J04

Proceedings of the 5th International Conference on Inverse Problems in Engineering: Theory and Practice, Cambridge, UK, 11-15th July 2005

AN ITERATIVE METHOD FOR RECONSTRUCTION OF TEMPERATURE

T. JOHANSSON
Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK, (Visitor)
e-mail: amt02tj@maths.leeds.ac.uk

Abstract - An iterative method for the reconstruction of a stationary temperature field, from Cauchy data given on a part of the boundary, is presented. At each iteration step, a series of mixed well-posed boundary value problems are solved for the heat operator and its adjoint. A convergence proof of this method in a weighted L^2-space is included.

1. INTRODUCTION

Let us begin by giving a background to the problem that we shall study. Assume that we have a body, denoted by Ω, occupying a volume in $\mathbb{R}^3$. Moreover, assume that there is some obstacle present, so that it is only possible to reach a part of the boundary. For example, the body could be partly buried in the soil. The part of the boundary where it is possible to measure data is denoted by Γ_0, see figure 1.

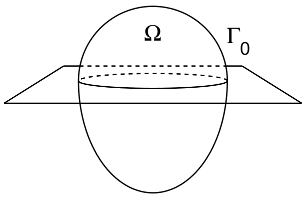

Figure 1. An example of a body Ω and boundary part Γ_0.

We wish to determine the temperature field inside the body from measurements of the temperature and heat flux on Γ_0. For a solid, the knowledge of the temperature field can be used to find structure properties through the determination of thermal stresses and deflections. If the temperature is independent of time, an approximation to the temperature field u can be obtained by solving the following so-called Cauchy problem:

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \Gamma_0, \\ Nu = \psi & \text{on } \Gamma_0. \end{cases} \tag{1}$$

Here, L is a linear elliptic operator of second-order and N is the co-normal derivative. For example, L can be the stationary heat operator $Lu = \nabla \cdot (k\nabla u)$, where k is the thermal conductivity. We consider a more general second-order linear elliptic operator which corresponds to the reconstruction of the temperature in a non-homogenous and non-isotropic medium. More generally, we can also have an energy source term in (1). Due to the linearity of the problem, the results obtained in this paper also cover that case.

There exist various methods for solving Cauchy problems for elliptic equations. One common approach is to use a Tikhonov type regularization which often leads to a change of the operator L of the problem, see Chapter 4 in [13]. Another way is to use iterative methods which preserves this operator. A method of this kind for a Cauchy problem for second-order elliptic equations in bounded plain domains is given in [7].

The aim of this paper is to show that the method in [7] can be applied to higher dimensional domains. The regularizing character of the procedure is achieved by appropriate change of boundary conditions, see Section 2.3. In each step, we solve mixed boundary value problems for the operators L and L^* with Neumann data on Γ_0 and Dirichlet data on Γ_1, where $\Gamma_1 = \Gamma \setminus \overline{\Gamma}_0$. We assume that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is a smooth $(n-2)$-dimensional manifold without boundary. Well-posedness of the problems used in the procedure are shown in weighted L^2-spaces with a weight of the form $r^{\beta-2}$, where the function r is the distance to $\overline{\Gamma}_0 \cap \overline{\Gamma}_1$, and β is a real number in a certain interval, see Lemma 3.4. Convergence of the method is proved in the above mentioned space, see Theorem 4.1.

Let us mention that iterative regularization methods for ill-posed boundary value problems have earlier been studied by Kozlov and Maz'ya, see [11]. They proposed iterative methods for solving some boundary value problems for elliptic, parabolic and hyperbolic equations. One of the advantages with these methods is that they preserve the original operator and that the regularizing character is achieved by appropriate change of boundary conditions. Most of the methods were suggested for differential operators which are self-adjoint. In [12], an alternating iterative method is applied to Cauchy problems for equations of anisotropic elasticity.

Numerical investigations and results for alternating iterative procedures can be found, for example, in [2], [4], [9], [14], and [15].

An iterative procedure for Cauchy problems for parabolic and elliptic equations, with not necessarily self-adjoint operators, was proposed in [1]. Numerical results were also presented (performed by the author of this paper). A similar method for the Laplace equation is investigated in [5]. The main restriction with these methods is that the boundary must consist of two separated parts, and data must be given on one of these parts. An iterative procedure which can handle both heat operators and non-separated boundary parts is presented in [8]. In the same paper, solvability results for the heat equation in weighted Sobolev spaces, with weights of the above type, are derived.

