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Fractional differential equations with causal operators

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Abstract

We study fractional differential equations with causal operators. The existence of solutions is obtained by applying the successive approximate method. Some applications are discussed including also the case when causal operator Q is a linear operator. Examples illustrate some results.

MSC: 26A33

Keywords: fractional differential equations; causal operators; Mittag-Leffler functions; existence of solutions

1 Introduction

Let $J = [0, T]$, $J_0 = (0, T]$, $E_0 = C(J_0, \mathbb{R})$, $E = C(J, \mathbb{R})$ and $Q \in C(E_0, E)$. We shall say that Q is a causal operator, or nonanticipative, if the following property holds: for each couple of elements of E such that $u(s) = v(s)$ for $0 \le s \le t$, there are the results $(Qu)(s) = (Qv)(s)$ for $0 \le s \le t$ with $t < T$ arbitrary; for details, see [1].

In this paper, we investigate the existence of solutions to fractional differential problems with causal operators Q of the form

$$\begin{cases} D^q x(t) = (Qx)(t), & t \in J_0, \\ \bar{x}(0) = k \in \mathbb{R}, \end{cases} \tag{1}$$

where $D^q x$ is the standard Riemann-Liouville derivative of x with $q \in (0,1]$ and $\bar{x}(0) = t^{1-q}x(t)|_{t=0}$. We introduce the space C_{1-q} by

$$C_{1-q}(J, \mathbb{R}) = \{u \in C(0,t] : t^{1-q}u \in C(J, \mathbb{R})\}, \quad q \in (0,1),$$

and $C_0(J, \mathbb{R}) = C(J, \mathbb{R})$ if $q = 1$.

Two significant examples of causal operators are: the Niemytzki operator

$$(Qu)(t) = f(t, u(t))$$

and the Volterra integral operator

$$(Qu)(t) = g(t) + \int_0^t k(t,s) f(s, u(s))\, ds.$$

For other concrete examples which serve to illustrate that the class of causal operators is very large, we refer the reader to the monographs [1, 2]. The study of functional equations with causal operators has seen a rapid development in the last few years. Various problems for functional differential equations with causal operators were considered, for example, in papers [1–10], in particular, the existence of solutions by using the monotone iterative method; see, for example, [2, 6–8]. Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The idea of fractional calculus has been a subject of interest not only among mathematicians, but also among physicists and engineers. Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, engineering and biological sciences; see, for example, [11–13]. To our knowledge, fractional differential equations with causal operators have not been studied extensively. Fractional differential equations of Caputo type with causal operators have been discussed in papers [2, 14] by using the monotone iterative technique.

This paper is organized as follows. In Section 2, the existence of solution for problem (1) is investigated by the successive approximate method. We showed it under the assumption that operator Q satisfies a Lipschitz condition. The error estimation is also given using a Mittag-Leffler function (Theorem 1). In Section 3, corresponding existence results are formulated for special cases of causal operators, using the main result of Section 2. The existence of a unique solution for the problem with a linear operator Q is also discussed and this solution is given by a corresponding formula with the Mittag-Leffler function (Theorem 5). Examples are also given to demonstrate some results.

2 Existence of solutions by the successive approximate method

Let us introduce the following assumptions:

H_1: Q is a causal operator, $Q \in C(E_0, E)$,

H_2: there exists $L \in \mathbb{R}_+ = [0, \infty)$ such that

$$\left|(Qu)(t) - (Q\bar{u})(t)\right| \le L|u - \bar{u}|_t,$$

where $|u|_t = \sup_{s\in[0,t]} |x(s)|$.

Fractional differential problem (1) is equivalent to

$$x(t) = kt^{q-1} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}(Qx)(s)\,ds \tag{2}$$

by [11]. We can also write (2) in the form $x(t) = kt^{q-1} + I^q(Qx)(t)$.

