

Lüders Instruments, Generalised Lüders Theorem, and Some Aspects of Sufficiency

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Abstract A class of Lüders instruments representing quantum measurement is defined, and some their properties are investigated. A generalisation of Lüders theorem is shown to hold for these instruments. It is also shown that the fixed-point algebra of the generalised Lüders operation is sufficient for the family of states determined by the observable associated with the instrument.

Keywords Quantum statistics · Weak sufficiency · Von Neumann algebra · Fixed-points · Instrument

1 Introduction

In 1951 G. Lüders [11] proved that for a complex separable Hilbert space \mathcal{H} and self-adjoint operator A with discrete spectrum and the spectral decomposition $A = \sum_i a_i E_i$, the relation

$$B = \sum_i E_i B E_i$$

holds for a self-adjoint operator B if and only if B commutes with all E_i . This result can be interpreted as follows: if A and B commute, then the outcomes of measurement of an observable represented by B do not depend on whether A has been measured first.

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In 1998 this result was generalised in [2] for unsharp observable A represented by a semispectral measure and for observable B of a special form. Other attempts of generalisation concerned only finite dimensional Hilbert space.

In the present paper, guided by the Lüders theorem, we define a Lüders instrument, compare it with another known classes of instruments such as ideal and strongly repeatable, and obtain as a corollary the Lüders theorem for these instruments. Moreover, we show a result about sufficiency of some important subalgebra for a family of states determined by the observable associated with the instrument.

2 Instruments in Quantum Measurement Theory

In a mathematical model of quantum measurement, a central object describing the process of changing the states of the system under measurement is an instrument. This object was introduced in [3] by E.B. Davies and J.T. Lewis, and some of its properties relevant to our work were further investigated e.g. in [1, 12, 15, 16].

Let (Ω, \mathcal{F}) be a measurable space of values of the bounded observables of a physical system which form a von Neumann algebra \mathcal{M} . By \mathcal{M}^+ we shall denote the positive elements in \mathcal{M} . \mathcal{M}_* will stand for the predual of \mathcal{M} , while \mathcal{M}_*^+ will denote the positive elements in \mathcal{M}_* .

An *instrument* on (Ω, \mathcal{F}) is a map

$$\mathcal{E}: \mathcal{F} \rightarrow \mathcal{L}^+(\mathcal{M}_*)$$

from the σ -field \mathcal{F} into the set of all positive linear transformations on the predual \mathcal{M}_* such that (for the sake of clarity we write the argument at \mathcal{E} as an index)

- (i) $(\mathcal{E}_\Omega \varphi)(\mathbb{1}) = \varphi(\mathbb{1})$ for all $\varphi \in \mathcal{M}_*$,
- (ii) $\mathcal{E}_{\bigcup_{n=1}^\infty \Delta_n} \varphi = \sum_{n=1}^\infty \mathcal{E}_{\Delta_n} \varphi$
for any $\varphi \in \mathcal{M}_*$ and pairwise disjoint sets Δ_n from \mathcal{F} , where the series on the right hand side is convergent in the $\sigma(\mathcal{M}_*, \mathcal{M})$ -topology on \mathcal{M}_* .

In the classical von Neumann theory of measurement we have $\mathcal{M} = \mathbb{B}(\mathcal{H})$ — the algebra of all bounded operators on a Hilbert space \mathcal{H} , and the observable A has the spectral decomposition

$$A = \sum_{n=1}^\infty \lambda_n P_{[\xi_n]},$$

where (ξ_n) is an orthonormal basis of \mathcal{H} , λ_n are distinct real numbers, and $P_{[\xi_n]}$ denotes the projection on the space spanned by vector ξ_n .

The (normalised) state φ of the system corresponds to a density matrix T according to the formula

$$\varphi(B) = \text{tr } TB, \quad B \in \mathbb{B}(\mathcal{H}).$$

If the outcome of measurement was λ_n , then it is assumed that the initial state φ of the system has transformed to the one described by non-normalised density matrix $P_{[\xi_n]} T P_{[\xi_n]}$.