2. AN ITERATIVE PROCEDURE FOR PROBLEM (1)

2.1 Assumptions

Let Ω be a bounded domain in $\mathbb{R}^n$, where $n \geq 3$, of class C^2 with boundary Γ. We assume that the boundary of Ω is the union of three non-empty and disjoint pieces, Γ_0, Γ_1, and M, such that $M = \overline{\Gamma}_0 \cap \overline{\Gamma}_1$ is a smooth $(n-2)$-dimensional manifold without boundary of class C^2. This in particular implies that $\overline{\Gamma}_0$ and $\overline{\Gamma}_1$ are $(n-1)$-dimensional manifolds of class C^2 with boundary equal to M.

We use the notation

$$\begin{aligned} Lu &= \sum_{i,j=1}^{n} \partial_{x_i}(a_{i,j}(x)\partial_{x_j} u) + \sum_{i=1}^{n} b_i(x)\partial_{x_i} u + c(x)u, \\ Nu &= \sum_{i,j=1}^{n} \nu_i a_{i,j}(x)\partial_{x_j} u, \end{aligned}$$

where the symbol ν denotes the outward unit normal to the boundary Γ. Here, the matrix $(a_{i,j})_{i,j=1,2}$, is symmetric and all the coefficients in the operator L are real-valued functions with $a_{i,j}$, $b_i \in C^1(\overline{\Omega})$ and $c \in C(\overline{\Omega})$. Moreover, we suppose that

$$\int_\Omega \left(\sum_{i,j=1}^{n} a_{i,j}\partial_{x_i} u \partial_{x_j} u - \sum_{i=1}^{n} b_i u \partial_{x_i} u + cu^2 \right) dx \geq C\|\nabla u\|_{L^2(\Omega)}, \quad \text{where } \; C > 0, \tag{2}$$

for all functions $u \in L^2(\Omega)$ with $|\nabla u| \in L^2(\Omega)$ and $u|_{\Gamma_1} = 0$. This implies, in particular, that the operator L is elliptic, i.e., there exists a constant C such that

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq C|\xi|^2$$

for every $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$.

2.2 Functional spaces

Put $r(x) = \inf_{y \in M} |x - y|$. The space $V_\beta^k(\Omega)$, where k is an integer with $0 \leq k \leq 2$ and β is a real number, consists of functions u with generalized derivatives of order $\leq k$ in $L^2_{\text{loc}}(\Omega)$, such that

$$\|u\|_{V_\beta^k(\Omega)} = \left(\int_\Omega \sum_{|\alpha| \leq k} r^{2(\beta - k + |\alpha|)} |\partial_x^\alpha u|^2 \, dx \right)^{1/2} < \infty. \tag{3}$$

This space was introduced by Kondrat'ev for elliptic boundary value problems in domains with conical points, see [10]. We let $L^2_\beta(\Omega) = V^0_\beta(\Omega)$. Due to the smoothness assumption on Ω, functions in $V_\beta^k(\Omega)$, where $1 \leq k \leq 2$, have traces on $\Gamma \backslash M$. The trace space is denoted by $V_\beta^{k-1/2}(\Gamma)$. The norm in $V_\beta^{k-1/2}(\Gamma)$ is defined by

$$\|u\|_{V_\beta^{k-1/2}(\Gamma)} = \inf \left\{ \|v\|_{V_\beta^k(\Omega)} : \, v \in V_\beta^k(\Omega), \, v|_{\Gamma \backslash M} = u \right\}. \tag{4}$$

If $u \in V_\beta^{k-1/2}(\Gamma)$, then we have the estimate $\|u\|_{L^2_{\beta-k+1/2}(\Gamma)} \le C\|u\|_{V_\beta^{k-1/2}(\Gamma)}$. The norm in the space $V_\beta^{k-1/2}(\Gamma_0)$ is defined by

$$\|u\|_{V_\beta^{k-1/2}(\Gamma_0)} = \inf\left\{\|v\|_{V_\beta^k(\Omega)} : v \in V_\beta^k(\Omega),\, v|_{\Gamma_1} = 0 \text{ and } v|_{\Gamma_0} = u\right\}. \tag{5}$$

Analogously, the space $V_\beta^{k-1/2}(\Gamma_1)$ is introduced. The definition of these spaces on the boundary gives the decomposition $V_\beta^{k-1/2}(\Gamma) = V_\beta^{k-1/2}(\Gamma_0) \oplus V_\beta^{k-1/2}(\Gamma_1)$.

