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The Elusive Slope

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Abstract In understanding upper secondary school students' interpretations of information in graphical representations of a distance–time graph and an ECG graph, little attention has been paid to the analysis of the condition of the conceptual development related to their utterances. Understanding this better can help improve the teaching of interpretations of information in graphical representations of different situations. This paper integrates results from 2 studies and 3 theoretical perspectives: Tall and Vinner's theoretical perspectives on learning, Chi's ontological perspectives on conceptual development and Friel's theoretical perspectives on interpretation of graphical information. The findings provide evidence to support the conjecture that iconic interpretations could be stimulated and generated as a result of student categorisation of a distance–time graph as an event, when in fact the graph is being used to describe and communicate a process. The outcome further indicates that students found a resemblance between the ECG diagram and the periodic function of $f(x) = \sin(x)$.

Introduction

Mathematical representations such as diagrams, histograms, functions, graphs, tables and symbols facilitate understanding and communication of abstract mathematical concepts or other situations described in mathematical terms (Elby, 2000; Leinhardt, Zaslavsky, & Stein, 1990). Nevertheless, humans of today are facing a world that is shaped by increasingly complex, dynamic and powerful systems of information through various media. Being able

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to interpret, understand and work with graphical representations involves mathematical processes the student needs to appreciate, comprehend and be able to address when facing interpretation challenges (Friel, Curcio & Bright, 2001).

For mathematics education in an elementary, middle, lower secondary and upper secondary perspective, teachers use different representations in order to make it possible for students to gradually understand more and more complex mathematical objects and concepts. Geometrical constructions, graphs of functions and a variety of diagrams of different kinds are used to introduce new concepts and to study relations, dependency and change (Trigueros & Martínez-Planell, 2010). Mathematical representations, structures and constructions are also used in different scientific branches, such as biology, chemistry, physics or social science. It is of major importance that students learn how to interpret graphical representations in a scientific and successful way.

Understanding a graphical representation of a situation requires different concepts be incorporated in the specific representation. The critical problem of transition between and within representations has been addressed in several studies (Breidenbach, Hawks, Nichols & Dubinsky, 1992; Janvier, 1987; Sfard, 1992). They claim that bridging the gap between algebraic and graphic representations depends highly on how students encapsulated relevant concepts involved in the representation.

We acknowledge that slope is a universal topic in every country's mathematics curricula. It is usually introduced with linear functions. It is central for describing the behaviour of a curve and has an essential role in the development of calculus (Lobato & Thanheiser, 2002; Stump, 1997, 1999, 2001a, b; Zaslavsky, Sela & Leron, 2002). These investigations on the understanding of slope have made valuable contributions to understanding what makes this concept so complex to learn.

Research Questions

- How do students interpret and understand graphical representations of a situation?
- How do students use their interpretation in order to investigate special features of the situation at hand?

Theoretical Framework

A good deal of research has been conducted to investigate student's alternative conception about scientific concepts (diSessa, Hammer, Sherin & Kolpakowski, 1991; diSessa, 1993; Elby, 2000; Hammer, 2000; McDermott & Schaffer, 2005). In our study, we use the term *alternative conception* instead of misconception, which is associated with inaccuracy and mistakes. diSessa (1993) claimed that humans gradually acquire an elaborate sense of mechanism—a sense of how things work in dealing with the physical world, what sorts of events are necessary, likely, possible or impossible. Control of the physical world is one function for the sense of mechanism and, in addition, of humans being capable of taking actions with appropriate consequences. diSessa et al. (1991) refer to the present view of the sense of mechanism as "knowledge in pieces".

This view of physics understanding and physics learning is knowledge-based. It assumes only a few very simple cognitive mechanisms, although the resulting knowledge system is conjectured to be large and complex. The central focus in addressing

these issues is a hypothetical knowledge structure called *phenomenological primitive*, p-prim for short, having these properties (diSessa, 1993). P-prims are rather small knowledge structures, typically involving configurations of only a few parts, which act largely by being recognised in a physical system or in the system's behaviour or hypothesized behaviour.

Kohl (2001) claims that interpretations of graphical representations are sometimes influenced by intuitive alternative ideas. There have been many studies on students' interpretations and conceptions of graphical information. diSessa et al. (1991) studied how students in compulsory school "reinvented" graphical representation of motion. Nemirovsky & Rubin (1992) studied student's interpretations of height and slope and how students confound these concepts when asked to articulate the relation between graphs and reality. Nemirovsky (1994) found that negative velocity is especially difficult since it is often interpreted as lower velocity.

Many mathematical concepts are encountered in some form or other before they are formally defined. Tall (2004) claims that mathematical thinking is strongly related to the cognitive processes that give rise to mathematical knowledge. Learning or conceptual development in mathematics is seen as a change in the individual's concept image.

> 23We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. …The concept definition [is] a form of words used to specify that concept. We shall call the portion of the concept image which is activated at a particular time the evoked concept image. At different times, seemingly conflicting images may be evoked. (Tall & Vinner, 1981, p. 152)

Tall (2004) suggests a possible categorisation of cognitive growth into three worlds of mathematics or three distinct but interacting developments. Three worlds of mathematics are founded on the assumption that the learning of mathematical concepts is individual and develops at different stages: through perception, through symbols or through axioms.