In general, measurement leads to a change of state of the form

$$\varphi \sim T \mapsto \varphi' \sim T' = \sum_{n=1}^\infty P_{[\xi_n]} T P_{[\xi_n]} = \sum_{n=1}^\infty \langle \xi_n | T \xi_n \rangle P_{[\xi_n]}.$$

The above formula leads to a definition of a specific instrument by

$$\mathcal{E}_\Delta \varphi \sim \sum_{\lambda_n \in \Delta} P_{[\xi_n]} T P_{[\xi_n]} = \sum_{\lambda_n \in \Delta} \langle \xi_n | T \xi_n \rangle P_{[\xi_n]}.$$

In particular, the map

$$\mathcal{J}_L : T \mapsto \sum_{n=1}^\infty P_{[\xi_n]} T P_{[\xi_n]},$$

or more general

$$\mathcal{J}_L : T \mapsto \sum_{n=1}^\infty E_n T E_n,$$

where $\sum_{n=1}^\infty E_n = \mathbb{1}$ is a resolution of the identity, is the Lüders operation on the density matrices.

The coefficients $\langle \xi_n | T \xi_n \rangle$ are interpreted as the probabilities of transition of the system from the initial state described by the density matrix T to the state described by the density matrix $P_{[\xi_n]}$. According to our previous description, we have

$$\langle \xi_n | T \xi_n \rangle = \text{tr} P_{[\xi_n]} T P_{[\xi_n]} = (\mathcal{E}_{\{\lambda_n\}} \varphi)(\mathbb{1}).$$

Considering now for each \mathcal{E}_Δ its dual map $\mathcal{E}_\Delta^* : \mathcal{M} \rightarrow \mathcal{M}$ defined by

$$\varphi (\mathcal{E}_\Delta^*(x)) = (\mathcal{E}_\Delta \varphi)(x), \quad \varphi \in \mathcal{M}_*, \quad x \in \mathcal{M},$$

we come to a notion of a *dual instrument* which is defined as a map $\mathcal{E}^* : \mathcal{F} \rightarrow \mathcal{L}_n^+(\mathcal{M})$ from \mathcal{F} into the set of all positive normal linear transformations on \mathcal{M} such that

(i*) $\mathcal{E}_\Omega^*(\mathbb{1}) = \mathbb{1},$

(ii*) $\mathcal{E}_{\bigcup_{n=1}^\infty \Delta_n}^*(x) = \sum_{n=1}^\infty \mathcal{E}_{\Delta_n}^*(x)$

for any $x \in \mathcal{M}$ and pairwise disjoint sets Δ_n from \mathcal{F} , where the series on the right hand side is convergent in the $\sigma(\mathcal{M}, \mathcal{M}_*)$ -topology on \mathcal{M} .

For a given instrument \mathcal{E} , its *associated observable* is defined as a map $e : \mathcal{F} \rightarrow \mathcal{M}$ by the formula

$$e(\Delta) = \mathcal{E}_\Delta^*(\mathbb{1}), \tag{1}$$

thus e is a positive operator valued measure (POVM, semispectral measure). If $e(\Delta)$ is a projection for any Δ , then e is a projection valued measure (PVM, spectral measure).

If the measured system with observable e is in the normalised state φ , we want $\varphi(e(\Delta))$ to be the probability that the observed value is in set Δ which should be equal to $(\mathcal{E}_\Delta \varphi)(\mathbb{1})$. This leads to the equality

$$\varphi(e(\Delta)) = (\mathcal{E}_\Delta \varphi)(\mathbb{1}) = \varphi(\mathcal{E}_\Delta^*(\mathbb{1}))$$

equivalent to (1), and thus justifying the definition of observable.

By \mathcal{N} we denote the W^* -algebra generated by e , that is

$$\mathcal{N} = W^*({e(\Delta) : \Delta \in \mathcal{F}}) = {e(\Delta) : \Delta \in \mathcal{F}}''.$$

The notion of non-degeneracy was introduced in [3] for a class of instruments. A natural generalisation of this notion is as follows. Instrument \mathcal{E} is said to be *non-degenerate*

(faithful) if the family $\{\mathcal{E}_\Omega\varphi : \varphi \in \mathcal{M}_*^+\}$ of normal positive functionals is faithful, i.e. for any $x \in \mathcal{M}^+$ the equality

$$(\mathcal{E}_\Omega\varphi)(x) = 0$$

for all $\varphi \in \mathcal{M}_*^+$, implies $x = 0$. It is easily seen that \mathcal{E} is non-degenerate if and only if \mathcal{E}_Ω^* is a faithful map, i.e. for any $x \in \mathcal{M}^+$ the equality $\mathcal{E}_\Omega^*(x) = 0$ implies $x = 0$.