We let $H^k(\Omega)$ stand for the usual Sobolev spaces, i.e., functions $u \in L^2(\Omega)$ whose weak derivatives of order $\le k$, are in $L^2(\Omega)$.

2.3 An iterative regularizing procedure for problem (1)

The iterative method described below involves the following problems:

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = \eta & \text{on } \Gamma_1, \\ Nu = \psi & \text{on } \Gamma_0, \end{cases} \tag{6}$$

and

$$\begin{cases} L^*v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ N^*v = \xi & \text{on } \Gamma_0, \end{cases} \tag{7}$$

where

$$\begin{aligned} L^*v &= \sum_{i,j=1}^n \partial_{x_i}(a_{i,j}(x)\partial_{x_j} v) - \sum_{i=1}^n \partial_{x_i}(b_i(x)v) + cv, \\ N^*v &= \sum_{i,j=1}^n \nu_i a_{i,j}(x)\partial_{x_j} v - \sum_{i=1}^n \nu_i b_i v. \end{aligned}$$

Let β be a real number such that $1/2 < \beta < 3/2$ and let the data $\varphi \in L^2_{\beta-3/2}(\Gamma_0)$ and $\psi \in L^2_{\beta-1/2}(\Gamma_0)$ be given. We shall study the following iterative procedure for problem (1):

- Choose an arbitrary function $\eta_0 \in L^2_{\beta-3/2}(\Gamma_1)$.
- The first approximation u_0 to the solution u is obtained by solving problem (6) with $\eta = \eta_0$ on Γ_1.
- Then we find v_0 by solving problem (7) with

  $$\xi = r^{2(\beta-3/2)}(u_0 - \varphi) \quad \text{on } \Gamma_0.$$

- When the solutions u_{j-1} and v_{j-1} have been constructed, the approximation u_j is the solution to problem (6) with data $\eta = \eta_j$, where

  $$\eta_j = u_{j-1} + \gamma_\beta\, r^{2(3/2-\beta)} N^* v_{j-1} \quad \text{on } \Gamma_1$$

  and γ_β is a fixed positive number.
- Then v_j is the solution to problem (7) with data

  $$\xi = r^{2(\beta-3/2)}(u_j - \varphi) \quad \text{on } \Gamma_0.$$

3. WELL-POSEDNESS AND TRACES OF SOLUTIONS TO PROBLEMS (6) AND (7)

First, we define what we mean by a solution to the above problems.

Definition 3.1. Let $\eta \in L^2_{\beta-3/2}(\Gamma_1)$ and $\psi \in L^2_{\beta-1/2}(\Gamma_0)$, where β is a real number. Then $u \in L^2_{\beta-2}(\Omega)$ is a weak solution to problem (6), if u satisfies

$$\int_\Omega uL^*w\,dx + \int_{\Gamma_0} \psi w\,dS - \int_{\Gamma_1} \eta N^* w\,dS = 0 \tag{8}$$

for every $w \in V^2_{2-\beta}(\Omega)$ subject to

$$\begin{cases} w = 0 & \text{on } \Gamma_1, \\ N^*w = 0 & \text{on } \Gamma_0. \end{cases} \tag{9}$$

Observe that all the integrals are well-defined in this definition. For example, if $w \in V^2_{2-\beta}(\Omega)$ then $w|_{\Gamma_0} \in V^{3/2}_{2-\beta}(\Gamma_0) \subset L^2_{1/2-\beta}(\Gamma_0)$. Hence,

$$\int_{\Gamma_0} \psi w\,dS \leq \|\psi\|_{L^2_{\beta-1/2}(\Gamma_0)} \|w\|_{L^2_{1/2-\beta}(\Gamma_0)}.$$

In a similar manner, we define a solution to the adjoint problem (7).