The first world is the *conceptual–embodied world*, the world we meet through perception, the visual and spatial mathematical world. Most of us have a concept image of a circle. A circle is round, it may be large or small, and it may be red or blue. We have not learned this through educational efforts; instead, we have learned this through the physical world and through observations. The first mathematical world consists of objects we have discovered and observed in the real world, knowledge we have gained through our senses. It also contains mental conceptions of non-existing objects such as a point with no size and lines with no thickness. This is in alignment with Lakoff and Nunez's (2000) claims that mathematics draws on and is founded in bodily experience.

The second world is the *proceptual–symbolic world*. In this world, we find symbols and actions that we have to perform when we, for example, are dealing with manipulations in algebra. In this mathematical world is the concept of *procept* central, which consists of the first part of *process* and the end of the word *concept*. Gray and Tall (1994) introduced the concept procept to describe a central part of the learning of mathematical concepts. Gray and Tall (1994) underlined that it is important to learn how to apprehend mathematical symbols both as concepts and as parts of a process at the same time.

An elementary procept is the amalgam of three components: a process which produces a mathematical object, and a symbol which is used to represent either process or object. (Gray & Tall, 1994, p. 12)

According to Gray and Tall (1994) 2 + 3 may be perceived as a process (addition) or as a concept (sum). When the individual is in this symbolic world, he/she may use and reflect over the mathematical symbolic language and its function, meaning and application.

The third mathematical world is the *formal mathematical world*. Is this world are axioms, theorems and proofs in focus. Based on given assumptions regarding the proportion and relation between mathematical objects are axiom-based structures built and used as foundations for mathematical theorems. Mathematical thinking is thereby based on perception developing subtly in sophistication through the mental world of conceptual embodiment, operations developing through actions that become mathematical operations in a world of operational symbolism and increasingly subtle use of verbal reasoning that leads to formal aspects of embodiment and symbolism and, eventually, to a world of axiomatic formalism. The development takes account of the individual's previous experience which may operate successfully in one context yet remain supportive or become problematic in another, giving rise to emotional reactions to mathematics, leading to a spectrum of success and failure over the longer term (Tall, 2004).

The theories about cognitive development of mathematical knowledge articulated in Tall and Vinner (1981) and Gray and Tall (1994) are in many ways quite comparable with the historical development of mathematics as an axiomatic science.

When we face a graph in a coordinate system, we need to identify attributes of the construction of the coordinate system. We are able to make sense of the graph by referring to the coordinate system if we manage to decode the meaning of the units for the axis and the behaviour of the graph. For example, Friel et al., (2001) claim that there are three different cognitive conditions of individual understanding when interacting with graphical representations: to read the features of the graph, to read between the features of the graph and to read beyond the features of the graph. The three aspects might be seen as "interpreting the axes individually", "seeking a relationship between the axes as indicated by the graph" and "re-interpreting graphical relationships in some contexts that are associated with the graph".

In order to understand underlying information presented in a graph, the interpreter also needs to understand the underlying concepts of the presented situation. According to Chi, Slotta and de Leeuw, (1994), & Chi, (2013), many of student's alternative conceptions about scientific concept depend on inappropriate categorisation of existing concepts in an ontological sense. Figure 1 illustrates how our understanding of the world around us is related to three ontologically "lateral" categories: mental state, entities and processes. This is a lateral structure and not a hierarchic structure. We do not know what structure is established first in our mind, for instance, animals or vegetables.

Meaning about concepts results from observations and active engagement. Additional characteristics evolve throughout the concept formation process. For example, toys have characteristics from the category artefacts that are mutual for all toys: they can be broken, they are not alive, they can be thrown away, etcetera.

We encounter a concept by assigning the concept to an existing ontological category, what Piaget (1970) called assimilation. For example, the weight of the world series in
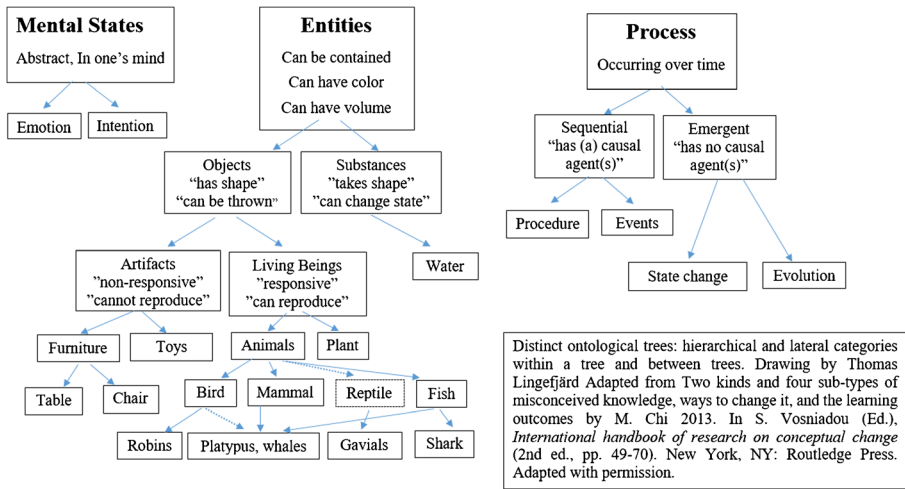
Mental States
Abstract, In one's mind

Emotion | Intention

Entities
Can be contained
Can have color
Can have volume

Objects
"has shape"
"can be thrown"

Substances
"takes shape"
"can change state"

Artifacts
"non-responsive"
"cannot reproduce"

Living Beings
"responsive"
"can reproduce"

Water

Furniture | Toys

Animals | Plant

Table | Chair

Bird | Mammal | Reptile | Fish

Robins | Platypus, whales | Gavials | Shark

Process
Occurring over time

Sequential
"has (a) causal
agent(s)"

Emergent
"has no causal
agent(s)"

Procedure | Events

State change | Evolution

Distinct ontological trees: hierarchical and lateral categories within a tree and between trees. Drawing by Thomas Lingefjärd Adapted from Two kinds and four sub-types of misconceived knowledge, ways to change it, and the learning outcomes by M. Chi 2013. In S. Vosniadou (Ed.), *International handbook of research on conceptual change* (2nd ed., pp. 49-70). New York, NY: Routledge Press. Adapted with permission.