To find a solution of (2) we use the method of successive approximations:

$$\begin{cases} x_0(t) = kt^{q-1}, \\ x_n(t) = kt^{q-1} + I^q(Qx_{n-1})(t), \quad n = 1, 2, \ldots. \end{cases} \tag{3}$$

Let

$$\frac{1}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1}\left|(Qx_0)(r)\right|dr \le M. \tag{4}$$

Indeed, $x_n - x_0 \in C(J,\mathbb{R})$, $x_n \in C_{1-q}(J,\mathbb{R})$, $n = 0,1,\ldots$. Put

$$w_n(t) = |x_n - x_{n-1}|_t, \quad n = 1,2,\ldots.$$

Then

$$w_1(t) = |x_1 - x_0|_t \le \frac{1}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1} \bigl|(Qx_0)(r)\bigr| \, dr \le M \equiv u_1(t).$$

Moreover, in view of assumption H_2, we obtain

$$\begin{aligned} w_{n+1}(t) &= \frac{1}{\Gamma(q)} \sup_{s\in[0,t]} \left| \int_0^s (s-r)^{q-1} \bigl[(Qx_n)(r) - (Qx_{n-1})(r)\bigr] \, dr \right| \\ &\le \frac{L}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1} w_n(r) \, dr \equiv u_{n+1}(t), \quad n = 1,2,\ldots. \end{aligned}$$

Hence

$$u_2(t) = \frac{L}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1} w_1(r) \, dr \le \frac{LM}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1} \, dr = \frac{LM}{\Gamma(q+1)} t^q.$$

Using the method of mathematical induction, we can show that

$$u_n(t) \le \frac{ML^{n-1}}{\Gamma((n-1)q+1)} t^{q(n-1)}, \quad n = 1,2,\ldots. \tag{5}$$

Now we have to show that the sequence $\{x_n\}$ is convergent. Note that

$$x_n(t) = x_0(t) + \sum_{j=1}^{n} \bigl[x_j(t) - x_{j-1}(t)\bigr], \quad n = 1,2,\ldots. \tag{6}$$

In view of (5), we see that

$$\begin{aligned} \sum_{j=1}^{\infty} u_j(t) &\le \sum_{j=1}^{\infty} \frac{ML^{j-1}}{\Gamma((j-1)q+1)} t^{(j-1)q} = M \sum_{j=0}^{\infty} \frac{L^j}{\Gamma(jq+1)} t^{jq} \\ &\le M \sum_{j=0}^{\infty} \frac{L^j}{\Gamma(jq+1)} T^{jq} = ME_{q,1}\bigl(LT^q\bigr), \end{aligned}$$

where $E_{q,1}$ is the Mittag-Leffler function defined by

$$E_{q,1}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(jq+1)}.$$

Using the Weierstrass test, this shows that the series

$$\sum_{j=1}^{\infty} \bigl[x_j(t) - x_{j-1}(t)\bigr]$$

is uniformly convergent. This asserts that the sequence $\{x_n - x_0\}$ is uniformly convergent too. It proves that $\lim_{n\to\infty} x_n(t)$ exists, so $x(t) = \lim_{n\to\infty} x_n(t)$. Indeed, $x - x_0$ is a continuous function on J and x is a continuous function on J_0. Taking the limit $n \to \infty$ in (3), we see that $x \in C_{1-q}(J, \mathbb{R})$ is a solution of problem (2), it is also a solution of problem (1).

Now we have to prove that x is a unique solution of (2). Let $v \in C_{1-q}(J, \mathbb{R})$ be another solution of (2). Put $U(t) = |x - v|_t$, $A_0 = U(T)$. Then

$$U(t) = \sup_{s\in[0,t]} \left|I^q(Qx)(s) - I^q(Qv)(s)\right| \le \frac{L}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1} U(r)\, dr,$$

by assumption H_2. This shows that

$$U(t) \le \frac{A_0 L}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1}\, dr = \frac{A_0 L}{\Gamma(q+1)} t^q \le \frac{A_0 L}{\Gamma(q+1)} T^q \equiv D.$$

Repeating it, we can show that

$$U(t) \le \frac{DL^n}{\Gamma(nq+1)} t^{nq}, \quad n = 0, 1, \ldots,$$

so

$$U(t) \le \frac{DL^n}{\Gamma(nq+1)} T^{nq}, \quad n = 0, 1, \ldots.$$

Indeed,

$$\lim_{n\to\infty} \frac{L^n}{\Gamma(nq+1)} T^{nq} = 0.$$

This shows that x is the unique solution of (2). This also proves that x is the unique solution of (1).