Definition 3.2. Let $\xi \in L^2_{\beta-1/2}(\Gamma_0)$, where β is a real number. Then $v \in L^2_{\beta-2}(\Omega)$ is a weak solution to problem (7), if v satisfies

$$\int_\Omega vLw\,dx + \int_{\Gamma_0} \xi w\,dS = 0 \tag{10}$$

for every $w \in V^2_{2-\beta}(\Omega)$ with

$$\begin{cases} w = 0 & \text{on } \Gamma_1, \\ Nw = 0 & \text{on } \Gamma_0. \end{cases} \tag{11}$$

To be able to prove well-posedness of the problems used in the above procedure. i.e., there exists a unique weak solution that depends continuously on the data, we need the following lemma.

Lemma 3.3. *Assume that* $1/2 < \beta < 3/2$.

(*i*) *Let* $f \in L^2_\beta(\Omega)$, $\eta \in V^{3/2}_\beta(\Gamma_1)$, *and* $\psi \in V^{1/2}_\beta(\Gamma_0)$. *Then there exists a unique solution* $u \in V^2_\beta(\Omega)$ *to the problem*

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = \eta & \text{on } \Gamma_1, \\ Nu = \psi & \text{on } \Gamma_0. \end{cases} \tag{12}$$

This solution satisfies the estimate

$$\|u\|_{V^2_\beta(\Omega)} \leq C(\|f\|_{L^2_\beta(\Omega)} + \|\psi\|_{V^{1/2}_\beta(\Gamma_0)} + \|\eta\|_{V^{3/2}_\beta(\Gamma_1)}). \tag{13}$$

(*ii*) *Let* $g \in L^2_\beta(\Omega)$, $\zeta \in V^{3/2}_\beta(\Gamma_1)$, *and* $\xi \in V^{1/2}_\beta(\Gamma_0)$, *then there exists a unique solution* $v \in V^2_\beta(\Omega)$ *to the problem*

$$\begin{cases} L^*u = g & \text{in } \Omega, \\ u = \zeta & \text{on } \Gamma_1, \\ N^*u = \xi & \text{on } \Gamma_0. \end{cases} \tag{14}$$

This solution satisfies the estimate

$$\|v\|_{V^2_\beta(\Omega)} \leq C(\|g\|_{L^2_\beta(\Omega)} + \|\xi\|_{V^{1/2}_\beta(\Gamma_0)} + \|\zeta\|_{V^{3/2}_\beta(\Gamma_1)}). \tag{15}$$

Proof. Since the proof of (ii) is literally the same as (i), we only prove (i). We begin by proving uniqueness. Let f, η and ψ be equal to zero and let $v \in V^2_\beta(\Omega)$ be a solution to (12), where $1/2 < \beta < 3/2$. We intend to show that this implies that $u \in V^2_\gamma(\Omega)$ for every γ in the interval $(1/2, 3/2)$. Let x be a point on the manifold M, and let $\mathbb{R}^n_+ = \{(y,z) \in \mathbb{R}^n : y = (y_1, y_2) \in \mathbb{R}^2_+ \text{ and } z = (z_1, \dots, z_{n-2}) \in \mathbb{R}^{n-2}\}$, where $n \geq 3$ and $\mathbb{R}^2_+ = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_2 > 0\}$. We can assume that Ω coincides with $\mathbb{R}^n_+ \cap B_r$, where B_r is a ball in $\mathbb{R}^n$ centered at 0 and with radius $r > 0$. To the point x, one can then assign the operator pencil $\mathcal{P}(\lambda) = \partial^2_\omega + \lambda^2$, where ω is the polar angle, with domain $u \in H^2((0,\pi))$ such that $u(\pi) = 0$, and $u'_\omega(0) = 0$, see Section 8.2 in [16]. The spectrum of this pencil is $\lambda_k = (k + 1/2)$, where $k = 0, \pm 1, \pm 2, \dots$, are integers. Thus, the strip between the lines $\operatorname{Re}\lambda = -1/2$ and $\operatorname{Re}\lambda = 1/2$ is free of eigenvalues of $\mathcal{P}(\lambda)$. Since $-1/2 < -\beta + 2 - 1 < 1/2$, we find from Proposition 2.7 in [16, p. 307] that $v \in V^2_\gamma(\mathbb{R}^n_+ \cap B_r)$ for every $\gamma \in (1/2, 3/2)$. Thus, since the manifold M is a compact set, this implies that $v \in V^2_\gamma(\Omega)$ for every γ in the interval $(1/2, 3/2)$. We can then multiply the equality $Lv = 0$, by v and use integration by parts. Inequality (2) then implies that $v = 0$. In a similar manner uniqueness can be proved for the adjoint problem (14).