Fig. 1 Conceptual development in an ontological structure (Chi, 2013)

soccer is an awkward statement since the weight of something belongs to "Entities", whilst the world series in soccer is categorized as an "Event", found under "Process". These two concepts belong to different categories with different properties, and they cannot be construed within the same context (Fig. 1).

Even close sub-categories under the same branch might be ontologically different lateral categories. For example, the difference between artefacts and living beings is quite transparent for most young children. The act of assigning concepts to different categories is an act of conceptual reconsidering. This is a constantly ongoing cognitive activity and strives for the interpretations of new situations and to make them understandable. Sometimes concepts are assigned to categories to which they do not belong. Conceptual development occurs when a concept has to be reassigned from one subcategory to another subcategory.

Learning of scientific concepts can be challenging since the process sometimes requires that a concept be reassigned to an ontologically distinct category (across trees). One such well-known situation is when students view light and electrical current as material substances based on an earlier encounter. Scientific theories describe electrical current and light as sequential processes (Chi, 2013).

Imagine that we want to model the situation of selling coffee cups in a coffee bar during a day from 08:00 AM and forward. This selling can be expressed as sold cups (y) over time (x) and expressed as $(x, y) = \{(10, 10), (12, 21), (14, 40), (16, 78)$, and so forth. When we construct a model from these values, we might arrive at the graphical representation of $f(x) = 0.34 \cdot e^{0.34 \cdot x}$. The table of values consists of discrete entities, whilst the graphical representation describes a continuous process, and they belong to different ontological categories.

There is a distinct difference between an event and a process. An event has a beginning and an end. For example, if we take a worldwide trip. A process has no beginning and no end, as, for example, electricity. Sometimes when we are telling about an event, it is unnecessary to tell about sub-events in the right order. For example, you can tell about a trip to Rome and to Paris without always keeping in mind what city

you visited first. But for a process, it is important that all sub-processes come in the right order. An event depends on time and space as it is almost impossible to describe an event without time and space. A process, on the other hand, is independent of time and space.

Methodology

This study relied on qualitative approaches regarding methodology and analysis. The qualitative research method created the opportunity to make a detailed analysis of how students perceived a certain situation or a certain concept. Our aim was to find a credible explanation that describes student's interpretations and explanations of a situation or a context in the world together with conceptual development related to their utterances.

Study 1

Our first study took place in the fall of 2013 with 17 upper secondary school students at the natural science programme (nine girls and eight boys) in year 2. The students were all 18 years old or older and had studied at least three different mathematics courses at upper secondary school, calculus inclusive. All students' volunteered to take part in the study. They were grouped into small groups of two or three students each group; a video camera was mounted to record the discussion. No observer was present since we wanted the students to be comfortable in the discussion and in their comments and attitudes.

Our intention was to record and analyse individual student's responses, and we considered that this was most easily done by encouraging group discussions in which students were trying to interpret the concept of motion. The group discussions were open and sometimes became intense as students discussed and tried to convince each other with arguments and responded to four separate questions. The students took notes and did their calculations on paper, and each session was recorded for about 1 h. The transcription was based on the group discussion, but the focus of the analysis was on the individually expressed interpretations of the situation. Our citations from students' reasoning where transcribed directly from the video film we analysed. When we categorized students' answers, we were not mainly interested in right or wrong answers, but more in the state of conceptual development.

As we engaged in analysis of the transcripts, we were guided by the notation that the transcription has to be selective in order to facilitate analysis of data (Linell, 1994). However, we chose not to take into account some variables, for example, body gestures and tone of voice. Transcription on the other hand has been designed as written linguistic constructions, and great importance has been given to accurate presentation of students' statements. Following Linell (1994), we decided to view the transcriptions as open units of analysis, which means that we have the opportunity to omit or include conventions, symbols or pauses based on the nature of the study.

An important note on the problem we gave the students is that it is a graphical representation of a theoretical physical concept of motion. In our study, we required the students to interpret the graphical representations of an imaginary train and thereafter use that interpretation as the base for their analysis.

Situation and Questions

When we face a graphical description of a real-life event, we need a mathematical interpretation of the graph since the graph belongs to a mathematical world. But we also try to understand the real-life event that took place, and sometimes we do this with non-mathematical references. We wanted the students to analyse and make interpretations of a real-life situation in mathematical terms. Students were presented with Fig. 2, which illustrated a graphical representation of a 4-h train journey. It is a distance–time graph, thereby requiring a coordinate system. We assumed that most students had either travelled on a train or at least knew what travelling on a train might be like. We also assumed that the students had experienced linear motion in different forms. In teaching mathematics and physics, linear motion is an important topic. In order to make useful interpretations of the given graph of the train's motion, the students needed to use accurate interpretations of how linear motion is expressed in a graph. In this representation, the train path is a continuous line.

