorems, we need the following lemmas.

Lemma 3.1 (Lakshmikantham et al. [3]). *Assume the following.*

(H_0) $m \in PC'(R^+, R)$ *and* $m(t)$ *is left-continuous at* $t_k, k = 1, 2, \ldots$.

(H_1) *For* $t_k, k = 1, 2, \ldots$ *and* $t \geq t_0$,

$$
\begin{aligned}
m'(t) &\leq p(t)m(t) + q(t), \quad t \neq t_k, \\
m(t_k^+) &\leq d_k m(t_k) + b_k,
\end{aligned}
\tag{3.1}
$$

where $p, q \in PC(R^+, R)$, $d_k \geq 0$, *and* b_k *are real constants. Then for* $t \geq t_0$,

$$
\begin{aligned}
m(t) \leq{}& m(t_0) \prod_{t_0<t_k<t} d_k \exp\left(\int_{t_0}^{t} p(s)ds\right) + \sum_{t_0<t_k<t} \left(\prod_{t_k<t_j<t} d_j \exp\left(\int_{t_k}^{t} p(s)ds\right)\right) b_k \\
&+ \int_{t_0}^{t} \left(\prod_{s<t_k<t} d_k\right) \exp\left(\int_{s}^{t} p(\sigma)d\sigma\right) q(s)ds.
\end{aligned}
\tag{3.2}
$$

Lemma 3.2. *Suppose that conditions (A)–(C) hold and* $x(t)$ *is a solution of* (1.1). *One has the following statements.*

(a) *If there exists some* $T \geq t_0$ *such that* $x''(t) > 0$ *and* $(r(t)x''(t))' \geq 0$ *for* $t \geq T$, *then there exists some* $T_1 \geq T$ *such that* $x'(t) > 0$ *for* $t \geq T_1$.

(b) *If there exists some* $T \geq t_0$ *such that* $x'(t) > 0$ *and* $x''(t) \geq 0$ *for* $t \geq T$, *then there exists some* $T_1 \geq T$ *such that* $x(t) > 0$ *for* $t \geq T_1$.

Proof. First of all, we will prove that (a) is true. Without loss of generality, we may assume that $x''(t) > 0$ and $(r(t)x''(t))' \geq 0$ for $t \geq t_0$. We assert that there exists some j such that $x'(t_j) > 0$ for $t_j \geq t_0$. If this is not true, then for any $t_k \geq t_0$, we have $x'(t_k) \leq 0$. Since $x'(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that $x'(t) \leq 0$ for $t \geq t_0$. Since $r(t)x''(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that for $(t_1, t_2]$,

$$
r(t)x''(t) \geq r(t_1)x''(t_1^+),
\tag{3.3}
$$

that is,

$$
x''(t) \geq \frac{r(t_1)}{r(t)} x''(t_1^+).
\tag{3.4}
$$

In particular,

$$x''(t_2) \geq \frac{r(t_1)}{r(t_2)} x''(t_1^+). \tag{3.5}$$

Similarly, for $(t_2, t_3]$, we have

$$x''(t) \geq \frac{r(t_2)}{r(t)} x''(t_2^+) \geq \frac{r(t_2)}{r(t)} a_2^{[2]} x''(t_2) \geq \frac{r(t_1)}{r(t)} a_2^{[2]} x''(t_1^+). \tag{3.6}$$

By induction, we know that for $t > t_1$,

$$x''(t) \geq \frac{r(t_1)}{r(t)} \prod_{t_1 < t_k < t} a_k^{[2]} x''(t_1^+), \quad t \neq t_k. \tag{3.7}$$

From condition (B), we have

$$x'(t_k^+) \geq b_k^{[1]} x'(t_k), \quad k = 2, 3, \ldots. \tag{3.8}$$

Set $m(t) = -x'(t)$. Then from (3.7) and (3.8), we see that for $t > t_1$,

$$\begin{aligned} m'(t) &\leq -\frac{r(t_1)}{r(t)} \prod_{t_1 < t_k < t} a_k^{[2]} x''(t_1^+), \quad t \neq t_k \\ m(t_k^+) &\leq b_k^{[1]} m(t_k), \quad k = 2, 3, \ldots. \end{aligned} \tag{3.9}$$

It follows from Lemma 3.1 that

$$\begin{aligned} m(t) &\leq m(t_1^+) \prod_{t_1 < t_k < t} b_k^{[1]} - x''(t_1^+) r(t_1) \int_{t_1}^{t} \frac{1}{r(s)} \prod_{s < t_k < t} b_k^{[1]} \prod_{t_1 < t_k < s} a_k^{[2]} ds \\ &= \prod_{t_1 < t_k < t} b_k^{[1]} \left\{ m(t_1^+) - x''(t_1^+) r(t_1) \int_{t_1}^{t} \frac{1}{r(s)} \prod_{t_1 < t_k < s} \frac{a_k^{[2]}}{b_k^{[1]}} ds \right\}. \end{aligned} \tag{3.10}$$

That is,

$$x'(t) \geq \prod_{t_1 < t_k < t} b_k^{[1]} \left\{ x'(t_1^+) + x''(t_1^+) r(t_1) \int_{t_1}^{t} \frac{1}{r(s)} \prod_{t_1 < t_k < s} \frac{a_k^{[2]}}{b_k^{[1]}} ds \right\}. \tag{3.11}$$