Now, existence of a solution will be proved. The operator of (12) has the Fredholm property, see Proposition 3.1 in [16, p. 308]. Combining this with the uniqueness result for the adjoint problem (14), which we proved above, the existence of a solution $u \in V^2_\beta(\Omega)$ to problem (12) and the estimate (13) follows. $\square$

Now, we can prove the well-posedness of the problems used in the above procedure.

Lemma 3.4. *Let* $1/2 < \beta < 3/2$.

(i) *Assume that* $\eta \in L^2_{\beta-3/2}(\Gamma_1)$ *and* $\psi \in L^2_{\beta-1/2}(\Gamma_0)$. *Then there exists a unique weak solution* $u \in L^2_{\beta-2}(\Omega)$ *to problem* (6). *This solution satisfies the estimate*

$$\|u\|_{L^2_{\beta-2}(\Omega)} \leq C(\|\psi\|_{L^2_{\beta-1/2}(\Gamma_0)} + \|\eta\|_{L^2_{\beta-3/2}(\Gamma_1)}). \tag{16}$$

(ii) *Assume that* $\xi \in L^2_{\beta-1/2}(\Gamma_0)$. *Then there exists a unique weak solution* $v \in L^2_{\beta-2}(\Omega)$ *to problem* (7) *and it satisfies the estimate*

$$\|v\|_{L^2_{\beta-2}(\Omega)} \leq C\|\xi\|_{L^2_{\beta-1/2}(\Gamma_0)}. \tag{17}$$

Proof. It suffices to prove (i). We start by proving the estimate (16). Let the function $u \in L^2_{\beta-2}(\Omega)$ be a weak solution to problem (6) and let $w \in V^2_{2-\beta}(\Omega)$ in (8) satisfy

$$\begin{cases} L^* w = r^{2(\beta-2)} u & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_1, \\ N^* w = 0 & \text{on } \Gamma_0. \end{cases} \tag{18}$$

Since $r^{2(\beta-2)} u \in L^2_{2-\beta}(\Omega)$ and $1/2 < \beta < 3/2$, Lemma 3.3 (ii) guarantees that there exists a solution $w \in V^2_{2-\beta}(\Omega)$ to problem (18). This solution satisfies the estimate

$$\|w\|_{V^2_{2-\beta}(\Omega)} \leq C\|u\|_{L^2_{\beta-2}(\Omega)}. \tag{19}$$

Now, if we let w in (8) be the solution to problem (18), we arrive at

$$\int_\Omega r^{2(\beta-2)} u^2 \, dx = -\int_{\Gamma_0} \psi w \, dS + \int_{\Gamma_1} \eta N^* w \, dS. \tag{20}$$

Cauchy's inequality, together with trace estimates, implies

$$\int_\Omega r^{2(\beta-2)} u^2 \, dx \leq C(\|\psi\|_{L^2_{\beta-1/2}(\Gamma_0)} + \|\eta\|_{L^2_{\beta-3/2}(\Gamma_1)}) \|w\,|_{V^2_{2-\beta}(\Omega)}. \tag{21}$$

Using (19), we derive that

$$\|u\|_{L^2_{\beta-2}(\Omega)} \leq C(\|\psi\|_{L^2_{\beta-1/2}(\Gamma_0)} + \|\eta\|_{L^2_{\beta-3/2}(\Gamma_1)}),$$

which proves inequality (16), and this implies, in particular, the uniqueness of a solution.

Now, we prove the existence of a weak solution to problem (6). We choose sequences $\psi_j \in C^\infty_0(\Gamma_0)$ and $\eta_j \in C^\infty_0(\Gamma_1)$, such that $\lim_{j\to\infty} \psi_j = \psi \in L^2_{\beta-1/2}(\Gamma_0)$ and $\lim_{j\to\infty} \eta_j = \eta \in L^2_{\beta-3/2}(\Gamma_1)$. If we take ψ_j and η_j as boundary data in problem (6), it follows from Lemma 3.3 (i), that there exists a solution $u_j \in V^2_\beta(\Omega)$. This solution satisfies (8). Then the estimate (16) shows that u_j is a Cauchy sequence in $L^2_{\beta-2}(\Omega)$. Put $\lim_{j\to\infty} u_j = u$. Then $u \in L^2_{\beta-2}(\Omega)$ and satisfies (8). $\square$