Figure 2 shows the motion of a train during a time interval. The distance s is in kilometres and the time t is in hours. The train starts from a train station at $t = 0$. The distance s is therefore a measurement of the distance the train has travelled from the station at a given time t.

Questions:

1. When does the train run at maximal velocity? Please explain your answer.
2. When does the train run at minimum velocity? Please explain your answer.
3. What could be said about the train's direction by interpretation of the graph?
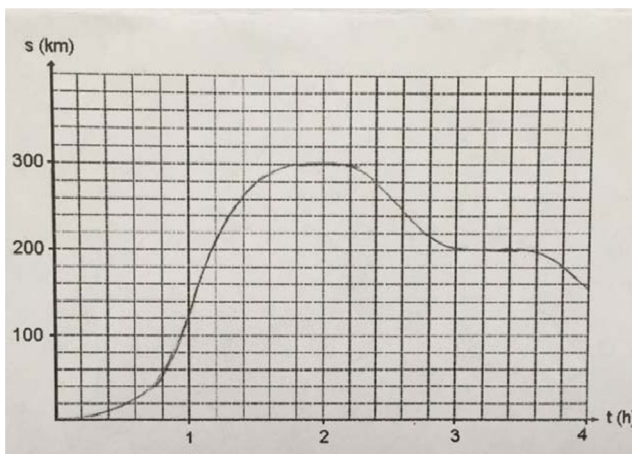4. Sketch an appropriate velocity–time graph of the motion during these 4 h.

Fig. 2 A train in motion

Results: Study 1

We searched for the most developed and least developed interpretations, which all were interesting. We were interested in finding similarities and differences amongst utterances. Three main categories of conceptual development were identified: intuitive and iconic interpretations, scientific interpretations and a category with influences from these two. We labelled these categories as least, more and most developed interpretations.

Least Developed Interpretation

Our description of student's utterances in this category is that the student's interpretations mainly were based on intuitive and iconic interpretations. In our analysis, we found six student's utterances that we considered belonging in this category. Below are quotes from some of the student's utterances in this category.

Question 1: Maximal Velocity

Student 1: The train has maximal velocity at 2 h. The train is at the highest point when the time is 2 h.
Student 2: Here is the train going down from a hill, here is the train going up at a hill, here is the train going up for a hill again, check it out …he he…

Question 2: Minimal Velocity

Student 1: The first 12 min.
Student 2: The train is at the lowest value during the first 12 min.

Question 3: Direction of Motion

Student 1: From the start it is a straight way so the train is accelerating up to a velocity of 300 km/h. The train is breaking in the curves, in the first curve is the velocity 300 km/h, in the next curve is the velocity 200 km/h.

Question 4: Velocity–Time Graph
The Figs. 3, 4, and 5 are computer constructed graphical representations that are true copies of the students' work.

Student 1: The student sketched this graph, but offered no rationale for its construction (Fig. 3).

More Developed Interpretation

Students in this category were students who initially started their interpretations and discussions based on scientific reasoning, but later abandoned that thinking and instead used intuitive and iconic interpretations. In our analysis, we found three students with
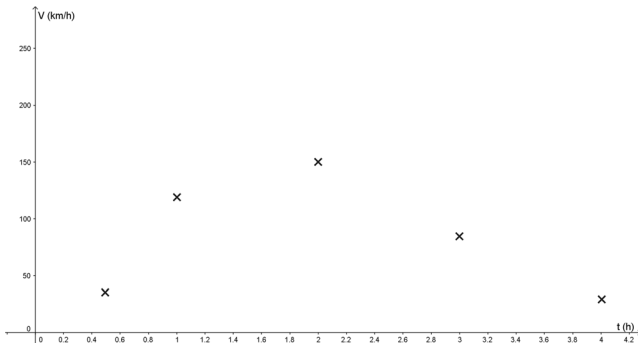
Fig. 3 An incorrect velocity–time graph

utterances that we considered belonging in this category. Below are quotes from some of the students' discussions in this category.

Question 1: Maximal Velocity

Student 3: After 1 h because it is the largest slope.

Question 2: Minimal Velocity

Student 4: During the interval 3.6 – 4.0 h since that is where the slope is large.

Question 3: Direction of Motion

Student 3: …if it goes up a hill, if possible, then the train driver must step on the accelerator. Is that even possible for a train?

Question 4: Velocity–Time Graph

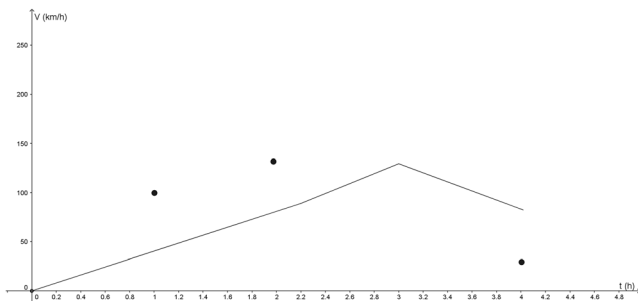Student 4: The student sketched this graph, but offered no rationale for its construction (Fig. 4).

Fig. 4 An incorrect velocity–time graph

Most Developed Interpretation

Students in this category used utterances with scientifically grounded interpretations from the start and maintained them during their discussion. Consequently, they constructed an accurate velocity–time graph. In our analysis, we found eight students with utterances that we considered belonging in this category. Below are quotes from some of the students' discussions in this category.