Now, we need to obtain the error estimation. Put $Z_{n+1}(t) = |x - x_{n+1}|_t$, $n = 0, 1, \ldots$. In view of (6) and (5), we obtain

$$\left|x_n(t) - x_0(t)\right| \le \sum_{j=1}^{\infty} u_j(t) \le M E_{q,1}\left(LT^q\right) \equiv B.$$

Because, $x_n \to x$, so $Z_0(t) \le B$. Moreover,

$$Z_{n+1}(t) = \sup_{s\in[0,t]} \left|I^q(Qx)(s) - I^q(Qx_n)(s)\right| \le \frac{L}{\Gamma(q)} \sup_{s\in[0,t]} \int_0^s (s-r)^{q-1} Z_n(r)\, dr,$$

by assumption H_2. Repeating it we obtain the result

$$U_n(t) \le \frac{BL^n}{\Gamma(nq+1)} t^{nq}, \quad n = 0, 1, \ldots,$$

by the mathematical induction method.

In this way we proved the following theorem.

Theorem 1 *Let assumptions* H_1, H_2 *hold and let* $q \in (0,1]$. *Assume that there exists a constant* $M \in \mathbb{R}_+$ *such that condition* (4) *holds. Then the sequence* $\{x_n\}$ *converges to the unique solution* $x \in C_{1-q}(J,\mathbb{R})$ *of problem* (1). *Moreover, we have the error estimation*

$$|x - x_n|_t \le \frac{BL^n}{\Gamma(nq+1)} t^{nq}, \quad n = 0,1,\ldots,$$

where $B = E_{q,1}(LT^q)$ *is the Mittag-Leffler function.*

Remark 1 If $|(Qu_0)(t)| \le M_1$, then $M = \frac{M_1}{\Gamma(q+1)} T^q$. Indeed, we have

$$\begin{aligned}\frac{1}{\Gamma(q)} \int_0^t (s-r)^{q-1} \big|(Qu_0)(r)\big|\, dr &\le \frac{M_1}{\Gamma(q)} \int_0^t (s-r)^{q-1}\, dr \\ &= \frac{M_1}{\Gamma(q+1)} t^q \le \frac{M_1}{\Gamma(q+1)} T^q = M.\end{aligned}$$

3 Some applications

1. Let operator Q be defined by

$$(Qx)(t) = f\big(t, x(\alpha_1(t)), \ldots, x(\alpha_p(t))\big). \tag{7}$$

We introduce the following assumption:

H_3: $f \in C(J \times \mathbb{R}^p, \mathbb{R})$, $\alpha_i \in C(J,J)$, $\alpha_i(t) \le t$, $i = 1,2,\ldots,p$, and there exist constants $L_i \in \mathbb{R}_+$ such that

$$\big|f(t,u_1,\ldots,u_p) - f(t,v_1,\ldots,v_p)\big| \le \sum_{i=1}^p L_i |u_i - v_i|.$$

Indeed, Q is a causal operator. Put $\bar{\alpha}(t) = \max\{\alpha_i(t) : i = 1,2,\ldots,p\}$, so $\bar{\alpha}(t) \le t$, $t \in J_0$. We see that

$$\begin{aligned}\big|(Qu)(t) - (Qv)(t)\big| &\le \sum_{i=1}^p L_i \big|u(\alpha_i(t)) - v(\alpha_i(t))\big| \le \sum_{i=1}^p L_i \sup_{s \in [0,\alpha_i(t)]} \big|u(s) - v(s)\big| \\ &\le \sum_{i=1}^p L_i \sup_{s \in [0,\bar{\alpha}(t)]} \big|u(s) - v(s)\big| \le |u - v|_t \sum_{i=1}^p L_i.\end{aligned}$$

It shows that assumption H_2 holds with $L = \sum_{i=1}^p L_i$. Basing on Theorem 1, we have the following.

Theorem 2 *Let assumption* H_3 *hold and let* $q \in (0,1]$. *Assume that there exists a constant* $M \in \mathbb{R}_+$ *such that condition* (4) *holds with operator* Q *defined by* (7). *Then the sequence* $\{x_n\}$ *converges to the unique solution* $x \in C_{1-q}(J,\mathbb{R})$ *of problem* (1) *with* Q *defined by* (7). *Moreover, we have the error estimation*

$$|x - x_n|_t \le \frac{BL^n}{\Gamma(nq+1)} t^{nq}, \quad n = 0,1,\ldots,$$

where $L = \sum_{i=1}^p L_i$.