We also have to prove that the various restrictions of weak solutions to the boundary that appear in the iterative procedure are well-defined. Clearly, away from the manifold M, the restrictions are well-defined. The question is what happens near the manifold M. Let x be a point on the manifold. One can assume that Ω locally coincides with $\mathbb{R}^n_+ \cap B_r$, where B_r is a ball in $\mathbb{R}^n$ with center at zero and radius $r > 0$. Verbatim from the proof of Lemma 6.2 in [8], we readily obtain

Lemma 3.5. *Let $1/2 < \beta < 3/2$ and let $u \in L^2_{\beta-2}(\Omega)$ be a weak solution to problem* (6)*.*

(*i*) *If $\eta = 0$, then $u|_{\Gamma_1} \in V^{3/2}_\beta(\Gamma_1)$ and $\|u\|_{V^{3/2}_\beta(\Gamma_1)} \le C\|\psi\|_{L^2_{\beta-1/2}(\Gamma_0)}$.*

(*ii*) *$u|_{\Gamma_0} \in L^2_{\beta-3/2}(\Gamma_0)$ and $\|u\|_{L_{\beta-3/2}(\Gamma_0)} \le C(\|\psi\|_{L^2_{\beta-1/2}(\Gamma_0)} + \|\eta\|_{L^2_{\beta-3/2}(\Gamma_1)})$.*

For the sake of completeness, we also state trace results for the adjoint problem.

Lemma 3.6. *Let $1/2 < \beta < 3/2$ and let $v \in L^2_{\beta-2}(\Omega)$ be a weak solution to problem* (7)*, where $\xi \in L^2_{\beta-1/2}(\Gamma_0)$, then*

(*i*) *$v|_{\Gamma_1} \in V^{3/2}_\beta(\Gamma_1)$ and $\|v\|_{V^{3/2}_\beta(\Gamma_1)} \le C\|\xi\|_{L^2_{\beta-1/2}(\Gamma_0)}$.*

(*ii*) *$v|_{\Gamma_0} \in L^2_{\beta-3/2}(\Gamma_0)$ and $\|v\|_{L^2_{\beta-3/2}(\Gamma_0)} \le C\|\xi\|_{L^2_{\beta-1/2}(\Gamma_0)}$.*

4. PROOF OF CONVERGENCE OF THE ITERATIVE PROCEDURE

To be able to formulate the main result, let the operator $K : L^2_{\beta-3/2}(\Gamma_1) \to L^2_{\beta-3/2}(\Gamma_0)$, where $1/2 < \beta < 3/2$, be defined by

$$K\eta = u|_{\Gamma_0} \quad \text{for} \quad \eta \in L^2_{\beta-3/2}(\Gamma_1), \tag{22}$$

where u is a weak solution to problem (6) with $\psi = 0$. Since the operator K depends on the number β, we also use the notation K_β.

It is possible to calculate the adjoint of K by solving a certain boundary value problem. To see this, let $\xi \in L^2_{\beta-3/2}(\Gamma_0)$. The adjoint operator $K^* : L^2_{\beta-3/2}(\Gamma_0) \to L^2_{\beta-3/2}(\Gamma_1)$, to the operator K defined in (22), is given by

$$K^*\xi = -r^{2(3/2-\beta)}N^*v|_{\Gamma_1} \quad for \quad \xi \in L^2_{\beta-3/2}(\Gamma_0),$$

where $v \in L^2_{-\beta}(\Omega)$ satisfies (7), with $N^*v = r^{2(\beta-3/2)}\xi$ on Γ_0. Indeed, assume first that $\eta \in C^\infty_0(\Gamma_1)$ and $\xi \in C^\infty_0(\Gamma_0)$. Let u be a weak solution to (6) with $\psi = 0$ and let v be a weak solution to (7) with data $N^*v = r^{2(\beta-3/2)}\xi$ on Γ_0. We observe that $r^{2(\beta-3/2)}\xi \in L^2_{3/2-\beta}(\Gamma_0)$. Since η and ξ are smooth, it follows that $u \in V^2_\beta(\Omega)$ and $v \in V^2_{2-\beta}(\Omega)$. Green's formula then gives

$$\int_\Omega (uL^*v - vLu)\,dx = -\int_\Gamma vNu\,dS + \int_\Gamma uN^*v\,dS. \tag{23}$$