Question 1: Maximal Velocity

> Student 5: When the curve has the largest slope.
> Student 6: …and one square upwards is a fifth, so it is 20 km, he is travelling 20 km in a short time, while here he is travelling 20 km during a long time …

Question 2: Minimal Velocity:

> Student 5: When the slope is zero. Since the velocity is not changing.

Question 3: Direction of Motion

> Student 6: We get a negative segment which meant that the train is going back but only about half the way.

Question 4: Velocity–Time Graph

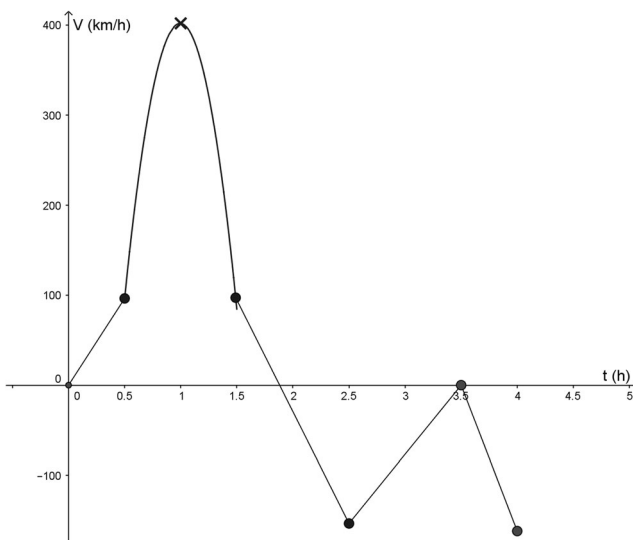> Student 5: The student sketched this graph, but offered no rationale for its construction.

Fig. 5 A correct velocity–time graph

Discussion: Study 1

Students 1 and 2 claim that the highest velocity occurs after 2 h, with the explanation that the graph is illustrating a train going "up a hill" and "down a hill". These students interpret the graph as a true image and make an iconic interpretation. Student 1 shares the opinion that the velocity is "highest at the top point up here" and that the train is going up and down. Students 1 and 2 also connect the graph to memories and events such as the roller coaster in the amusement park Liseberg in Gothenburg or to trams in San Francisco. These students select an everyday interpretation of the train's motion, not a scientific interpretation.

Students' alternative iconic interpretations of graphical representations are well known from research (Duval, 2006; diSessa, 1993; Elby, 2000; Friel et al., 2001; Leinhardt et al., 1990) and also obvious in this study.

When the students are asked to sketch a velocity–time graph, it seems as if they are strongly affected by the distance–time graph. Therefore, they sketched something almost identical to the graph in Fig. 2. Maybe this was derived from a wish to at least present something.

Students sometimes interpret a representation in the simplest, most literal way possible, such as a bump on a graph corresponding to a hill. This knowledge element is a representational analog of the phenomenological primitives (or p-prim) described by diSessa (1993), which include such basic reasoning elements. Some representations also include special attributes that quickly capture our attention. These attributes reference all the visual characteristics that are embedded in graphical representations such as corners, edges, contours, etc. We claim that the intuitive interpretation could be seen as a part of the individual's perception of a representation, such as a graph. Iconic views are partly due to the activation of a cognitive structure, a special intuitive knowledge element. These interpretations can be highlighted when they are expressed verbally or in writing (see Elby, 2000).

Student 1 and 2's interpretation of the distance–time graph could be seen as an alternative descriptive model. The students talk about different places, refer to different entities and give examples in different ways. One thing they have in common is that they all give iconic-based interpretations. The interpretation process, together with the need to create a context for the train, seems to be more important for the students in this group than to think about the explanation model's validity and reliability. Once they have selected an everyday discourse in their arguments, it seems difficult for them to explain the motion scientifically.

Students 1 and 2 did not manage to apply any scientific concepts for describing the motion of the train. In the students' interpretations of the graph, we find references to place and time, which characterize the descriptions of events. It is almost impossible to describe an event without referring to places and time, whilst a process in general is independent of time and place (Chi, 2013). Many of students' alternative conceptions about science concepts can be explained by an inappropriate ontological category to which the concepts belong, according to Chi (2013).

However, the empirical data indicates that students' initial conceptions of distance–time are categorized as "event" since students' interpretations link their thinking to amusement parks in Gothenburg or to trams in San Francisco, which contain attributes of the category event. However, the distance–time graph in Fig. 2 is a mathematical representation of a sequential "process".

In this case, the interpretation of the graphical representation has to be reassigned to an appropriate ontologically distinct subcategory, from subcategory "event" to

subcategory "procedure". Learning of science concepts in general and motion graph in particular could be facilitated by training students in reassigning the concept to an appropriate ontological category.

Our results indicate that these students eventually favour an event in their explanations. The challenging issue is that a train trip and the mathematical representation used to describe the motion of the train belong to two different ontological categories. This makes it sometimes more difficult to learn and understand mathematical and science concepts. The conceptual change in Fig. 1 occurs when the concept is reassigned from the subcategory "event" to the subcategory "procedure". We do not of course know how robust this shift might be over time or in other situations.

In question 1, student 3 and student 4 identified the interval where the train's velocity was maximal by using the slope of the graph. Later on, the student claims that the velocity is lowest in an interval where the graph is decreasing. What characterizes the interpretations in this category is that they start with relevant mathematical concepts in their reasoning, whilst they later on change into iconic interpretations. Student 3 identifies the interval 3.6 to 4 h where the graph has its lowest slope. Then, student 3 relates this to the least velocity. Student 3 and student 4 construct a velocity–time graph that is nearly identical to the distance–time graph given in the situation. Some of these students' interpretations and utterances seem to be mainly based on mathematical concepts, whilst iconic and intuitive ideas also come forward and take place in the discussion. Some intuitive ideas are resistant and remain in our thinking long after we have encountered them.