2. Let

$$(Qx)(t) = g\left(t, x(\alpha_1(t)), \max_{s\in[0,\alpha_2(t)]} x(s), \int_0^t h(t,r)x(\alpha_3(r))\,dr\right). \tag{8}$$

We introduce the following assumption:

H_4: $g \in C(J \times \mathbb{R}^3, \mathbb{R})$, $h \in C(J \times J, \mathbb{R})$, $\alpha_i \in C(J, J)$, $\alpha_i(t) \le t$, $i = 1, 2, 3$ and there exist constants $L_1, L_2, L_3 \in \mathbb{R}_+$ such that

$$\left|g(t, u_1, u_2, u_3) - g(t, v_1, v_2, v_3)\right| \le \sum_{i=1}^{p} L_i |u_i - v_i|.$$

Indeed, Q is a causal operator. Put $z(t) = |u(t) - v(t)|$. In this case, we have

$$\begin{aligned}\left|(Qu)(t) - (Qv)(t)\right| &\le L_1 z(\alpha_1(t)) + L_2 \max_{s\in[0,\alpha_2(t)]} z(s) + L_3 \int_0^t \left|h(t,r)\right| z(\alpha_3(r))\,dr \\ &\le \left(L_1 + L_2 + L_3 \max_{s\in J} \int_0^s \left|h(s,r)\right|dr\right)|z|_t,\end{aligned}$$

so assumption H_2 holds with $L = L_1 + L_2 + L_3 \max_{s\in J} \int_0^s |h(s,r)|\,dr$.

Basing on Theorem 1, we have the following.

Theorem 3 *Let assumption* H_4 *hold and let* $q \in (0,1]$. *Assume that there exists a constant* $M \in \mathbb{R}_+$ *such that condition* (4) *holds with operator* Q *defined by* (8). *Then the sequence* $\{x_n\}$ *converges to the unique solution* $x \in C_{1-q}(J, \mathbb{R})$ *of problem* (1) *with* Q *defined by* (8). *Moreover, we have the error estimation*

$$|x - x_n|_t \le \frac{BL^n}{\Gamma(nq+1)} t^{nq}, \quad n = 0, 1, \ldots,$$

where $L = L_1 + L_2 + L_3 \max_{s\in J} \int_0^s |h(s,r)|\,dr$.

3. Let

$$(Qx)(t) = \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(t, s, x(s))\,ds, \quad q_1 > 0. \tag{9}$$

We see that Q is also a causal operator. Assume that

H_5: $f \in C(J \times J \times \mathbb{R}, \mathbb{R})$, $q_1 > 0$ and there exists a constant $D \in \mathbb{R}_+$ such that

$$\left|f(t,s,u) - f(t,s,v)\right| \le D|u - v|.$$

Under assumption H_5 we see that

$$\left|(Qu)(t) - (Qv)(t)\right| \le \frac{D}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1}\left|u(s) - v(s)\right|ds \le \frac{DT^{q_1}}{\Gamma(q_1+1)}|u - v|_t,$$

so assumption H_2 holds with $L = \frac{DT^{q_1}}{\Gamma(q_1+1)}$.

Basing on Theorem 1, we have the following.

Theorem 4 *Let assumption* H_5 *hold and let* $q \in (0,1]$. *Assume that there exists a constant* $M \in \mathbb{R}_+$ *such that condition* (4) *holds with operator* Q *defined by* (9). *Then the sequence* $\{x_n\}$ *converges to the unique solution* $x \in C_{1-q}(J,\mathbb{R})$ *of problem* (1) *with* Q *defined by* (9). *Moreover, we have the error estimation*

$$|x - x_n|_t \le \frac{BL^n}{\Gamma(nq+1)} t^{nq}, \quad n = 0,1,\ldots,$$

where $L = \frac{DT^{q_1}}{\Gamma(q_1+1)}$.