Note that $a_k^{[i]} > 0$, $b_k^{[i]} > 0$, and the second equality of condition (C) holds. Thus we get $x'(t) > 0$ for all sufficiently large t. The relation $x'(t) \leq 0$ leads to a contradiction. Thus, there exists some j such that $t_j \geq t_0$ and $x'(t_j) > 0$. Since $x'(t)$ is increasing on intervals of the form $(t_{j+\lambda}, t_{j+\lambda+1}]$ for $\lambda = 0, 1, 2, \ldots$, thus for $t \in (t_j, t_{j+1}]$, we have

$$x'(t) \geq x'\left(t_j^+\right) \geq a_j^{[1]} x'(t_j) > 0. \tag{3.12}$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$,

$$x'(t) \geq x'\left(t_{j+1}^+\right) \geq a_{j+1}^{[1]} x'(t_{j+1}) \geq a_j^{[1]} a_{j+1}^{[1]} x'(t_j) > 0. \tag{3.13}$$

We can easily prove that, for any positive integer $\lambda \geq 2$ and $t \in (t_{j+\lambda}, t_{j+\lambda+1}]$,

$$x'(t) \geq a_j^{[1]} a_{j+1}^{[1]} \cdots a_{j+\lambda}^{[1]} x'(t_j) > 0. \tag{3.14}$$

Therefore, $x'(t) > 0$ for $t \geq t_j$. Thus, (a) is true.

Next, we will prove that (b) is true. Without loss of generality, we may assume that $x'(t) > 0$ and $x''(t) \geq 0$ for $t \geq t_0$. We assert that there exists some j such that $x(t_j) > 0$ for $t_j \geq t_0$. If this is not true, then for any $t_k \geq t_0$, we have $x(t_k) \leq 0$. Since $x(t)$ is increasing on intervals of the form $(t_k, t_{k+1}]$, we see that $x(t) \leq 0$ for $t \geq t_0$. By $x'(t) > 0, x''(t) \geq 0, t \in (t_k, t_{k+1}]$, we have that $x'(t)$ is nondecreasing on $(t_k, t_{k+1}]$. For $t \in (t_1, t_2]$, we have

$$x'(t) \geq x'\left(t_1^+\right). \tag{3.15}$$

In particular,

$$x'(t_2) \geq x'\left(t_1^+\right). \tag{3.16}$$

Similarly, for $t \in (t_2, t_3]$, we have

$$x'(t) \geq x'\left(t_2^+\right) \geq a_2^{[1]} x'(t_2) \geq a_2^{[1]} x'\left(t_1^+\right). \tag{3.17}$$

By induction, we know that for $t > t_1$,

$$x'(t) \geq \prod_{t_1 < t_k < t} a_k^{[1]} x'\left(t_1^+\right), \quad t \neq t_k. \tag{3.18}$$

From condition (B), we have

$$x\left(t_k^+\right) \geq b_k^{[0]} x(t_k), \quad k = 2, 3, \ldots. \tag{3.19}$$

Set $u(t) = -x(t)$. Then from (3.18) and (3.19), we see that for $t > t_1$,

$$u'(t) \le -\prod_{t_1<t_k<t} a_k^{[1]} x'(t_1^+), \quad t \ne t_k,$$
$$u(t_k^+) \le b_k^{[0]} u(t_k), \quad k = 2,3,\ldots. \tag{3.20}$$

It follows from Lemma 3.1 that

$$\begin{aligned} u(t) &\le u(t_1^+) \prod_{t_1<t_k<t} b_k^{[0]} - x'(t_1^+) \int_{t_1}^{t} \prod_{s<t_k<t} b_k^{[0]} \prod_{t_1<t_k<s} a_k^{[1]} ds \\ &= \prod_{t_1<t_k<t} b_k^{[0]} \left\{ u(t_1^+) - x'(t_1^+) \int_{t_1}^{t} \prod_{t_1<t_k<s} \frac{a_k^{[1]}}{b_k^{[0]}} ds \right\}. \end{aligned} \tag{3.21}$$

That is,

$$x(t) \ge \prod_{t_1<t_k<t} b_k^{[0]} \left\{ x(t_1^+) + x'(t_1^+) \int_{t_1}^{t} \prod_{t_1<t_k<s} \frac{a_k^{[1]}}{b_k^{[0]}} ds \right\}. \tag{3.22}$$

Note that $a_k^{[i]} > 0$, $b_k^{[i]} > 0$, and the first equality of condition (C) holds. Thus we get $x(t) > 0$ for all sufficiently large t. The relation $x(t) \le 0$ leads to a contradiction. So there exists some j such that $t_j \ge t_0$ and $x(t_j) > 0$. Then

$$x\left(t_j^+\right) \ge a_j^{[0]} x(t_j) > 0. \tag{3.23}$$

Since $x'(t) > 0$, we see that $x(t)$ is strictly monotonically increasing on $(t_{j+m}, t_{j+m+1}]$ for $m = 0, 1, 2, \ldots$. For $t \in (t_j, t_{j+1}]$, we have

$$x(t) \ge x\left(t_j^+\right) > 0. \tag{3.24}$$

In particular,

$$x(t_{j+1}) \ge x\left(t_j^+\right) > 0. \tag{3.25}$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$x(t) \ge x\left(t_{j+1}^+\right) \ge a_{j+1}^{[0]} x(t_{j+1}) > 0. \tag{3.26}$$

By induction, we have $x(t) > 0$ for $t \in (t_{j+m}, t_{j+m+1}]$. Thus, we know that $x(t) > 0$, for $t \ge t_j$. The proof of Lemma 3.2 is complete. □

Remark 3.3. We may prove in similar manners the following statements.