Student 5 and student 6's discussion uses a formal mathematical discourse. One qualitative aspect of understanding of a *distance–time* graph is found in student 5's reasoning. She illustrates a useful technique when she is looking at proportions for determination of slope of the graph. The discussion reveals that the students manage to distinguish the meanings of distance and velocity.

In this conversation, the students validate and develop their reasoning concerning the meaning of the graph's slope. The students' responses show that they are not confused by the shape of the graph; instead, they are arguing about the relation between the slope of the graph and the state of motion. Neither student 5 nor student 6 is making iconic interpretations of the graph, which allows them to give a scientific description of the meaning of the graph. These students are comfortable with scientific discourse, both in speech and in written language, which they use for explaining interpretations or to explain concepts. The students are using concepts from mathematics such as "direction", "positive" or "negative".

Student 5 and student 6 could also draw a velocity–time graph, the most developed interpretation in this study. According to Friel et al. (2001), "reinterpreting graphical relationships in some contexts that are associated with the graph" is generally more challenging for students. Our study confirms this claim as only two students demonstrated the ability to recognise, identify, interpret and draw conclusions about the concept slope and thus were able to do "reinterpreting graphical relationships in some contexts that are associated with the graph".

Study 2

Our second study took place in the fall of 2014 with 18 upper secondary school students, ten girls and eight boys, at a natural science programme in year 3 in a local upper secondary

school in Gothenburg. The students had all studied at least four mathematics courses and two courses in physics. The students took a test in physics with ten questions to be answered within 2.5 h. To assess students' interpretations, we used a graphical representation of an ECG diagram presented to them as one of the ten questions. The empirical data consists of short written answers from 18 students. Categorisation was done based on the condition of conceptual development. Conceptual development is often described as a causal process in which changes in an embraced system of assumptions result in a new system of assumptions. We considered it important to use the same categorisation as in study 1.

Situation and Questions

The graphical representation of an ECG diagram is probably one of the most well-known representations for describing a function of the body, namely the work of the heart. Since all humans have a heart, it is interesting per se. We assumed that all students in the study would be able to recognise the shape of an ECG diagram and therefore be able to find interest in the underlying mathematics. Our aim was to investigate how the students would be able to interpret and relate the mathematical model expressed by the ECG diagram. The students were also asked to calculate the frequency and the period for the ECG signal in the task, concepts included in both trigonometry and wave physics instruction. We consider that the ECG task furthermore also highlights several mathematical competencies in the Swedish curriculum, such as communication, conceptual understanding, problem-solving, procedures and relevance. The following ECG diagram was given to the group of students.

Figure 6 represents a graph of the behaviour of an ECG signal from an arbitrary patient. ECG measurement is a technically difficult problem since a rather weak signal of about 1 mV has to be discerned in an environment with strong electricity disturbance from fluorescent lamps, electrical fans, electrical heaters and so forth. An ECG diagram is a graphical representation of the activity of a heart as a process (over time). An event in this context would be something that happens just once.
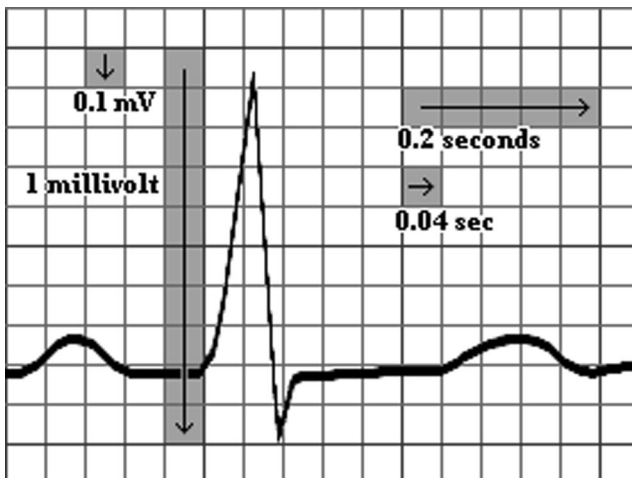

Fig. 6 An ECG diagram over one period

Questions:

(a)   Determine the ECG signal's period and frequency.
(b)   How many heartbeats per minute is the ECG diagram showing?
(c)   When is the signal's rate of change maximal? Explain your interpretation.

Results: Study 2

When we categorized students' answers, we were not mainly interested in right or wrong answers, but more in the condition of conceptual development. As in the previous study, three main categories could be identified for answers: intuitive and iconic interpretations, scientific interpretations and a third category with influences from these two categories. We have labelled these categories as least, more and most developed interpretations, respectively.

Question (a)

All 18 students responded to question (a), which investigated students' concept images of period and frequency. Twelve students managed to calculate the period. Six students managed to calculate the frequency correctly.

Student 1: 360° and 900 mV/s

Student 2: 0.64 s (draws a sine curve).

Question (b)

All 18 students responded to question (b). A question of heartbeats per minute is directly related to something the students were expected to know something about. A larger number of students determined the number of heartbeats per minute by using period instead of frequency. Twelve students managed to calculate the pulse as 94 beats per minute, which is correct. One student arrived at a pulse of 6; two students got a pulse of 150.