3 Repeatable and Ideal Measurements

Weakly repeatable and repeatable measurements are important classes of measurements. Roughly speaking, they express the celebrated von Neumann repeatability hypothesis which says: *if the physical quantity is measured twice in succession in a system, then we get the same value each time* (cf. [3, 14]). Following [3], the measurement is said to be *weakly repeatable* if the instrument describing it satisfies the condition:

$$(\mathcal{E}_{\Delta_1}\mathcal{E}_{\Delta_2}\varphi)(\mathbb{1}) = (\mathcal{E}_{\Delta_1 \cap \Delta_2}\varphi)(\mathbb{1}),$$

for all $\Delta_1, \Delta_2 \in \mathcal{F}$ and any $\varphi \in \mathcal{M}_*$.

A measurement is said to be *repeatable* if the instrument describing it satisfies the condition

$$\mathcal{E}_{\Delta_1}\mathcal{E}_{\Delta_2}\varphi = \mathcal{E}_{\Delta_1 \cap \Delta_2}\varphi,$$

for all $\Delta_1, \Delta_2 \in \mathcal{F}$ and any $\varphi \in \mathcal{M}_*$, i.e.

$$\mathcal{E}_{\Delta_1}\mathcal{E}_{\Delta_2} = \mathcal{E}_{\Delta_1 \cap \Delta_2}.$$

Instead of repeatable (weakly repeatable) measurements we shall speak of repeatable (weakly repeatable) instruments. In terms of the dual instrument, weak repeatability is described by the condition

$$\mathcal{E}_{\Delta_1}^*(e(\Delta_2)) = \mathcal{E}_{\Delta_2}^*(e(\Delta_1)) = e(\Delta_1 \cap \Delta_2), \quad \Delta_1, \Delta_2 \in \mathcal{F}. \tag{2}$$

In particular, we have for each $\Delta \in \mathcal{F}$, and $\Delta' = \Omega \setminus \Delta$,

$$\mathcal{E}_{\Delta'}^*(e(\Delta)) = e(\Delta' \cap \Delta) = 0. \tag{3}$$

In terms of the dual instrument repeatability means that

$$\mathcal{E}_{\Delta_1}^*\mathcal{E}_{\Delta_2}^* = \mathcal{E}_{\Delta_1 \cap \Delta_2}^* \quad \Delta_1, \Delta_2 \in \mathcal{F}.$$

It is obvious that every repeatable instrument is weakly repeatable. Our first result shows that the observables of weakly repeatable non-degenerate instruments are spectral measures.

Theorem 1 *Let \mathcal{E} be a weakly repeatable instrument such that $\mathcal{E}_\Omega^*|\mathcal{N}$ is a faithful map. Then the observable of \mathcal{E} is a spectral measure.*

Proof Let e be the observable of \mathcal{E} . For each $\Delta \in \mathcal{F}$ we have

$$\mathbb{1} = e(\Delta) + e(\Delta'),$$

and multiplying both sides of the above equality by $e(\Delta)$ we obtain

$$e(\Delta) = e(\Delta)^2 + e(\Delta)e(\Delta') = e(\Delta)^2 + e(\Delta')e(\Delta). \tag{4}$$

In particular, $e(\Delta)e(\Delta') \geq 0$, and the weak repeatability of \mathcal{E} yields on account of equality (3)

$$0 = \mathcal{E}_{\Delta'}^*(e(\Delta)) = \mathcal{E}_{\Delta'}^*(e(\Delta)^2) + \mathcal{E}_{\Delta'}^*(e(\Delta)e(\Delta')).$$