(a$'$) If we replace the condition (a) in Lemma 3.2 "$x''(t) > 0$ and $(r(t)x''(t))' \geq 0$ for $t \geq T$" with "$x''(t) < 0$ and $(r(t)x''(t))' \leq 0$ for $t \geq T$", then there exists some $T_1 \geq T$ such that $x'(t) < 0$ for $t \geq T_1$.

(b$'$) If we replace the condition (b) in Lemma 3.2 "$x'(t) > 0$ and $x''(t) \geq 0$ for $t \geq T$" with "$x'(t) < 0$ and $x''(t) \leq 0$ for $t \geq T$", then there exists some $T_1 \geq T$ such that $x(t) < 0$ for $t \geq T_1$.

Lemma 3.4. *Suppose that conditions (A)–(C) hold and $x(t)$ is a solution of* (1.1) *such that $x(t) > 0$ for $t \geq T$, where $T \geq t_0$. Then there exists $T' \geq T$ such that either (a) $x''(t) > 0$, $x'(t) < 0$ for $t \geq T'$ or (b) $x''(t) > 0$, $x'(t) > 0$ for $t \geq T'$.*

Proof. Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0$. By (1.1) and condition (A), we have for $t \geq t_0$.

$$(r(t)x''(t))' = -f(t,x) \leq -p(t)\varphi(x) < 0. \tag{3.27}$$

We assert that for any $t_k \geq t_0, x''(t_k) > 0$. If this is not true, then there exists some j such that $x''(t_j) \leq 0$, so $x''(t_j^+) \leq a_j^{[2]} x''(t_j) \leq 0$. Since $r(t)x''(t)$ is decreasing on $(t_{j+k-1}, t_{j+k}]$ for $k = 1, 2, \ldots$, we see that for $t \in (t_j, t_{j+1}]$,

$$x''(t) < \frac{r(t_j)}{r(t)} x''\left(t_j^+\right) \leq 0. \tag{3.28}$$

In particular,

$$x''(t_{j+1}) < \frac{r(t_j)}{r(t_{j+1})} x''\left(t_j^+\right) \leq 0. \tag{3.29}$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$x''(t) < \frac{r(t_{j+1})}{r(t)} x''\left(t_{j+1}^+\right) \leq \frac{r(t_{j+1})}{r(t)} a_{j+1}^{[2]} x''(t_{j+1}) \leq \frac{r(t_j)}{r(t)} a_{j+1}^{[2]} x''\left(t_j^+\right) \leq 0. \tag{3.30}$$

In particular,

$$x''(t_{j+2}) < \frac{r(t_j)}{r(t_{j+2})} a_{j+1}^{[2]} x''\left(t_j^+\right) \leq 0. \tag{3.31}$$

By induction, for any $t \in (t_{j+n-1}, t_{j+n}]$ for $n = 2, 3, \ldots$, we have

$$x''(t) < \frac{r(t_j)}{r(t)} \prod_{k=1}^{n-1} a_{j+k}^{[2]} x''\left(t_j^+\right) \leq 0. \tag{3.32}$$

Hence, $x''(t) < 0$ for $t \geq t_j$. By Remark 3.3(a′), there exists $T_1 \geq t_j$ such that $x'(t) < 0$ for $t \geq T_1$; by Remark 3.3(b′), we get $x(t) < 0$ for $t \geq T_1$, which is contrary to $x(t) > 0$ for $t \geq t_0$. Hence, for any $t_k \geq t_0, x''(t_k) > 0$, since $r(t)x''(t)$ is decreasing on $(t_{j+k-1}, t_{j+k}]$ for $k = 1, 2, \ldots$, therefore $x''(t) > 0$ for $t \geq t_0$. It follows that $x'(t)$ is strictly increasing on $(t_k, t_{k+1}]$ for $k = 1, 2, \ldots$. Furthermore, note that $a_k^{[1]} > 0$, $k = 1, 2, \ldots$. we see that if for any $t_k, x'(t_k) < 0$, then $x'(t) < 0$ for $t \geq t_0$. If there exists some t_j such that $x'(t_j) \geq 0$, then $x'(t) > 0$ for $t > t_j$. The proof of Lemma 3.4 is complete. □

Lemma 3.5 (see [12]). *Suppose that $x(t)$ is continuous at $t > 0$ and $t \neq t_k$, it is left-continuous at $t = t_k$ and $\lim_{t \to t_k^+} x(t)$ exists for $k = 1, 2, \ldots$. Further assume that*

(H$_2$) *there exists $\bar{t} \in R^+$, such that $x(t) > 0(< 0)$ for $t \geq \bar{t}$;*

(H$_3$) *$x(t)$ is nonincreasing (resp., nondecreasing) on $(t_k, t_{k+1}]$ for $k = 1, 2, \ldots$;*

(H$_4$) *$\sum_{k=1}^{+\infty} [x(t_k^+) - x(t_k)]$ is convergent.*

Then $\lim_{t \to +\infty} x(t) = r$ exists and $r \geq 0$ (resp., ≤ 0).

