Scaling Laws and Intermittency in Highly Compressible Turbulence

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Abstract. We use large-scale three-dimensional simulations of supersonic Euler turbulence to get a better understanding of the physics of a highly compressible cascade. Our numerical experiments describe non-magnetized driven turbulent flows with an isothermal equation of state and an rms Mach number of 6. We find that the inertial range velocity scaling deviates strongly from the incompressible Kolmogorov laws. We propose an extension of Kolmogorov's K41 phenomenology that takes into account compressibility by mixing the velocity and density statistics and preserves the K41 scaling of the density-weighted velocity $v \equiv \rho^{1/3}u$. We show that low-order statistics of v are invariant with respect to changes in the Mach number. For instance, at Mach 6 the slope of the power spectrum of v is -1.69 and the third-order structure function of v scales linearly with separation. We directly measure the mass dimension of the "fractal" density distribution in the inertial subrange, $D_m \approx 2.4$, which is similar to the observed fractal dimension of molecular clouds and agrees well with the cascade phenomenology.

INTRODUCTION

In the late 1930's, Kolmogorov clearly realized that chances to develop a closed purely mathematical theory of turbulence are extremely low [1].[1] Therefore, the basic approach in [2, 3] (usually referred to as the K41 theory) was to rely on physical intuition and formulate two general statistical hypotheses which describe the universal equilibrium regime of small-scale fluctuations in arbitrary turbulent flow at high Reynolds number. Following the Landau (1944) remark on the lack of universality in turbulent flows [4], and with information extracted from new experimental data, the original similarity hypotheses of K41 were then revisited and refined to account for intermittency effects [5, 6, 7]. While the K41 phenomenology became the cornerstone for all subsequent developments in incompressible turbulence research [e.g., 8], there was no similar result established for compressible flows yet [9, 10]. Historically, compressible turbulence research, preoccupied with a variety of specific engineering applications, was generally

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[1] "An understanding of solutions to the [incompressible] Navier-Stokes equations" yet remains one of the six unsolved grand challenge problems nominated by the Clay Mathematics Institute in 2000 for a $1M *Millennium Prize* [http://www.claymath.org/millennium/].
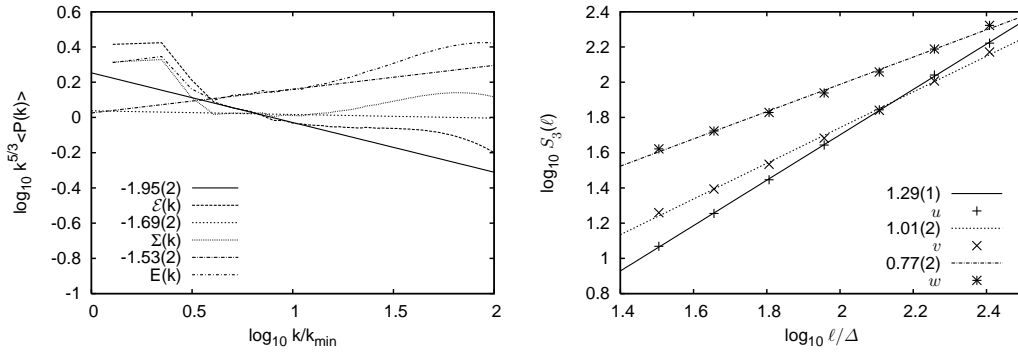
FIGURE 1. Time average compensated power spectra (*left*) and third-order transverse structure functions (*right*) for velocity u and mass-weighted velocities $v \equiv \rho^{1/3}u$ and $w \equiv \rho^{1/2}u$. The statistics of v clearly demonstrate a K41-like scaling. Notice strong bottleneck contamination in the spectra at high wavenumbers.

lagging behind the incompressible developments.[2] The two major reasons for this time lag were an additional complexity of analytical treatment of compressible flows and a shortage in experimental data for super- and hypersonic turbulence. In this respect, although limited to relatively low Reynolds numbers, direct numerical simulations (DNS) of turbulence (pioneered by Orszag and Patterson [14]) have occupied the niche of experiments at least for the most simple flows. One particularly important advantage of DNS is an easy access to variables that are otherwise difficult to measure in the laboratory or treat analytically.

A traditionally straightforward approach to data analysis from DNS of compressible turbulence includes computation of the "standard" statistics of velocity fluctuations. In addition, the diagnostics for density fluctuations are also computed and discussed as the direct measures of compressibility. Quite naturally, both density and velocity statistics demonstrate strong dependence on the Mach number $\mathscr{M}$ in supersonic ($\mathscr{M} \in [1,3]$) and hypersonic ($\mathscr{M} > 3$) regimes, while the variations in turbulent diagnostics at sub- or transonic Mach numbers are rather small.