Question (c)

Thirteen students responded to question (c). The purpose of question (c) was to actualize the students' interpretations of the concept rate of change in relation to the ECG diagram, which, according to Friel et al. (2001), may be interpreted as to read between the features of the graph. This means that the students have to create a context about actual quantities and units.

Least Developed Interpretation

Question (c)

Six students responded with the top of the graph. Their concept image interprets the highest point of the graph as the maximum rate of change. This might very well be the result of an immediate reaction triggered by the word "maximum".

Student 3: The highest point is the maximum.

More Developed Interpretation

Question (c)

Five students gave responses that we considered more developed.

Student 8: Between 0.24 and 0.28.

Student 9: Arrow in the figure.

Most Developed Interpretation

Question (c)

Students in this category gave an implicit definition, thereby avoiding the possibility of pointing in the wrong direction. Two students managed to relate the rate of change to the steepness of the graph.

Student 10: Where the slope is steepest.

Discussion: Study 2

Question (a) is considered to be of the kind that students have to "seek a relationship between the axes as indicated by the graph" Friel et al., 2001, which in turn means that the students have to interpret the coordinate system and the features of the ECG diagram in terms of what the graph represents.

Question (b) requires that the students have a concept image of periodic processes of some situation or some phenomena, which in turn may involve the concept of frequency. It came as a surprise to us that only 12 students out of 18 answered with the correct period from interpretation of the ECG diagram. All 18 students used the same reasoning for calculation of the period, where 12 students managed the correct calculations. Even more surprising was that only six students could determine the frequency of the ECG diagram. One possibility, according to Tall & Vinner (1981), is that students who had not answered the question about frequency may perhaps not have a concept definition of frequency and therefore do not know how it should be interpreted in this context, or perhaps they just did not know how frequency and period are related.

Student 1's response to question (a) was 360° and 900 mV/s. This interpretation seems to be inspired by trigonometric function curves, for instance the function $f(x) = \sin(x)$. That, in turn, according to Tall & Vinner (1981), may be seen as an evoked concept image which is strongly associated with trigonometric curves. Another aspect is that the student's interpretation is independent of time. The student's evoked concept image results in the fact that he or she ignores the ECG diagram's time dependence. Student 1 seemed to need the periodic behaviour from trigonometric functions in order to handle the period when interpreting the ECG diagram. All students in this study have been introduced to basic concepts related to trigonometric functions. This student experienced difficulties in the interpretation of the ECG diagram and, hence, relates the period to 360°.

Student 2's response to question (a) was 0.64 s (draws a sine curve; see Fig. 7). This student sees a resemblance between the ECG diagram and the function $f(x) = \sin(x)$. The student used the sine curve to define one period and to set it equal to 0.64 s. The student has a developed concept image of periodic behaviour, which allows her/him to apply properties of this behaviour on a different situation. An important notation is that time as variable is integrated in this interpretation.

Then, the student calculates the frequency by 1/period and also calculates the number of heartbeats: 94 beats per minute. In this example, the evoked concept image was useful for the student's interpretation and analysis.

Even if an ECG diagram is used as a tool of medical expertise to do diagnostic work and to recommend suitable actions, it mediates a content that is constructed, interpreted
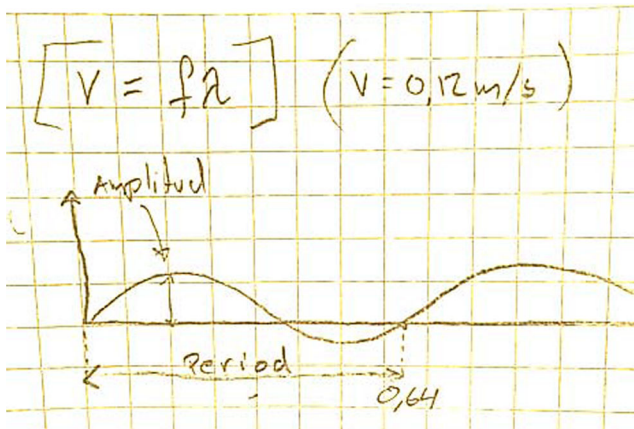
Fig. 7 An invoked concept image

and communicated in mathematical ways and with elementary mathematical explanation models. One such example is the description of a graph in terms of the graph's slope and how it should be interpreted within the frame of the actual scientific discipline. Our purpose with question (c) was to actualize the students' interpretations of the concept rate of change in relation to the ECG diagram, which, according to Friel et al. (2001), may be interpreted as "seeking a relationship between the axes as indicated by the graph". This means that students can create a context around actual quantities and units.

Some students responded with the top of the graph. Their concept image interprets the highest point of the graph as the maximum rate of change. It is well known that students sometimes have difficulty to distinguish height and slope in a distance–time graph (see Lobato & Thanheiser, 2002; Stump 1997, 1999, 2001a, b; Zaslavsky, Sela, & Leron 2002). We see this occurring also in a study with an ECG diagram. Students might respond similarly if we had a situation with the number of unemployed on the *y*-axis instead.

Students' interpretations of graphs and other representations can shed empirical light on a long-standing theoretical debate about learning of scientific concepts. Students' intuitive knowledge about science concepts may consist of unarticulated, loosely connected knowledge elements, the activation of which depends sensitively on context (see diSessa, 1993).

Learning is *not* a matter of replacing bad mini-generalizations with good ones. Instead, it is partly a matter of tweaking those mini-generalizations into a more articulate, unified, coherent structure. In our view, that is because the hill mistake and similar iconic interpretations spring from the activation of a cognitive structure, an intuitive knowledge element p-prim described by diSessa (1993).