4. Proofs of Main Theorems

We now turn to the proof of Theorem 2.1. Without loss of generality, we may assume that $k_0 = 1$. If (1.1) has a nonoscillatory solution $x = x(t)$, we first assume that $x(t) > 0$ for $t \geq t_0$. By (1.1) and the condition (A), for $t \geq T \geq t_0$, we get

$$\left(r(t)x''(t)\right)' = -f(t, x(t)) \leq -p(t)\varphi(x(t)), \quad t \neq t_k. \tag{4.1}$$

From the condition (B), we know that

$$r\left(t_k^+\right)x''\left(t_k^+\right) \leq b_k^{[2]} r(t_k)x''(t_k). \tag{4.2}$$

By Lemma 3.4, there exists a $T \geq t_0$ such that either (a) $x''(t) > 0$, $x'(t) < 0$ for $t \geq T$ or (b) $x''(t) > 0$, $x'(t) > 0$ for $t \geq T$.

Suppose that (a) holds. Then we see that the conditions (H$_2$) and (H$_3$) of Lemma 3.5 are satisfied. Furthermore, note that $\sum_{k=1}^{+\infty}(b_k^{[0]} - 1) < +\infty$ and $b_k^{[0]} \geq a_k^{[0]} \geq 1$. Then we have

$$\prod_{k=1}^{+\infty} b_k^{[0]} < +\infty. \tag{4.3}$$

Since $x'(t) < 0, t \geq T$, we obtain for any $t_k > T$,

$$x(t_k) \leq \prod_{T < t_j < t_k} b_j^{[0]} x(T^+). \tag{4.4}$$

By (4.3) and (4.4), we know that the sequence $\{x(t_k)\}$ is bounded. Thus there exists $M > 0$ such that $|x(t_k)| \leq M$. It follows from the condition (B) that

$$\left|x\left(t_k^+\right) - x(t_k)\right| \leq \left|b_k^{[0]} - 1\right| |x(t_k)| \leq M\left(b_k^{[0]} - 1\right). \tag{4.5}$$

From (4.5) and the fact that $\sum_{k=1}^{+\infty}(b_k^{[0]}-1)$ is convergent, we know that $\sum_{k=1}^{+\infty}[x(t_k^+)-x(t_k)]$ is convergent. Therefore, the condition (H_4) of Lemma 3.5 is also satisfied. By Lemma 3.5, we know that $\lim_{t\to+\infty}x(t)=r\geq 0$. We assert that $r=0$. If $r>0$, then there exists $T_1\geq T$ such that for any $t\geq T_1$, $x(t)>r/2>0$. Note further that $\varphi'(x)\geq 0$; so we obtain $\varphi(x(t))\geq\varphi(r/2)$ for $t\geq T_1$. Let $m(t)=r(t)x''(t)$ for $t\geq T_1$. By (4.1) and (4.2), we have

$$m'(t)\leq q(t),\quad t\geq T_1,\ t\neq t_k, \tag{4.6}$$

$$m(t_k^+)\leq b_k^{[2]}m(t_k),\quad t_k\geq T_1, \tag{4.7}$$

where $q(t)=-\varphi(r/2)p(t)$. From (4.6), (4.7), and Lemma 3.1, we get for $t\geq T_1$,

$$\begin{aligned} m(t)&\leq m(T_1^+)\prod_{T_1<t_k<t}b_k^{[2]}+\int_{T_1}^{t}\left(\prod_{s<t_k<t}b_k^{[2]}\right)q(s)ds\\ &=\prod_{T_1<t_k<t}b_k^{[2]}\left\{m(T_1^+)-\varphi\left(\frac{r}{2}\right)\int_{T_1}^{t}\left(\prod_{T_1<t_k<s}\frac{1}{b_k^{[2]}}\right)p(s)ds\right\}. \end{aligned} \tag{4.8}$$

It is easy to see from (2.2) and (4.8) that $m(t)<0$ for sufficiently large t. This is contrary to $m(t)>0$ for $t\geq T_1$. Thus $r=0$, that is, $\lim_{t\to+\infty}x(t)=0$.

Suppose that (b) holds. Let $\Psi(t)=(r(t)x''(t)/\varphi(x(t)))$ for $t\geq T$. Then $\Psi(t)>0$ for $t\geq T$. By (1.1) and the condition (A), we get, for $t\geq T$,

$$\Psi'(t)=\frac{-f(t,x(t))}{\varphi(x(t))}-\frac{r(t)x''(t)\varphi'(x(t))x'(t)}{\varphi^2(x(t))}\leq\frac{-f(t,x(t))}{\varphi(x(t))}\leq -p(t),\quad t\neq t_k. \tag{4.9}$$

From the conditions (A), (B) and $a_k^{[0]}\geq 1$, we know that

$$\Psi(t_k^+)=\frac{r(t_k)x''(t_k^+)}{\varphi(x(t_k^+))}\leq\frac{r(t_k)b_k^{[2]}x''(t_k)}{\varphi\left(a_k^{[0]}x(t_k)\right)}\leq b_k^{[2]}\frac{r(t_k)x''(t_k)}{\varphi(x(t_k))}\leq b_k^{[2]}\Psi(t_k),\quad t_k\geq T. \tag{4.10}$$

From (4.9), (4.10), and Lemma 3.1, we get, for $t\geq T$,

$$\begin{aligned} \Psi(t)&\leq\Psi(T^+)\prod_{T<t_k<t}b_k^{[2]}-\int_{T}^{t}\left(\prod_{s<t_k<t}b_k^{[2]}\right)p(s)ds\\ &=\prod_{T<t_k<t}b_k^{[2]}\left\{\Psi(T^+)-\int_{T}^{t}\left(\prod_{T<t_k<s}\frac{1}{b_k^{[2]}}\right)p(s)ds\right\}. \end{aligned} \tag{4.11}$$

It is easy to see from (2.2) and (4.11) that $\Psi(t)<0$ for sufficiently large t. This is contrary to $\Psi(t)>0$ for $t\geq T$, and hence we obtain a contradiction. Thus in case (b) $x(t)$ must be oscillatory. The proof of Theorem 2.1 is complete.