Based on the data from numerical experiments, it is well established that: (i) the velocity power spectra tend to get steeper as the Mach number increases, reaching the Burgers slope of -2 asymptotically [16, and references therein]; (ii) the density power spectra instead get shallower at high Mach numbers, approaching a slope of -1 or even shallower [15]; (iii) the density PDF in isothermal turbulent flows is well represented by a lognormal distribution [16, and references therein]; (iv) the dimensionality of the most singular velocity structures increases from $D_{s,u} \sim 1$ in a subsonic regime to $D_{s,u} \sim 2$ in highly supersonic [17]; (v) the mass dimension of the turbulent structures decreases from $D_m = 3$ in weakly compressible flows to $D_m \sim 2.5$ in highly compressible [18].

How can we combine these seemingly disconnected pieces of information into a

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[2] A reasonable measure of the delay is 60+ years passed between the appearance of incompressible Reynolds averaging [11] and mass-weighted Favre averaging for fluid flows with variable density [12], although see [13] for references to a few earlier papers that dealt with density-weighted averaging.

coherent physical picture to improve our understanding of compressible turbulence? One way to do this is to consider a phenomenological concept of a *lossy* compressible turbulent cascade that would asymptotically match the incompressible Kolmogorov-Richardson energy cascade [2, 19] in the limit of very low Mach numbers. Since incompressible turbulence represents a degenerate case where the density is uncorrelated with the velocity, the phenomenology of the compressible cascade must include this correlation. This essentially means that instead of velocity, which is a single key ingredient of the K41 laws, one needs to consider a set of mixed variables, $\rho^{1/\eta}u$, where η can take values 1, 2, or 3 depending on the statistical measure of interest [18]. For instance, if one is studying the scale-by-scale kinetic energy budget in a compressible turbulent flow, a mixed variable power spectrum with $\eta = 2$ would be an appropriate choice. To deal with the kinetic energy flux through the hierarchy of scales within the inertial range, the key mixed variable would be the one with $\eta = 3$.

How will these mixed statistics scale in the inertial range of highly compressible turbulent flows? Will their scaling depend on the Mach number? Can the K41 phenomenology be extended to cover hypersonic turbulent flows? These and other related questions are in detail discussed in [18] based on large scale simulations of isotropic supersonic turbulence with resolution up to 2048^3 grid zones. In this paper we present the highlights of the compressible cascade phenomenology verified in [18].

SCALING, STRUCTURES, AND INTERMITTENCY

Nonlinear interactions transfer kinetic energy supplied to the system at large scales through the inertial range with little dissipation. Let us assume that the mean *volume* energy transfer rate in a compressible fluid, $\rho u^2 u/\ell$, is constant in a statistical steady state [e.g., 20]. If this is true, then

$$v^p \equiv (\rho^{1/3}u)^p \sim \ell^{\,p/3} \qquad (1)$$

for an arbitrary power p and, with the standard assumption of self-similarity of the cascade, the structure functions (SFs) of mixed variable v for compressible flows should scale in the inertial range as

$$\mathscr{S}_p(\ell) \equiv \langle |v(r+\ell) - v(r)|^p \rangle \sim \ell^{\,p/3}. \qquad (2)$$

In the limit of weak compressibility, the scaling laws (2) will reduce to the K41 results for the velocity structure functions. The scaling laws $\mathscr{S}_p(\ell) \sim \ell^{\zeta_p}$, where $\zeta_p = p/3$ are not necessarily exact. As the incompressible K41 scaling, they are subject to "intermittency corrections", e.g. $\zeta_p = p/3 + \tau_p/3$ [5]. The only exception is, perhaps, the third order relation for the longitudinal velocity SFs, which is exact in the incompressible case and is known as the *four-fifth law* [3]. Our focus here is mostly on the low order statistics ($p \leq 3$) for which the corrections are small. Since the power spectrum slope is related to the exponent of the second order structure function, the K41 slope of $5/3$ is expected to hold for $v \equiv \rho^{1/3}u$ in the compressible case.