The boundary conditions and that the left hand side is zero in the above equality, imply that

$$\int_{\Gamma_0} r^{2(\beta-3/2)}(K\eta)\xi\,dS = -\int_{\Gamma_1} \eta N^*v\,dS.$$

This can be written as

$$\int_{\Gamma_0} r^{2(\beta-3/2)}(K\eta)\xi\,dS = -\int_{\Gamma_1} r^{2(\beta-3/2)}\eta\,(r^{2(3/2-\beta)}N^*v)\,dS.$$

Since $r^{2(3/2-\beta)}N^*v \in L^2_{\beta-3/2}(\Gamma_1)$, we have $K^*\xi = -r^{2(3/2-\beta)}N^*v|_{\Gamma_1}$.

For arbitrary $\eta \in L^2_{\beta-3/2}(\Gamma_1)$ and $\xi \in L^2_{\beta-3/2}(\Gamma_0)$, we approximate these functions with smooth functions and use the inequalities (16) and (17).

Moreover, the kernel of K consists of only zero. Assume on the contrary that the kernel contains a non-zero element $\eta \in L^2_{\beta-3/2}(\Gamma_1)$. Let $u \in L^2_{\beta-2}(\Omega)$ be the weak solution to problem (6), where $\psi = 0$. Such a solution exists by Lemma 3.4. Let $x \in \Gamma_0$ and let $B_r(x)$ be a ball with center at x with radius $r > 0$. Here, r is chosen such that $\bar{B}_r(x)$ does not contain any of the points of M. Since both $u = 0$ and $Nu = 0$ on Γ_0, the function u can be extended by zero in $B_r(x) \setminus \Omega$. The main result on local regularity for elliptic equations implies that $u \in H^2(B_{r/2})$. From Theorem 17.2.6 in [6, p. 14], it follows that $u = 0$ in $\bar{B}_{r/2}(x)$. This implies that $u = 0$ in $\overline{\Omega}$. Thus, the kernel consists of zero only.

We now state and give a proof of the main result.

Theorem 4.1. *Let β be a real number with $1/2 < \beta < 3/2$ and let $\varphi \in L^2_{\beta-3/2}(\Gamma_0)$ and $\psi \in L^2_{\beta-1/2}(\Gamma_0)$. Assume that problem* (1) *has a solution $u \in L^2_{\beta-2}(\Omega)$ and that γ_β satisfies $0 < \gamma_\beta < 1/\|K_\beta\|^2$. Let u_j be the j–th approximation in the iterative procedure above. Then*

$$\lim_{j\to\infty} \|u - u_j\|_{L^2_{\beta-2}(\Omega)} = 0 \tag{24}$$

for every function $\eta_0 \in L^2_{\beta-3/2}(\Gamma_1)$.

Proof. Let $\beta \in (1/2, 3/2)$. By the linearity of problem (1), it suffices to consider the case when $\psi = 0$. To find a solution to the Cauchy problem (1) is then equivalent to finding $\eta \in L^2_{\beta-3/2}(\Gamma_1)$ such that

$$K\eta = \varphi. \tag{25}$$

The function $\eta = u|_{\Gamma_1}$ is a solution to problem (25) and this solution is unique. From the procedure given in Section 2.3, and using the above expression for calculating the adjoint operator, it follows that

$$\begin{aligned} \eta_j &= u_{j-1}|_{\Gamma_1} + \gamma_\beta\, r^{2(3/2-\beta)} N^* v_{j-1}|_{\Gamma_1} = \eta_{j-1} - \gamma_\beta\, K^*\,(u_{j-1}|_{\Gamma_0} - \varphi) \\ &= \eta_{j-1} - \gamma_\beta\, K^*\,(K\eta_{j-1} - \varphi). \end{aligned}$$

This procedure is the Landweber scheme for solving equation (25), see [3, p. 155]. Now, the sequence η_j converges to η in $L^2_{\beta-3/2}(\Gamma_1)$, since, by assumption, $0 < \gamma_\beta < 1/\|K\|^2$. Note that u satisfies problem (6). The inequality (16) then implies that u_j converges to u in $L^2_{\beta-2}(\Omega)$. □

Using this connection with the Landweber method, it is possible to propose appropriate stopping rules. Thus, the method presented in this paper can handle the case when there is some error in the data.