Now we consider the following linear problem:

$$\begin{cases} D^q x(t) = \lambda I^{q_1} x(t) + \sigma(t), & q_1 > 0, \lambda \in \mathbb{R}, \sigma \in C_{1-q}(J,\mathbb{R}), \\ \bar{x}(0) = k \in \mathbb{R}. \end{cases} \tag{10}$$

Note that problem (10) is a special case of problem (1) with Q defined by (9). Problem (10) has a unique solution, and we can write this solution by a corresponding formula. The next theorem concerns this fact.

Theorem 5 *Let* $q \in (0,1]$, $q_1 > 0$, $\lambda, k \in \mathbb{R}$, $\sigma \in C_{1-q}(J,\mathbb{R})$. *Then problem* (10) *has a unique solution given by the formula*

$$x(t) = k\Gamma(q)t^{q-1}E_{q+q_1,q}\big(\lambda t^{q+q_1}\big) + \int_0^t (t-s)^{q-1}E_{q+q_1,q}\big(\lambda(t-s)^{q+q_1}\big)\sigma(s)\,ds, \tag{11}$$

where $E_{\nu,\beta}(\zeta) = \sum_{r=0}^{\infty} \frac{\zeta^r}{\Gamma(\nu r+\beta)}$ *is the Mittag-Leffler function.*

Proof Indeed, problem (10) is equivalent in the space $C_{1-q}(J,\mathbb{R})$ to the following fractional integral equation:

$$x(t) = x_0(t) + \lambda I^{q+q_1}x(t) + I^q\sigma(t), \quad t \in J_0 \tag{12}$$

with $x_0(t) = kt^{q-1}$.

To find the solution of problem (12) we use the method of successive approximations. For $n = 0,1,\ldots,$ we have

$$x_{n+1}(t) = x_0(t) + \lambda I^{q+q_1}x_n(t) + I^q\sigma(t).$$

Hence,

$$\begin{aligned} x_1(t) &= x_0(t) + \lambda I^{q+q_1}x_0(t) + I^q\sigma(t), \\ x_2(t) &= x_0(t) + \lambda I^{q+q_1}x_1(t) + I^q\sigma(t) \\ &= x_0(t) + \lambda I^{q+q_1}\big[x_0(t) + \lambda I^{q+q_1}x_0(t) + I^q\sigma(t)\big] + I^q\sigma(t) \\ &= x_0(t) + \lambda I^{q+q_1}x_0(t) + \lambda^2 I^{2(q+q_1)}x_0(t) + \lambda I^{2q+q_1}\sigma(t) + I^q\sigma(t), \end{aligned}$$

using the relation $I^rI^m x(t) = I^{r+m}x(t)$, $r,m > 0$.

Thus, in general, we get by induction x_n as follows:

$$x_n(t) = x_0(t) + \sum_{i=1}^{n} \lambda^i I^{i(q+q_1)} x_0(t) + \sum_{i=1}^{n} \lambda^{i-1} I^{(i-1)(q+q_1)+q} \sigma(t), \quad n = 1, 2, \ldots. \tag{13}$$

Using the following formula

$$I^\delta x_0(t) = x_0(t) \frac{\Gamma(q)}{\Gamma(\delta + q)} t^\delta, \quad \delta > 0$$

to (13), we obtain

$$\begin{aligned} x_n(t) &= x_0(t) \left[1 + \Gamma(q) \sum_{i=1}^{n} \lambda^i \frac{1}{\Gamma(i(q+q_1)+q)} t^{i(q+q_1)} \right] \\ &\quad + \sum_{i=1}^{n} \lambda^{i-1} \frac{1}{\Gamma((i-1)(q+q_1)+q)} \int_0^t (t-s)^{(i-1)(q+q_1)+q-1} \sigma(s)\, ds \\ &= x_0(t) \Gamma(q) \sum_{i=0}^{n} \lambda^i \frac{1}{\Gamma(i(q+q_1)+q)} t^{i(q+q_1)} \\ &\quad + \int_0^t (t-s)^{q-1} \left[\sum_{i=0}^{n-1} \lambda^i \frac{1}{\Gamma(i(q+q_1)+q)} (t-s)^{i(q+q_1)} \right] \sigma(s)\, ds \end{aligned}$$

for $n = 0, 1, \ldots$. Taking the limit as $n \to \infty$, we obtain the unique solution x in terms of the Mittag-Leffler function given by formula (11). □