From the positivity of $e(\Delta)^2$ and $e(\Delta)e(\Delta')$ we get

$$\mathcal{E}_{\Delta'}^*(e(\Delta)e(\Delta')) = 0.$$

Analogously, we find that

$$\mathcal{E}_{\Delta}^*(e(\Delta)e(\Delta')) = 0,$$

thus

$$\mathcal{E}_{\Omega}^*(e(\Delta)e(\Delta')) = \mathcal{E}_{\Delta}^*(e(\Delta)e(\Delta')) + \mathcal{E}_{\Delta'}^*(e(\Delta)e(\Delta')) = 0.$$

Since \mathcal{E}_{Ω}^* is faithful on \mathcal{N} and $e(\Delta)e(\Delta') \geq 0$, we obtain that

$$e(\Delta)e(\Delta') = 0,$$

and equality (4) yields

$$e(\Delta) = e(\Delta)^2,$$

that is $e(\Delta)$ is a projection. □

Another class of instruments considered in [3] consists of strongly repeatable instruments defined as follows. Let $\Omega = \{\lambda_1, \lambda_2, \dots\}$ be a countable set. Instrument \mathcal{E} defined on Ω is said to be *strongly repeatable* if it is repeatable, faithful, and satisfies the following condition of *minimal disturbance*: for each $\varphi \in \mathcal{M}_*^+$, and each n

$$(\mathcal{E}_n\varphi)(\mathbb{1}) = \varphi(\mathbb{1}) \quad \text{implies} \quad \mathcal{E}_n\varphi = \varphi, \tag{5}$$

where \mathcal{E}_n stands for $\mathcal{E}_{\{\lambda_n\}}$. In terms of the dual instrument condition (5) takes the form: for each $\varphi \in \mathcal{M}_*^+$, and each n

$$\text{supp } \varphi \leq e_n \quad \text{implies} \quad \varphi \circ \mathcal{E}_n^* = \varphi, \tag{6}$$

where $e_n = e(\{\lambda_n\}) = \mathcal{E}_n^*(\mathbb{1})$ is the observable associated with \mathcal{E} , and $\text{supp } \varphi$ stands for the *support* of φ , i.e. the smallest projection $p \in \mathcal{M}$ such that $\varphi(p) = \varphi(\mathbb{1})$.

The idea of *minimal global disturbance* caused by measurement can be expressed as follows: suppose that a physical system is in an arbitrary state φ . Then after measurement its state is $\mathcal{E}_{\Omega}\varphi$. Now, if we have another observable $x \in \mathcal{M}$, $0 \leq x \leq \mathbb{1}$, compatible with the associated observable e , i.e. $x \in \mathcal{M} \cap \mathcal{N}'$, and such that

$$\varphi(x) = \varphi(\mathbb{1}),$$

then we want to also have

$$(\mathcal{E}_{\Omega}\varphi)(x) = (\mathcal{E}_{\Omega}\varphi)(\mathbb{1}).$$

Instruments satisfying this condition are called *ideal*, and were investigated in [1, 12]. The following result is a generalisation of the one proved in [1] for the full algebra $\mathbb{B}(\mathcal{H})$.

Theorem 2 *Let \mathcal{E} be an ideal instrument with the associated observable being a spectral measure. Then \mathcal{E} is repeatable.*

Proof On account of [12, Theorem 2] we have

$$\mathcal{E}_{\Delta}^*(x) = e(\Delta)\mathcal{E}_{\Omega}^*(x), \quad x \in \mathcal{M}, \Delta \in \mathcal{F}, \tag{7}$$

where \mathcal{E}_Ω^* is a normal conditional expectation onto $\mathcal{M} \cap \mathcal{N}'$. Thus for each $x \in \mathcal{M}$ and any $\Delta_1, \Delta_2 \in \mathcal{F}$ we obtain

$$\begin{aligned} \mathcal{E}_{\Delta_1}^*(\mathcal{E}_{\Delta_2}^*(x)) &= \mathcal{E}_{\Delta_1}^*(e(\Delta_2)\mathcal{E}_\Omega^*(x)) = e(\Delta_1)\mathcal{E}_\Omega^*(e(\Delta_2)\mathcal{E}_\Omega^*(x)) \\ &= e(\Delta_1)e(\Delta_2)\mathcal{E}_\Omega^*(\mathcal{E}_\Omega^*(x)) = e(\Delta_1 \cap \Delta_2)\mathcal{E}_\Omega^*(x) \\ &= \mathcal{E}_{\Delta_1 \cap \Delta_2}^*(x), \end{aligned}$$