5. FINAL REMARKS

(a) The parameter-free procedure presented in [7] can, after obvious changes, be applied to the problem of this paper.

(b) The method presented here can also be applied if Cauchy data are measured at several disjoint regions of the boundary. One only has to redefine the weight in an appropriate way. For example, assume that we measure data at the grey-shaded regions in figure 2 below. The weight should in this case be the distance to the union of the boundaries of these grey-shaded regions.

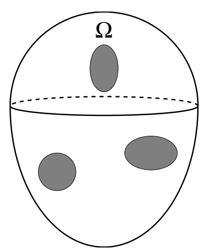

Figure 2. The grey-shaded regions indicate where Cauchy data can be measured.

Acknowledgement
The author would like to thank the School of Mathematics at the University of Leeds for their hospitality and kindness, and in particular L. Elliott, D. B. Ingham and D. Lesnic. The author also acknowledges grants and support from The Wenner-Gren Foundations.

REFERENCES

1. G. Bastay, V. A. Kozlov and B. O. Turesson, Iterative methods for an inverse heat conduction problem. *J. Inv. Ill-posed Probl.* (2001) **9**, 375–388.

2. J. Baumeister and A. Leitão, On iterative methods for solving ill-posed problems modeled by partial differential equations. *J. Inv. Ill-Posed Probl.* (2001) **9**, 13–29.

3. H. W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publishers Group, Dordrecht, 1996.

4. H. W. Engl and A. Leitão, A Mann iterative regularization method for elliptic Cauchy problems. *Numer. Funct. Anal. Optim.* (2001) **22**, 861–884.

5. D. H. Hào and D. Lesnic, The Cauchy problem for Laplace's equation via the conjugate gradient method. *IMA J. Appl. Math.* (2000) **65**, 199–217.

6. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. **III**, Springer-Verlag, Berlin, 1985.

7. T. Johansson, An iterative procedure for solving a Cauchy problem for second order elliptic equations. *Math. Nachr.* (2004) **272**, 46–54.

8. T. Johansson, An iterative method for a Cauchy problem for the heat equation. *To appear in IMA J. Appl. Math.*

9. M. Jourhmane and A. Nachaoui, An alternating method for an inverse Cauchy problem. *Numer. Algorithms* (1999) **21**, 247–260.

10. V. A. Kondrat'ev, Boundary problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat. Obshch.* (1967) **16**, 209–292. English transl.: *Trans. Moscow Mat. Soc.* (1967) **16**, 227–313.

11. V. A. Kozlov and V. G. Maz'ya, On iterative procedures for solving ill-posed boundary value problems that preserve differential equations. *Algebra i Analiz* (1989) **1**, 144–170. English transl.: *Leningrad Math. J.* (1990) **1**, 1207–1228.

12. V. A. Kozlov, V. G. Maz'ya and A. V. Fomin, An iterative method for solving the Cauchy problem for elliptic equations. *Zh. Vychisl. Mat. i Mat. Fiz.* (1991) **31**, 64–74. English transl.: *U.S.S.R. Comput. Math. and Math. Phys.* (1991) **31**, 45–52.

13. R. Lattès and J.-L. Lions, *The Method of Quasi-reversibility, Applications to Partial Differential Equations*, American Elsevier Publishing Co., New York, 1969.

14. L. Marin, L. Elliot, P. J. Heggs, D. B. Ingham, D. Lesnic and X. Wen, Comparisation of regularization methods for solving the Cauchy problem associated with the Helmoltz equation. *Int. J. Numer. Meth. Engng.* (2004) **60**, 1933–1947.

15. N. S. Mera, L. Elliot, D. B. Ingham and D. Lesnic, The boundary element solution of the Cauchy steady heat conduction problem in an anisotropic medium. *Int. J. Numer. Meth. Engng.* (2000) **49**, 481–499.

16. S. A. Nazarov and B. A. Plamenevskiĭ, *Elliptic Problems in Domains With Piecewise Smooth Boundaries,* Walter de Gruyter & Co., Berlin, 1994.