Example 1 Consider the following problem:

$$\begin{cases} D^q x(t) = I^{q_1} x(t) + \frac{4}{3\sqrt{\pi}} (\frac{3}{2} t^{\frac{1}{2}} - t^{\frac{3}{2}}) - 5\sqrt{\pi}, & t \in (0, T], \\ \bar{x}(0) = 5 \end{cases} \tag{14}$$

with $q = q_1 = \frac{1}{2}$. In view of formula (11), we have

$$\begin{aligned} x(t) &= 5\sqrt{\pi} t^{-\frac{1}{2}} E_{1,\frac{1}{2}}(t) + \int_0^t (t-s)^{-\frac{1}{2}} E_{1,\frac{1}{2}}(t-s) \left[\frac{4}{3\sqrt{\pi}} \left(\frac{3}{2} s^{\frac{1}{2}} - s^{\frac{3}{2}} \right) - 5\sqrt{\pi} \right] ds \\ &= 5\sqrt{\pi} t^{-\frac{1}{2}} \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i+\frac{1}{2})} + \sum_{i=0}^{\infty} \frac{1}{\Gamma(i+\frac{1}{2})} \left\{ \frac{4}{3\sqrt{\pi}} \left[\frac{3}{2} \frac{\Gamma(i+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(i+2)} t^{i+1} \right. \right. \\ &\quad \left. \left. - \frac{\Gamma(i+\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(i+3)} t^{i+2} \right] - 5\sqrt{\pi} \frac{\Gamma(i+\frac{1}{2})}{\Gamma(i+\frac{3}{2})} t^{i+\frac{1}{2}} \right\} \\ &= 5t^{-\frac{1}{2}} + \frac{4}{3\sqrt{\pi}} \left\{ \frac{3}{2} \Gamma\left(\frac{3}{2}\right) \sum_{i=0}^{\infty} \frac{t^{i+1}}{\Gamma(i+2)} - \Gamma\left(\frac{5}{2}\right) \left[\sum_{i=0}^{\infty} \frac{t^{i+1}}{\Gamma(i+2)} - t \right] \right\} \\ &= 5t^{-\frac{1}{2}} + t. \end{aligned}$$

It shows that $x(t) = 5t^{-\frac{1}{2}} + t$ is the unique solution of problem (14).

Theorem 6 *Let $q \in (0,1]$, $q_1 > 0$, $\lambda_1, \lambda_2, k_1, k_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in C_{1-q}(J, \mathbb{R})$. Then the system*

$$\begin{cases} D^q x(t) = \lambda_1 I^{q_1} x(t) + \lambda_2 x(t) - \lambda_1 I^{q_1} y(t) + \lambda_2 y(t) + \sigma_1(t), & t \in (0,T], \\ D^q y(t) = -\lambda_1 I^{q_1} x(t) + \lambda_2 x(t) + \lambda_1 I^{q_1} y(t) + \lambda_2 y(t) + \sigma_2(t), & t \in (0,T], \\ \bar{x}(0) = k_1, \qquad \bar{y}(0) = k_2, \quad k_1, k_2 \in \mathbb{R} \end{cases} \tag{15}$$

has a unique solution given by

$$\begin{cases} x(t) = \frac{1}{2}[k\Gamma(q)t^{q-1}E_{q,q}(2\lambda_2 t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(2\lambda_2(t-s)^q)\sigma(s)\,ds \\ \qquad + \bar{k}\Gamma(q)t^{q-1}E_{q+q_1,q}(2\lambda_1 t^{q+q_1}) + \int_0^t (t-s)^{q-1}E_{q+q_1,q}(2\lambda_1(t-s)^{q+q_1})\bar{\sigma}(s)\,ds], \\ y(t) = \frac{1}{2}[k\Gamma(q)t^{q-1}E_{q,q}(2\lambda_2 t^q) + \int_0^t (t-s)^{q-1}E_{q,q}(2\lambda_2(t-s)^q)\sigma(s)\,ds \\ \qquad - \bar{k}\Gamma(q)t^{q-1}E_{q+q_1,q}(2\lambda_1 t^{q+q_1}) + \int_0^t (t-s)^{q-1}E_{q+q_1,q}(2\lambda_1(t-s)^{q+q_1})\bar{\sigma}(s)\,ds]. \end{cases}$$