Next, we give the proof of Theorem 2.2. Without loss of generality, we may assume that $k_0 = 1$. If (1.1) has an eventually positive solution $x = x(t)$ for $t \geq t_0$. By (1.1) and conditions (A) and (B), we have that (4.1) and (4.2) hold. By Lemma 3.4, there exists a $T \geq t_0$ such that either (a) $x''(t) > 0$, $x'(t) < 0$ for $t \geq T$ or (b) $x''(t) > 0$, $x'(t) > 0$ for $t \geq T$.

Suppose that (a) holds. Note that $b_k^{[0]} \leq 1$ and for $t_j \geq T$ and each $l = 0, 1, 2, \ldots$, $x(t)$ is decreasing on $(t_{j+l}, t_{j+l+1}]$; we have for $t \in (t_j, t_{j+1}]$

$$x(t) < x\left(t_j^+\right) \leq b_j^{[0]} x\left(t_j\right) \leq x\left(t_j\right). \tag{4.12}$$

Similarly, for $t \in (t_{j+1}, t_{j+2}]$, we have

$$x(t) < x\left(t_{j+1}^+\right) \leq b_{j+1}^{[0]} x\left(t_{j+1}\right) \leq x\left(t_{j+1}\right) \leq x\left(t_j\right). \tag{4.13}$$

By induction, for each $l = 0, 1, 2, \ldots$, we have

$$x(t) < x\left(t_{j+l}\right) \leq \cdots \leq x\left(t_{j+1}\right) \leq x\left(t_j\right), \quad t \in \left(t_{j+l}, t_{j+l+1}\right] \tag{4.14}$$

so that $x(t)$ is decreasing on $(t_j, +\infty)$. We know that $x(t)$ is convergent as $t \to +\infty$. Let $\lim_{t \to +\infty} x(t) = r$. Then $r \geq 0$. We assert that $r = 0$. If $r > 0$, then there exists $T_1 \geq t_0$, such that for $t \geq T_1, x(t) > r/2 > 0$. Since $\varphi'(x) \geq 0$, then $\varphi(x(t)) \geq \varphi(r/2)$. Let $m(t) = r(t)x''(t)$ for $t \geq T_1$. Then By (4.1) and (4.2), we have that (4.6) and (4.7) hold. From (4.6), (4.7), and Lemma 3.1, we get for $t \geq T_1$,

$$m(+\infty) \leq m(t) \prod_{t<t_k<+\infty} b_k^{[2]} - \varphi\left(\frac{r}{2}\right) \int_t^{+\infty} \prod_{s<t_k<\infty} b_k^{[2]} p(s) ds. \tag{4.15}$$

That is,

$$0 \leq \lim_{t \to +\infty} r(t)x''(t) \leq r(t)x''(t) \prod_{t<t_k<+\infty} b_k^{[2]} - \varphi\left(\frac{r}{2}\right) \int_t^{+\infty} \prod_{s<t_k<\infty} b_k^{[2]} p(s) ds. \tag{4.16}$$

It is easy to see from (4.16) that the following inequality holds:

$$x''(t) \geq \frac{\varphi(r/2)}{r(t)} \int_t^{+\infty} \prod_{t<t_k<s} \frac{1}{b_k^{[2]}} p(s) ds, \quad t \geq T_1. \tag{4.17}$$

Note that $a_k^{[1]} \geq 1$; it follows from integrating (4.17) from t_0 to t and by using the condition (B) that

$$\begin{aligned}
x'(t) - x'(t_0^+) &\geq x'(t) - x'(t_0^+) + \sum_{t_0<t_k<t} \left(a_k^{[1]} - 1\right) x'(t_k) \\
&\geq x'(t) - x'(t_0^+) + \sum_{t_0<t_k<t} \left[x'(t_k^+) - x'(t_k)\right] \\
&\geq \varphi\left(\frac{r}{2}\right) \int_{t_0}^{t} \frac{1}{r(s)} \left(\int_{s}^{+\infty} \prod_{s<t_k<u} \frac{1}{b_k^{[2]}} p(u) du \right) ds.
\end{aligned} \tag{4.18}$$

It is easy to see from (2.4) and (4.18) that $x'(t) > 0$ for sufficiently large t. This is contrary to $x'(t) < 0$ for $t \geq T_1$. Thus $r = 0$, that is, $\lim_{t\to+\infty} x(t) = 0$.