since from the fact that e is a spectral measure it follows that

$$e(\Delta_1)e(\Delta_2) = e(\Delta_1 \cap \Delta_2).$$

Thus

$$\mathcal{E}_{\Delta_1}^* \mathcal{E}_{\Delta_2}^* = \mathcal{E}_{\Delta_1 \cap \Delta_2}^*,$$

showing that \mathcal{E} is repeatable. □

As an interesting consequence of the result above we obtain

Corollary 1 *For ideal instruments weak repeatability and repeatability coincide.*

Indeed, if an ideal instrument is weakly repeatable, then on account of [12, Theorem 4] its observable is a spectral measure, thus Theorem 2 yields its repeatability.

Let $\Omega = \{\lambda_1, \lambda_2, \dots\}$, and let (e_n) be a discrete spectral measure on Ω , i.e.

$$e(\{\lambda_n\}) = e_n.$$

In line with our earlier considerations, we define the Lüders instrument by the formula

$$\mathcal{E}_\Delta \varphi = \sum_{\lambda_n \in \Delta} e_n \varphi e_n, \quad \varphi \in \mathcal{M}_*, \Delta \subset \Omega,$$

where

$$(e_n \varphi e_n)(x) = \varphi(e_n x e_n), \quad x \in \mathcal{M}, \varphi \in \mathcal{M}_*.$$

In terms of the dual instrument this reads

$$\mathcal{E}_\Delta^*(x) = \sum_{\lambda_n \in \Delta} e_n x e_n, \quad x \in \mathcal{M}, \Delta \subset \Omega,$$

or equivalently

$$\mathcal{E}_n^*(x) = e_n x e_n, \quad x \in \mathcal{M}.$$

For the Lüders operation \mathcal{J}_L , we clearly have $\mathcal{J}_L = \mathcal{E}_\Omega^*$, and the Lüders theorem mentioned in the Introduction says that

$$\text{Fix } \mathcal{J}_L = \mathcal{M} \cap \mathcal{N}'.$$

Remark The Lüders instrument defined above was under the name of *Lüders-von Neumann* or *strongly repeatable instrument* introduced in [12]. However, the name *strongly repeatable* used there was a little misleading since it was not shown that the Lüders-von Neumann instrument is indeed strongly repeatable in the sense of the general definition presented above. In a theorem that follows, we show this. □

Now we are going to compare the three classes of instruments: strongly repeatable, ideal and Lüders. Observe first that by definition both strongly repeatable and Lüders instruments are discrete (i.e. defined on a countable space Ω), and have as their associated observables spectral measures (for strongly repeatable instruments this follows from Theorem 1). However, this is not the case for ideal instruments since they can be defined on an arbitrary

space Ω and can have as observable an arbitrary semispectral measure. Still, if we restrict attention to discrete instruments with spectral measures as the observables, then we have

Theorem 3 *Let \mathcal{E} be a discrete instrument with the associated observable being a spectral measure. Then the following conditions are equivalent.*

- (i) \mathcal{E} is ideal,
- (ii) \mathcal{E} is strongly repeatable,
- (iii) \mathcal{E} is Lüders.

Proof We employ the notation used before, so let (e_n) be the observable of the instrument:

$$e_n = \mathcal{E}_n^*(\mathbb{1}).$$

(i) \implies (ii). From Theorem 2 it follows that \mathcal{E} is repeatable.

Suppose that $\mathcal{E}_\Omega^*(x) = 0$ for $x \in \mathcal{M}^+$. We have $e_n x e_n \in \mathcal{M} \cap \mathcal{N}'$, so

$$0 = e_n \mathcal{E}_\Omega^*(x) e_n = \mathcal{E}_\Omega^*(e_n x e_n) = e_n x e_n = \left(e_n x^{\frac{1}{2}}\right) \left(e_n x^{\frac{1}{2}}\right)^*,$$

which yields

$$e_n x^{\frac{1}{2}} = 0,$$

for each n . Summing up on n we get

$$x^{\frac{1}{2}} = 0.$$

Consequently, $x = 0$, showing that \mathcal{E}_Ω^* is faithful.