Proof Put $u = x + y$, $v = x - y$. Then from system (15) we have two following systems for solving

$$\begin{cases} D^q u(t) = 2\lambda_2 u(t) + \sigma(t), & \sigma(t) = \sigma_1(t) + \sigma_2(t), t \in (0,T], \\ \bar{u}(0) = k_1 + k_2 \equiv k, \end{cases} \tag{16}$$

$$\begin{cases} D^q v(t) = 2\lambda_1 I^{q_1} v(t) + \bar{\sigma}(t), & \bar{\sigma}(t) = \sigma_1(t) - \sigma_2(t), t \in (0,T], \\ \bar{v}(0) = k_1 - k_2 \equiv \bar{k}. \end{cases} \tag{17}$$

The solution of (16) is given by

$$u(t) = k\Gamma(q)t^{q-1}E_{q,q}\bigl(2\lambda_2 t^q\bigr) + \int_0^t (t-s)^{q-1}E_{q,q}\bigl(2\lambda_2 t^q\bigr)\sigma(s)\,ds.$$

The solution of (17) has the form

$$v(t) = \bar{k}\Gamma(q)t^{q-1}E_{q+q_1,q}\bigl(2\lambda_1 t^{q+q_1}\bigr) + \int_0^t (t-s)^{q-1}E_{q+q_1,q}\bigl(2\lambda_1(t-s)^{q+q_1}\bigr)\bar{\sigma}(s)\,ds$$

by Theorem 6. Solving the system

$$x + y = u, \qquad x - y = v$$

we have the solution of x and y. This ends the proof. □

Example 2 Consider the system

$$\begin{cases} D^q x(t) = \frac{1}{2}[I^{q_1}x(t) + x(t) - I^{q_1}y(t) + y(t)] + \sigma_1(t), & t \in (0,T], \\ D^q y(t) = \frac{1}{2}[-I^{q_1}x(t) + x(t) + I^{q_1}y(t) + y(t)] + \sigma_2(t), & t \in (0,T], \\ \bar{x}(0) = 2, \qquad \bar{y}(0) = 0, \end{cases} \tag{18}$$

with $q = \frac{1}{2}$, $q_1 = \frac{3}{2}$ and

$$\sigma_1(t) = -t^{-\frac{1}{2}} + \frac{10}{\sqrt{\pi}}t^{\frac{1}{2}} - \left(\sqrt{\pi} + \frac{5}{2}\right)t - \frac{3}{2}t^2 - \frac{4}{3\sqrt{\pi}}t^{\frac{5}{2}} + \frac{16}{35\sqrt{\pi}}t^{\frac{7}{2}},$$
$$\sigma_2(t) = -t^{-\frac{1}{2}} + \left(\sqrt{\pi} - \frac{5}{2}\right)t + \frac{8}{\sqrt{\pi}}t^{\frac{3}{2}} - \frac{3}{2}t^2 + \frac{4}{3\sqrt{\pi}}t^{\frac{5}{2}} - \frac{16}{35\sqrt{\pi}}t^{\frac{7}{2}}.$$