Suppose (b) holds. Without loss of generality, we may assume that $T = t_0$. Then we see that $x'(t) > 0, t \geq t_0$. Since $x(t)$ is nondecreasing on $(t_k, t_{k+1}]$, for $t \in (t_0, t_1]$, we have

$$x(t) \geq x(t_0^+). \tag{4.19}$$

In particular,

$$x(t_1) \geq x(t_0^+). \tag{4.20}$$

Similarly, for $t \in (t_1, t_2]$, we have

$$x(t) \geq x(t_1^+) \geq a_1^{[0]} x(t_1) \geq a_1^{[0]} x(t_0^+). \tag{4.21}$$

By induction, we know that

$$x(t) \geq \prod_{t_0<t_k<t} a_k^{[0]} x(t_0^+), \quad t > t_0. \tag{4.22}$$

That is, $x(t) \geq \prod_{t_0<t_k<t} a_k^{[0]} x(t_0^+)$ for $t > t_0$. Note that $b_k^{[0]} \leq 1$ and $\prod_{t_0\leq t_k<+\infty} a_k^{[0]} \geq \sigma > 0$. From the condition (B), we have $x(t) \geq \sigma x(t_0^+)$. Since $\varphi'(x) \geq 0$, we have $\varphi(x(t)) \geq \varphi(\sigma x(t_0^+))$. Let $m(t) = r(t)x''(t)$; by (4.1) and (4.2), we have, for $t \geq t_0$, that

$$\begin{aligned}
m'(t) &\leq -\varphi(\sigma x(t_0^+)) p(t), \quad t \neq t_k, \\
m(t_k^+) &\leq b_k^{[2]} m(t_k), \quad t_k > t_0.
\end{aligned} \tag{4.23}$$

Similar to the proof of (4.17), we obtain

$$x''(t) \geq \frac{\varphi(\sigma x(t_0^+))}{r(t)} \int_{t}^{+\infty} \prod_{t<t_k<s} \frac{1}{b_k^{[2]}} p(s) ds, \quad t \geq t_0. \tag{4.24}$$

Let $s(t) = -x'(t)$ for $t \geq t_0$. Then $s(t) \leq 0$. By (4.24) and the condition (B), and noting that $a_k^{[1]} \geq 1$, we have for $t \geq t_0$,

$$s'(t) \leq -\frac{\varphi(\sigma x(t_0^+))}{r(t)} \int_t^{+\infty} \prod_{t<t_k<s} \frac{1}{b_k^{[2]}} p(s) ds, \quad t \neq t_k,$$
$$s(t_k^+) \leq a_k^{[1]} s(t_k) \leq s(t_k), \quad t_k \geq t_0. \tag{4.25}$$

By Lemma 3.1, we get

$$0 \leq s(+\infty) \leq s(t) - \varphi(\sigma x(t_0^+)) \int_t^{+\infty} \frac{1}{r(s)} \left(\int_s^{+\infty} \prod_{s<t_k<u} \frac{1}{b_k^{[2]}} p(u) du \right) ds. \tag{4.26}$$

It follows that

$$0 \geqslant x'(t) + \varphi(\sigma x(t_0^+)) \int_t^{+\infty} \frac{1}{r(s)} \left(\int_s^{+\infty} \prod_{s<t_k<u} \frac{1}{b_k^{[2]}} p(u) du \right) ds. \tag{4.27}$$

In view of (4.27), we have, for $t \geq t_0$,

$$x'(t) \leq -\varphi(\sigma x(t_0^+)) \int_t^{+\infty} \frac{1}{r(s)} \left(\int_s^{+\infty} \prod_{s<t_k<u} \frac{1}{b_k^{[2]}} p(u) du \right) ds. \tag{4.28}$$

It is easy to see from (2.4) and (4.28) that $x'(t) < 0$. This is contrary to $x'(t) > 0$ for $t \geq t_0$. Thus in case (b) $x(t)$ must be oscillatory. The proof of Theorem 2.2 is complete.

We now give the proof of Theorem 2.3. Without loss of generality, we may assume that $k_0 = 1$. If (1.1) has an eventually positive solution, $x = x(t)$ for $t \geq t_0$. By Lemma 3.4, there exists a $T \geq t_0$ such that either (a) $x''(t) > 0$, $x'(t) < 0$, $t \geq T$ or (b) $x''(t) > 0$, $x'(t) > 0$, $t \geq T$ holds.

Suppose that (a) holds. Note that $b_k^{[0]} \leq 1$, since for $t_j \geq T$ and each $l = 0, 1, 2, \ldots, x(t)$ is decreasing on $(t_{j+l}, t_{j+l+1}]$; then for $t \in (t_j, t_{j+1}]$, we have

$$x(t) < x\left(t_j^+\right) \leq b_j^{[0]} x(t_j) \leq x(t_j). \tag{4.29}$$

Similarly, for $t \in (t_j, t_{j+1}]$, we have

$$x(t) < x\left(t_{j+1}^+\right) \leq b_{j+1}^{[0]} x(t_{j+1}) \leq x(t_{j+1}) \leq x(t_j). \tag{4.30}$$