Finally, let $\text{supp } \varphi \leq e_n$. Then from formula (7) and the already employed equality $\mathcal{E}_\Omega^*(e_n x e_n) = e_n x e_n$ we obtain

$$\begin{aligned} \varphi(\mathcal{E}_n^*(x)) &= \varphi(e_n \mathcal{E}_\Omega^*(x)) = \varphi(e_n \mathcal{E}_\Omega^*(x) e_n) \\ &= \varphi(\mathcal{E}_\Omega^*(e_n x e_n)) = \varphi(e_n x e_n) = \varphi(x), \end{aligned}$$

showing that $\varphi \circ \mathcal{E}_n^* = \varphi$.

(ii) \implies (iii). For arbitrary $x \in \mathcal{M}$, $0 \leq x \leq \mathbb{1}$, we have

$$0 \leq \mathcal{E}_n^*(x) \leq \mathcal{E}_n^*(\mathbb{1}) = e_n,$$

and thus

$$\mathcal{E}_n^*(x) = e_n \mathcal{E}_n^*(x) e_n. \tag{8}$$

Since the elements x as above span the whole of \mathcal{M} , equality (8) holds for all $x \in \mathcal{M}$ which means that $\mathcal{E}_n^*(\mathcal{M}) \subset e_n \mathcal{M} e_n$. Let φ be an arbitrary normal state on \mathcal{M} such that $\text{supp } \varphi \leq e_n$. From relation (6) we obtain

$$\varphi(\mathcal{E}_n^*(x)) = \varphi(x) = \varphi(e_n x e_n), \quad x \in \mathcal{M}.$$

Since normal states on \mathcal{M} with the supports contained in e_n separate the points of $e_n \mathcal{M} e_n$, and since $\mathcal{E}_n^*(x)$ and $e_n x e_n$ are in $e_n \mathcal{M} e_n$, we get the formula

$$\mathcal{E}_n^*(x) = e_n x e_n, \quad x \in \mathcal{M},$$

which shows that \mathcal{E} is a Lüders instrument.

(iii) \implies (i). For arbitrary $x \in \mathcal{M} \cap \mathcal{N}'$ we have

$$\mathcal{E}_\Omega^*(x) = \sum_n e_n x e_n = \left(\sum_n e_n\right) x = x,$$

and [12, Lemma 1] yields that \mathcal{E} is ideal. □

Remark The equivalence (i) \iff (iii) was proved in [12, Theorem 3]. □

Remark It is interesting to compare a simple proof of the above theorem obtained by an application of algebraic methods with a long and complicated proof of the equivalence (ii) \iff (iii) for the particular case $\mathcal{M} = \mathbb{B}(\mathcal{H})$ in [3, Theorem 10], referring to concrete constructions in $\mathbb{B}(\mathcal{H})$. □

As seen from Theorem 3, the ideal instruments are a natural generalisation of Lüders instruments so it is not surprising that the following generalisation of the Lüders result holds true.

Generalised Lüders Theorem *Let \mathcal{E} be an ideal instrument with the observable being a spectral measure. Then*

$$\text{Fix } \mathcal{E}_\Omega^* = \mathcal{M} \cap \mathcal{N}'.$$

Proof Since the observable of \mathcal{E} is a spectral measure, [12, Theorem 1] yields the inclusion

$$\text{Fix } \mathcal{E}_\Omega^* \subset \mathcal{E}_\Omega^*(\mathcal{M}) \subset \mathcal{M} \cap \mathcal{N}',$$

and since \mathcal{E} is ideal, [12, Lemma 1] yields the inclusion

$$\mathcal{M} \cap \mathcal{N}' \subset \text{Fix } \mathcal{E}_\Omega^*,$$

showing the claim. □

4 Sufficiency

Let \mathcal{M} be a von Neuman algebra, let \mathcal{R} be a von Neumann subalgebra of \mathcal{M} , and let $\mathcal{K}_* \subset \mathcal{M}_*$ be a family of normal states. \mathcal{R} is said to be *sufficient* for the family \mathcal{K}_* if there exists a linear normal unital positive map $\alpha: \mathcal{M} \rightarrow \mathcal{R}$ such that for each $\varphi \in \mathcal{K}_*$ we have