Alternative interpretations have advantages; it correlates with daily life experience and with common sense (no major conflicts to start with). In daily life, often, higher means more; higher salary is better than lower salary.

The figural units of an original figure can be visually reconfigured without any recourse to a mathematical property. This purely visual operation of reconfiguring

an original figure underlies most of the examples of visual evidence that are used in teaching to give 'intuitive' explanations of certain mathematical results. But, in most cases it does not work because the visual processes of gestalt recognition do not run in the same way as required and expected from a mathematical point of view. (Duval, 2006, p. 112)

Some students compared different time intervals and their responses contained the time interval where the rate of change was maximal. Other students simply formulated it as: Where the slope is steepest. The students in this category made the interpretation that the slope of the graph is a relevant property of the graph. We interpret that these students' knowledge about graphical representation has been developed to a more integrated and more consistent concept image.

Conclusions of the Two Studies

Our questions in study 1 problematized a distance–time graph describing a train journey between stations. Interpretation of motion graphs covers a number of interesting concepts. The questions in study 2 problematized an ECG diagram. The slope of the graph is a central concept in both study 1 and study 2. The tasks were intended to highlight different aspects of students' conceptual understanding of graphical representations and its properties. Particular attention is put on students' interpretations and ability to read the features of the graph, to read between the features of the graph and to read beyond the features of the graph (Friel et al., 2001).

Stump (1997, 2001a, b) found that geometric ratios were the dominating representation teachers use to introduce slope. Stump (1999) noted that teachers expressed concern with students' understanding of slope. Students, texts and teachers sometimes put great attention on procedures for determining slope rather than developing conceptual notions of slope. Our studies focused on conceptual development and how interpretations of daily life situations of a train's motion or a heart beating can affect students' creative and scientific reasoning.

Study 1 revealed that some students interpret distance–time graphs as a mathematical description of an event. Students express statements with strong influences from places (such as San Francisco and Liseberg) which characterize the description of an event. However, a distance–time graph is a mathematical construction of a sequential process independent of time and places. We consider students' interpretations of the distance–time graph as a roller coaster ride as related to an event.

Another learning challenge is a train from the "material world" and distance–time graph as a mathematical construction of motion of the train belonging to a different ontological category. On the other hand, if a student conceives the distance–time graph as a roller coaster, then the graph will be interpreted as a distance–distance graph and time as a variable disappears in that interpretation.

Study 2 gave another result. When interpreting the ECG diagram, few students made utterances which related the ECG diagram to a realistic situation or daily life awareness about their heart. The findings from study 2 indicate that questions and information in relation to an ECG diagram, in some cases, reveal students' evoked concept images expressed as a sine curve (see Fig. 7).

To interpret students' interpretations of graphical representations is a complex issue. How would our theoretical sources be used in order to view and explain these situations? The three cognitive conditions in the framework of Friel et al. (2001)—to read the features of the graph, to read between the features of the graph and to read beyond the features of the graph—have been identified amongst student's responses in both studies. Reading between the features of the graph data requires more developed conceptual understanding than reading from the features of the graph, whilst reading beyond the features of the graph, for example the relation between the graph and reality, requires perhaps the most developed conceptual understanding.

Chi (2013) offers a framework based on an ontological assumption about the nature of scientific concepts. Some of the alternative conceptions in this study can be explained and understood by assigning the concept to an inappropriate ontological category. The ECG diagram represents the activity of the heart and belongs to a different ontological category. Given our results, this was a learning challenge for our students.

Tall and Vinner (1981) provide a theoretical framework to analyse and explain the condition of students' concept image of graphical representation. Both studies 1 and 2 showed that the students' evoked concept images are strongly context-dependent. The distance–time graph of a train's motion arose as an association to a roller coaster, whilst the ECG diagram resulted in a graph of $f(x) = \sin(x)$.

What is new in our two studies?

We have merged different theoretical perspectives together to explain students' interpretations. The theoretical framework in our studies includes an ontological perspective on learning of graphical information. Chi (2013) claims that students' alternative conceptions about light and electrical current emerge from an inappropriate ontological category to which the concept belongs. Students' reassigning of the concepts light and electrical current from matter to process promoted their learning process. We have used the ontological perspective to explain students' alternative conceptions about graphical representations of two different situations. Adding to Chi's framework, and as a way to develop this learning perspective, we identified two subcategories—"Procedure" and "Events"—in the category "Process", which is used to explain the nature of alternative conception and also as a goal for the learning of graphical representation.

We used Tall and Vinner's (1981) framework to explain why the student who draws a sine curve actually does that. The student sees a resemblance between the ECG diagram and the function $f(x) = \sin(x)$, and the sketch of $f(x) = \sin(x)$ helps the student to solve the problem. We understand this as an evoked concept image.

All three cognitive conditions in the framework of Friel et al. (2001)—to read the features of the graph, to read between the features of the graph and to read beyond the features of the graph—have been identified amongst students' responses in both studies. Our findings, based on two different studies, indicate that students sometimes lack coherent concept images and coherent concept definitions when they are challenged and asked to do interpretations of graphical representations. Consequently, discussing and teaching students about scientific concepts may help them learn and understand the mathematical construction of science processes such as distance–time

graph and ECG diagram. Helping students understand the challenging topics in science and mathematics through ontology training may facilitate the learning process.

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