By induction, for any $t \in (t_{j+l}, t_{j+l+1}]$ for $l = 0, 1, 2, \ldots$, we have

$$x(t) < x(t_{j+l}) \leq \cdots \leq x(t_{j+1}) \leq x(t_j). \tag{4.31}$$

So $x(t)$ is decreasing and bounded on $(t_j,+\infty)$; we know that $x(t)$ is convergent as $t \to +\infty$. Let $\lim_{t\to+\infty}x(t) = r$, then $r \geq 0$. We assert that $r = 0$. If $r > 0$, then there exists $T_1 \geq T$, such that for $t \geq T_1, x(t) > r/2 > 0$. Since $\varphi'(x) \geq 0$, then $\varphi(x(t)) \geq \varphi(r/2)$. By (1.1) and condition (A), we have for $t \geq T_1$

$$\left(r(t)x''(t)\right)' = -f(t,x) \leq -p(t)\varphi(x(t)) \leq -\varphi\left(\frac{r}{2}\right)p(t) < 0, \quad t \neq t_k, \tag{4.32}$$

From condition (B), and noting that $b_k^{[2]} \leq 1$, we have

$$r\left(t_k^+\right)x''\left(t_k^+\right) \leq b_k^{[2]}r(t_k)x''(t_k) \leq r(t_k)x''(t_k), \quad t_k \geq T_1. \tag{4.33}$$

Let $\Phi(t) = r(t)x''(t)$. Then $\Phi(t) > 0$ for $t \geq T_1$. By (4.32) and (4.33), we have for $t \geq T_1$, that

$$\Phi'(t) \leq -\varphi\left(\frac{r}{2}\right)p(t), \quad t \neq t_k, \tag{4.34}$$

$$\Phi\left(t_k^+\right) \leq \Phi(t_k), \quad t_k \geq T_1. \tag{4.35}$$

From (4.34), (4.35), and Lemma 3.1, we get, for $t \geq T_1$, that

$$\Phi(t) \leq \Phi\left(T_1^+\right) - \varphi\left(\frac{r}{2}\right)\int_{T_1}^{t} p(s)ds, \tag{4.36}$$

It is easy to see from (2.6) and (4.36) that $\Phi(t) \leq 0$ for sufficiently large t. This is contrary to $\Phi(t) > 0$ for $t \geq T_1$. Thus $r = 0$, that is, $\lim_{t\to+\infty}x(t) = 0$.

If (b) holds, let $\Psi(t) = (r(t)x''(t)/\varphi(x(t)))$ for $t \geq T$. We see that $\Psi(t) > 0$ for $t \geq T$. By (1.1) and the condition (A), we get for $t \geq T$

$$\Psi'(t) = \frac{-f(t,x(t))}{\varphi(x(t))} - \frac{r(t)x''(t)\varphi'(x(t))x'(t)}{\varphi^2(x(t))} \leq \frac{-f(t,x(t))}{\varphi(x(t))} \leq -p(t), \quad t \neq t_k. \tag{4.37}$$

From the conditions (A) and (B), we know that

$$\Psi\left(t_k^+\right) = \frac{r(t_k)x''\left(t_k^+\right)}{\varphi\left(x\left(t_k^+\right)\right)} \leq \frac{r(t_k)b_k^{[2]}x''(t_k)}{\varphi\left(a_k^{[0]}x(t_k)\right)} \leq \frac{b_k^{[2]}}{\varphi\left(a_k^{[0]}\right)}\frac{r(t_k)x''(t_k)}{\varphi(x(t_k))} \leq \Psi(t_k), \quad t_k \geq T. \tag{4.38}$$

From (4.37), (4.38), and Lemma 3.1, we get for $t \geq T$

$$\Psi(t) \leq \Psi\left(T^+\right) - \int_{T}^{t} p(s)ds. \tag{4.39}$$

It is easy to see from (2.6) and (4.39) that $\Psi(t) < 0$ for sufficiently large t. This is contrary to $\Psi(t) > 0$ for $t \geq T$. Thus in case (b) $x(t)$ must be oscillatory. The proof of Theorem 2.3 is complete.

Finally, we give the proof of Theorem 2.4. Without loss of generality, we may assume that $k_0 = 1$. If (1.1) has an eventually positive solution, $x = x(t)$ for $t \geq t_0$. By Lemma 3.4, there exists a $T \geq t_0$ such that either (a) $x''(t) > 0$, $x'(t) < 0$, $t \geq T$ or (b) $x''(t) > 0$, $x'(t) > 0$, $t \geq T$ holds.

Suppose that (a) holds. We may easily see that the conditions (H_2), (H_3) of Lemma 3.5 are satisfied. Furthermore, since $x'(t) < 0$, $t \geq T$, then there exists some $t_i \geq T$, such that for $t \in (t_i, t_{i+1}]$

$$x(t) \leq x\left(t_i^+\right). \tag{4.40}$$

In particular,

$$x(t_{i+1}) \leq x\left(t_i^+\right). \tag{4.41}$$

Similarly, we have for $t \in (t_{i+1}, t_{i+2}]$

$$x(t) \leq x\left(t_{i+1}^+\right) \leq b_{i+1}^{[0]} x(t_{i+1}) \leq b_{i+1}^{[0]} x\left(t_i^+\right). \tag{4.42}$$

In particular,

$$x(t_{i+2}) \leq b_{i+1}^{[0]} x\left(t_i^+\right). \tag{4.43}$$

By induction, we obtain for any $t_k > t_i$

$$x(t_k) \leq \prod_{t_i < t_j < t_k} b_j^{[0]} x\left(t_i^+\right). \tag{4.44}$$