Figure 1 shows the power spectra of u, v, and $w \equiv \rho^{1/2}u$ and the corresponding third-order transverse structure functions based on the simulations at Mach 6 [23, 18]. The
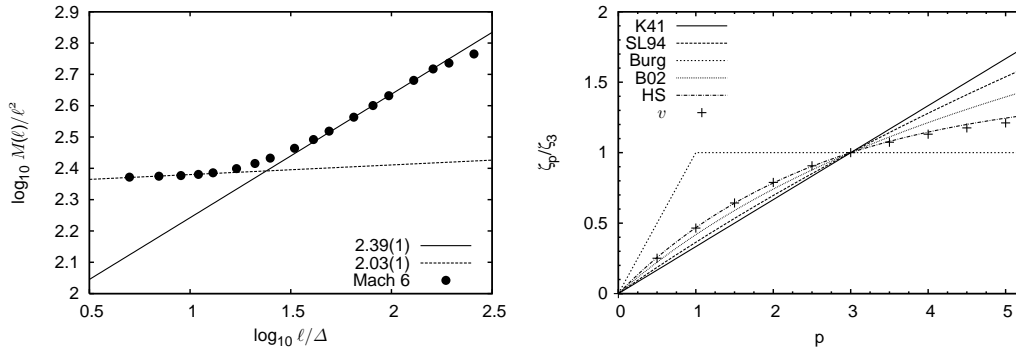
FIGURE 2. Gas mass $M(\ell)$ as a function of the box size ℓ (*left*). The mass dimension D_m is defined as the log-log slope of $M(\ell)$, see eq. (4). Relative exponents for structure functions of the transverse modified velocities v versus order p and the best-fit hierarchical structure model [HS, 6] for $p \in [0,3]$ (*right*). Also shown are model predictions for the Kolmogorov-Richardson cascade [K41, 2, 3], for intermittent incompressible turbulence [SL94, 6], for "burgulence" [Burg, 21], and for the velocity fluctuations in supersonic turbulence [B02, 22].

power spectrum $\Sigma(k)$ and the structure function of v clearly follow the K41 scaling: $\Sigma \sim k^{-1.69}$ and $\mathscr{S}_3 \sim \ell^{1.01}$ [18], while the velocity power spectrum $\mathscr{E}(k)$ and structure function have substantially steeper-then-K41 slopes: -1.95 and 1.29 [23]. At the same time, the kinetic energy spectrum $E \sim k^{-1.53}$ is shallow and both solenoidal and dilatational components of w have the same slope implying a single compressible energy cascade with strong interaction between the two components [18]. These results based on the high dynamic range simulations lend strong support to the scaling relations described by eq. (2) and to the conjecture from which they were inferred. Previous simulations at lower resolution did not allow to measure the absolute exponents reliably due to insufficient dynamic range and due to the bottleneck contamination [24].

In 1951, von Weiszäker [25] introduced a phenomenological model for scale-invariant hierarchy of density fluctuations in compressible turbulence described by a simple equation that relates the mass density at two successive levels to the corresponding scales through a universal measure of the degree of compression, α,

$$\rho_n/\rho_{n-1} = (\ell_n/\ell_{n-1})^{-3\alpha}. \tag{3}$$

The geometric factor α takes the value of 1 in a special case of isotropic compression in three dimensions, $1/3$ for a perfect one-dimensional compression, and zero in the incompressible limit. From equations (1) and (3), assuming mass conservation, Fleck [26] derived a set of scaling relations for the velocity, specific kinetic energy, density, and mass:

$$u \sim \ell^{1/3+\alpha}, \quad \mathscr{E}(k) \sim k^{-5/3-2\alpha}, \quad \rho \sim \ell^{-3\alpha}, \quad M(\ell) \sim \ell^{D_m} \sim \ell^{3-3\alpha}, \tag{4}$$