We see that

$$
\begin{aligned}
U(t) &\equiv 2\sqrt{\pi}t^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\big(t^{\frac{1}{2}}\big) + \int_0^t (t-s)^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\big((t-s)^{\frac{1}{2}}\big)\big[\sigma_1(s)+\sigma_2(s)\big]\,ds \\
&= 2\sqrt{\pi}t^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\big(t^{\frac{1}{2}}\big) \\
&\quad + \sum_{i=0}^{\infty}\frac{1}{\Gamma(\frac{1}{2}(i+1))}\int_0^t (t-s)^{\frac{1}{2}(i-1)}\bigg[-2s^{-\frac{1}{2}} + \frac{10}{\sqrt{\pi}}s^{\frac{1}{2}} - 5s - 3s^2 + \frac{8}{\sqrt{\pi}}s^{\frac{3}{2}}\bigg]\,ds \\
&= 2\sqrt{\pi}t^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\big(t^{\frac{1}{2}}\big) - 2\sqrt{\pi}\sum_{i=0}^{\infty}\frac{t^{\frac{1}{2}i}}{\Gamma(\frac{1}{2}(i+2))} + 5\sum_{i=0}^{\infty}\frac{t^{\frac{1}{2}i+1}}{\Gamma(\frac{1}{2}(i+4))} \\
&\quad - 5\sum_{i=0}^{\infty}\frac{t^{\frac{1}{2}i+\frac{3}{2}}}{\Gamma(\frac{1}{2}(i+5))} - 6\sum_{i=0}^{\infty}\frac{t^{\frac{1}{2}i+\frac{5}{2}}}{\Gamma(\frac{1}{2}(i+7))} + 6\sum_{i=0}^{\infty}\frac{t^{\frac{1}{2}i+2}}{\Gamma(\frac{1}{2}(i+6))} \\
&= 2\sqrt{\pi}t^{-\frac{1}{2}}E_{\frac{1}{2},\frac{1}{2}}\big(t^{\frac{1}{2}}\big) - 2\sqrt{\pi}\bigg[1 + \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} + t + \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} + \frac{t^2}{2} + t^{\frac{5}{2}}E_{\frac{1}{2},\frac{7}{2}}\big(t^{\frac{1}{2}}\big)\bigg] \\
&\quad + 5\bigg[t + \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} + \frac{1}{2}t^2 + t^{\frac{5}{2}}E_{\frac{1}{2},\frac{7}{2}}\big(t^{\frac{1}{2}}\big)\bigg] - 5\bigg[\frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} + \frac{1}{2}t^2 + t^{\frac{5}{2}}E_{\frac{1}{2},\frac{7}{2}}\big(t^{\frac{1}{2}}\big)\bigg] \\
&\quad - 6t^{\frac{5}{2}}E_{\frac{1}{2},\frac{7}{2}}\big(t^{\frac{1}{2}}\big) + 6\bigg[\frac{1}{2}t^2 + t^{\frac{5}{2}}E_{\frac{1}{2},\frac{7}{2}}\big(t^{\frac{1}{2}}\big)\bigg] = 2t^{-\frac{1}{2}} + 5t + 3t^2
\end{aligned}
$$

and

$$
\begin{aligned}
V(t) &\equiv 2\sqrt{\pi}t^{-\frac{1}{2}}E_{2,\frac{1}{2}}\big(t^2\big) + \int_0^t (t-s)^{-\frac{1}{2}}E_{2,\frac{1}{2}}\big((t-s)^2\big)\big[\sigma_1(s)-\sigma_2(s)\big]\,ds \\
&= 2\sqrt{\pi}t^{-\frac{1}{2}}E_{2,\frac{1}{2}}\big(t^2\big) + \sum_{i=0}^{\infty}\frac{1}{\Gamma(2i+\frac{1}{2})}\int_0^t (t-s)^{2i-\frac{1}{2}}\bigg[\frac{10}{\sqrt{\pi}}s^{\frac{1}{2}} - 2\sqrt{\pi}s - \frac{8}{\sqrt{\pi}}s^{\frac{3}{2}} \\
&\quad - \frac{8}{3\sqrt{\pi}}s^{\frac{5}{2}} + \frac{32}{35\sqrt{\pi}}s^{\frac{7}{2}}\bigg]\,ds \\
&= 2\sqrt{\pi}t^{-\frac{1}{2}}E_{2,\frac{1}{2}}\big(t^2\big) + 5\sum_{i=0}^{\infty}\frac{t^{2i+1}}{\Gamma(2i+2)} - 2\sqrt{\pi}\sum_{i=0}^{\infty}\frac{t^{2i+\frac{3}{2}}}{\Gamma(2i+\frac{5}{2})} - 6\sum_{i=0}^{\infty}\frac{t^{2i+2}}{\Gamma(2i+3)} \\
&\quad - 5\sum_{i=0}^{\infty}\frac{t^{2i+3}}{\Gamma(2i+4)} + 6\sum_{i=0}^{\infty}\frac{t^{2i+4}}{\Gamma(2i+5)} = 2t^{-\frac{1}{2}} + 5t - 3t^2.
\end{aligned}
$$

By Theorem 6, the pair of functions

$$x(t) = 2t^{-\frac{1}{2}} + 5t, \qquad y(t) = 3t^2$$

is the unique solution of system (18).

Competing interests
The author declares that he has no competing interests.

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