Since $\{\prod_{k=1}^{n} b_k^{[0]}\}$ is bounded and (4.44) holds, we know that $\{x(t_k)\}$ is bounded. Thus there exists $M_1 > 0$, such that $|x(t_k)| \leq M_1$. It follows from the condition (B) that

$$\left|x\left(t_k^+\right) - x(t_k)\right| \leq \max\left\{\left|a_k^{[0]} - 1\right|, \left|b_k^{[0]} - 1\right|\right\} |x(t_k)| \leq M_1 \max\left\{\left|a_k^{[0]} - 1\right|, \left|b_k^{[0]} - 1\right|\right\}. \tag{4.45}$$

By (4.45), we know that $\sum_{k=1}^{+\infty} [x(t_k^+) - x(t_k)]$ is convergent. Therefore, the condition (H_4) of Lemma 3.5 is also satisfied. By Lemma 3.5, we know that $\lim_{t \to +\infty} x(t) = r \geq 0$. We assert that $r = 0$. If $r > 0$, then there exists $T_1 \geq T$, such that for $t \geq T_1, x(t) > r/2 > 0$. Since $\varphi'(x) \geq 0$, we have $\varphi(x(t)) \geq \varphi(r/2)$. Since $(r(t)x''(t))' < 0$, $t \geq T_1$, there exists some $t_i \geq T_1$ such that for $t \in (t_i, t_{i+1}]$

$$r(t)x''(t) \leq r(t_i)x''\left(t_i^+\right). \tag{4.46}$$

In particular,

$$r(t_{i+1})x''(t_{i+1}) \leq r(t_i)x''\left(t_i^+\right). \tag{4.47}$$

Similarly, we have for $t \in (t_{i+1}, t_{i+2}]$

$$r(t)x''(t) \leq r(t_{i+1})x''(t_{i+1}^{+}) \leq b_{i+1}^{[2]} r(t_{i+1})x''(t_{i+1}) \leq b_{i+1}^{[2]} r(t_i)x''(t_i^{+}). \tag{4.48}$$

In particular,

$$r(t_{i+2})x''(t_{i+2}) \leq b_{i+1}^{[2]} r(t_i)x''(t_i^{+}). \tag{4.49}$$

By induction, we obtain for any $t_k > t_i$

$$r(t_k)x''(t_k) \leq \prod_{t_i < t_j < t_k} b_j^{[2]} r(t_i)x''(t_i^{+}). \tag{4.50}$$

By $b_k^{[2]} \leq a_k^{[0]}$ and the condition (B), we know that $\{\prod_{k=1}^{n} b_k^{[2]}\}$ is bounded, and from (4.50), we see that $\{r(t_k)x''(t_k)\}$ is bounded. There then exists $M_2 > 0$ such that $|r(t_k)x''(t_k)| \leq M_2$. Therefore, we have

$$\left|\left(b_k^{[2]} - 1\right) r(t_k)x''(t_k)\right| \leq M_2 \left|b_k^{[2]} - 1\right|. \tag{4.51}$$

By (1.1) and the condition (A), we have that (4.1) holds. Integrating (4.1) from T_1 to t, it follows from (4.51) and $\varphi(x(t)) \geq \varphi(r/2)$ for $t \geq T_1$ that

$$\begin{aligned}
r(t)x''(t) - r(T_1)x''(T_1^{+}) &\leq \sum_{T_1 < t_k < t} r(t_k)\left[x''(t_k^{+}) - x''(t_k)\right] - \int_{T_1}^{t} p(s)\varphi(x(s))ds \\
&\leq \sum_{T_1 < t_k < t} r(t_k)\left[x''(t_k^{+}) - x''(t_k)\right] - \varphi\left(\frac{r}{2}\right)\int_{T_1}^{t} p(s)ds \\
&\leq \sum_{T_1 < t_k < t} \left(b_k^{[2]} - 1\right) r(t_k)x''(t_k) - \varphi\left(\frac{r}{2}\right)\int_{T_1}^{t} p(s)ds \\
&\leq \sum_{T_1 < t_k < t} \left|\left(b_k^{[2]} - 1\right) r(t_k)x''(t_k)\right| - \varphi\left(\frac{r}{2}\right)\int_{T_1}^{t} p(s)ds \\
&\leq \sum_{T_1 < t_k < t} M_2\left|b_k^{[2]} - 1\right| - \varphi\left(\frac{r}{2}\right)\int_{T_1}^{t} p(s)ds.
\end{aligned} \tag{4.52}$$

Note that $\sum_{k=1}^{+\infty} |b_k^{[2]} - 1|$ is convergent. Thus it is easy to see from (2.8) and (4.52) that $x''(t) < 0$ for sufficiently large t. This is contrary to $x''(t) > 0$ for $t \geq T$. Thus $r = 0$, that is, $\lim_{t \to +\infty} x(t) = 0$.

Suppose that (b) holds. Let $\Psi(t) = (r(t)x''(t)/\varphi(x(t)))$ for $t \geq T$. We see that $\Psi(t) > 0$ for $t \geq T$. Similar to the proof of (4.39), we also obtain

$$\Psi(t) \leq \Psi(T^{+}) - \int_{T}^{t} p(s)ds. \tag{4.53}$$

It is easy to see from (2.8) and (4.53) that $\Psi(t) < 0$ for sufficiently large t. This is contrary to $\Psi(t) > 0$ for $t \geq T$. Thus in case (b) $x(t)$ must be oscillatory. The proof of Theorem 2.4 is complete.

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