$$\varphi \circ \alpha = \varphi.$$

If, moreover, α is two-positive, then \mathcal{R} is said to be *sufficient in Petz’s sense*, while if α is a conditional expectation onto \mathcal{R} , then \mathcal{R} is said to be *sufficient in Umegaki’s sense*. Obviously, sufficiency in Umegaki’s sense implies sufficiency in Petz’s sense which in turn implies sufficiency. Some aspects of sufficiency, Petz’s sufficiency, and Umegaki’s sufficiency were investigated in [13], [6, 7], and [17, 18], respectively, while in [8, 9] yet another notion of sufficiency was investigated.

Let e be a semispectral measure. Normal states φ and ρ are said to be *e-equivalent*, denoted by $\varphi \underset{e}{\sim} \rho$, if for each $\Delta \in \mathcal{F}$ we have

$$\varphi(e(\Delta)) = \rho(e(\Delta)),$$

i.e. if the probability measures $\varphi \circ e$ and $\rho \circ e$ coincide. It is clear that $\underset{e}{\sim}$ is an equivalence relation; by $[\varphi]_e$ we shall denote the equivalence class of the state φ determined by relation $\underset{e}{\sim}$.

The idea of distinguishing states by a semispectral measure was considered in [4]. Following it, the state φ is said to be *determined by e* if $[\varphi]_e = \{\varphi\}$. By \mathcal{D}_e we shall denote the set of all states determined by e , thus

$$\mathcal{D}_e = \{\varphi \in \mathcal{M}_* : \varphi - \text{state}, [\varphi]_e = \{\varphi\}\}.$$

It was shown in [10] that if e is a randomised semispectral measure, i.e.

$$e(\Delta_1)e(\Delta_2) = e(\Delta_2)e(\Delta_1), \quad \text{for all } \Delta_1, \Delta_2 \in \mathcal{F},$$

then the algebra $\mathcal{N} = W^*({e(\Delta) : \Delta \in \mathcal{F}})$ is sufficient in Petz's sense for the family \mathcal{D}_e .

For instruments we obtain

Theorem 4 *Let \mathcal{E} be an instrument with the associated observable e being a spectral measure.*

- (i) *If \mathcal{E} is weakly repeatable, then $\mathcal{M} \cap \mathcal{N}'$ is sufficient for \mathcal{D}_e .*
- (ii) *If \mathcal{E} is ideal, then $\mathcal{M} \cap \mathcal{N}'$ is sufficient in Umegaki's sense for \mathcal{D}_e .*

Proof In either case we have

$$\mathcal{E}_\Omega^*(e(\Delta)) = e(\Delta), \quad \Delta \in \mathcal{F},$$

where e is the associated observable of \mathcal{E} . For weakly repeatable instruments this follows from formula (2), while for ideal instruments this follows from the fact that \mathcal{E}_Ω^* is a conditional expectation onto $\mathcal{M} \cap \mathcal{N}'$. Consequently, for arbitrary normal state φ the following equality holds

$$(\varphi \circ \mathcal{E}_\Omega^*)(e(\Delta)) = \varphi(e(\Delta)), \quad \Delta \in \mathcal{F},$$

which means that $\varphi \circ \mathcal{E}_\Omega^* \underset{e}{\sim} \varphi$, i.e. $\varphi \circ \mathcal{E}_\Omega^* \in [\varphi]_e$. Now if $\varphi \in \mathcal{D}_e$, then $[\varphi]_e = \{\varphi\}$, and we obtain

$$\varphi \circ \mathcal{E}_\Omega^* = \varphi,$$

showing that φ is \mathcal{E}_Ω^* -invariant. By virtue of [12, Theorem 1], we have $\mathcal{E}_\Omega^*(\mathcal{M}) \subset \mathcal{M} \cap \mathcal{N}'$ which shows that $\mathcal{M} \cap \mathcal{N}'$ is sufficient, while for an ideal instrument \mathcal{E}_Ω^* is a conditional expectation onto $\mathcal{M} \cap \mathcal{N}'$ which shows that $\mathcal{M} \cap \mathcal{N}'$ is sufficient in Umegaki's sense. \square

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