where all the exponents depend on the compression measure α which is in turn a function of the rms Mach number of the turbulent flow. We can now use the data from numerical experiments to verify the scaling relations (4). Since the first-order velocity structure function scales as $\ell^{0.54}$ [18], we can estimate α for the Mach 6 flow, $\alpha \approx 0.21$.
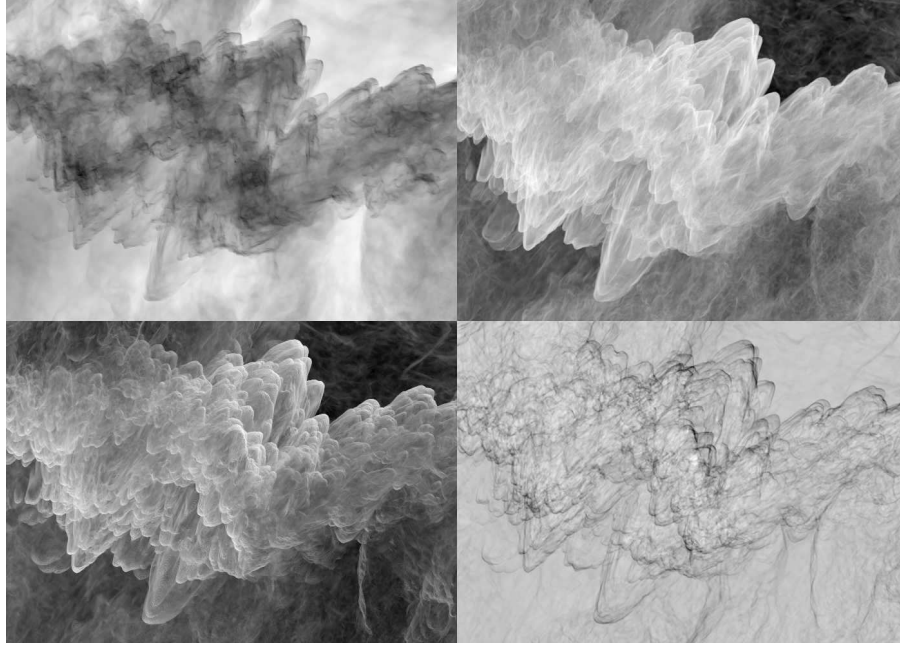
FIGURE 3. Coherent structures in Mach 6 turbulence at resolution of 1024^3. Projections along the minor axis of a subvolume of $700 \times 500 \times 250$ zones for the density (*upper left*), the enstrophy (*upper right*), the dissipation rate (*lower left*), and the dilatation (*lower right*). The logarithmic grey-scale ramp shows the lower values as dark in all cases except for the density. The inertial subrange structures correspond to scales between 40 and 250 zones and represent a fractal with $D_m \approx 2.4$. The dominant structures in the dissipation range ($\ell < 30\Delta$) are shocks with $D_m = 2$. [Reprinted from [18].]

Using the last relation in (4), we can calculate the mass dimension for the density distribution, $D_m \approx 2.38$. It is indeed consistent with our direct measurement of the mass dimension for the same range of scales, $D_m \approx 2.39$, see Fig. 2.

In strongly compressible turbulence at Mach 6, the density contrast between supersonically moving blobs and their more diffuse environment can be as high as 10^6. The most common structural elements in such highly fragmented flows are nested bow-shocks [15]. Figure 3 shows an extreme example of structures formed by a collision of counter-propagating supersonic flows. On small scales within the dissipative range, these structures are characterized by $D_m = 2$, while within the inertial range $D_m \approx 2.4$ (Fig. 2, *left*). The hierarchical structure (HS) model for the mass-weighted velocity v

$$\zeta_p/\zeta_3 = \gamma p + C(1 - \beta^p) \tag{5}$$

[6] predicts the codimension of the support of the most singular dissipative structures $C \equiv 3 - D_{s,v} = (1 - 3\gamma)/(1 - \beta^3) = 1.5$ (Fig. 2 *right*). The best-fit parameters of the model: $\beta^3 = 1/3$ (a measure of intermittency) and $\gamma = 0$ (a measure of singularity of structures) correspond to a hybrid between the B02 model for the velocity fluctuations ($\beta_{B02}^3 = 1/3$, $\gamma_{B02} = 1/9$) [22] and the Burgers' model ($\beta_{Burg} = 0$, $\gamma_{Burg} = 0$) [21]. Thus, Mach 6 turbulence appears to be more intermittent than incompressible turbulence ($\beta_{SL94}^3 = 1/9$) and has the same degree of singularity of structures as burgulence. The dissipative structures with fractal dimension $D_{s,v} = 1.5$ can be conceived as perforated

sheets reminiscent of the Sierpinski sieve. Since the level of noise in high order statistics remains high even at a resolution of 1024^3 grid points, larger dynamic range simulations are needed to determine the parameters of the HS model with better confidence limits.

CONCLUSION

Using large-scale Euler simulations of supersonic turbulence at Mach 6 we have demonstrated that there exists an analogue of the K41 scaling laws valid for both weakly and highly compressible flows. The mass-weighted velocity $v \equiv \rho^{1/3}u$ – the primary variable governing the energy transfer through the cascade – should replace the velocity in intermittency models for compressible flows at high Mach numbers.

ACKNOWLEDGMENTS

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