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Fractional boundary value problems with Riemann-Liouville fractional derivatives

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Abstract

In this paper, by employing two fixed point theorems of a sum operators, we investigate the existence and uniqueness of positive solutions for the following fractional boundary value problems: $-D^{\alpha}_{0+}x(t) = f(t,x(t),x(t)) + g(t,x(t))$, $0 < t < 1$, $1 < \alpha < 2$, where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative, subject to either the boundary conditions $x(0) = x(1) = 0$ or $x(0) = 0$, $x(1) = \beta x(\eta)$ with $\eta, \beta\eta^{\alpha-1} \in (0,1)$. We also construct an iterative scheme to approximate the solution. As applications of the main results, two examples are given.

Keywords: fractional differential equation; boundary value problem; fixed point theorem; mixed monotone operators

1 Introduction

Fractional differential equations are important mathematical models of some practical problems in many fields such as polymer rheology, chemistry physics, heat conduction, fluid flows, electrical networks, and many other branches of science (see [1–7]). Consequently, the fractional calculus and its applications in various fields of science and engineering have received much attention, and many papers and books on fractional calculus, fractional differential equations have appeared (see [8–15]). It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of the existence of positive solutions for nonlinear fractional differential equation boundary value problems (see [16–21]). However, there are few papers to deal with the existence and uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems.

In this paper, we study the existence and uniqueness of positive solutions to the following two nonlinear Riemann-Liouville fractional differential equation boundary value problems:

$$\begin{cases} D^{\alpha}_{0+}x(t) = f(t,x(t),x(t)) + g(t,x(t)), & 0 < t < 1, 1 < \alpha < 2, \\ x(0) = x(1) = 0, \end{cases} \tag{1}$$

and

$$\begin{cases} D^{\alpha}_{0+}x(t) = f(t,x(t),x(t)) + g(t,x(t)), & 0 < t < 1, 1 < \alpha < 2, \\ x(0) = 0, \quad x(1) = \beta x(\eta), \end{cases} \tag{2}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative and $\eta, \beta\eta^{\alpha-1} \in (0,1)$.

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In [22], by means of the Krasnoselskii fixed point theorem, Jiang and Yuan obtained the existence of positive solutions to the following nonlinear fractional differential equation Dirichlet-type boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 2, \\ u(0) = u(1) = 0. \end{cases} \tag{3}$$

Bai and Lü [23] obtained some existence results of positive solutions of the problem by (3) using some fixed point theorems on cone. A more general equation with f depending on derivatives was studied in [24]. In [25], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, u(t)), & 0 < t < 1, 1 < \alpha < 2, \\ u(0) = 0, \qquad u(1) = \beta u(\eta), \end{cases}$$

where $0 \leq \beta, \eta \leq 1$. They established some existence results of positive solutions. In [26], the boundary condition $D_{0+}^{\beta}u(1) = aD_{0+}^{\beta}u(\xi)$ was considered, where $0 \leq \beta \leq 1$, $0 \leq a \leq 1$, $0 < \xi < 1$. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

Motivated by the results mentioned above, in this paper, we study the existence and uniqueness of positive solutions for the BVP (1) and (2). We have found that no result has been established for the existence and uniqueness of positive solutions for the problem (1) and (2) of fractional differential equation. This paper aims to establish the existence and uniqueness of positive solutions for the problem (1) and (2). The technique relies on two fixed point theorems of a sum operator. The method used in this paper is different from the ones in the papers mentioned above. In addition, we also construct an iterative sequence to approximate the solution.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and facts. In Section 3, the main results are discussed by using the properties of the Green function and the fixed point theorem on mixed monotone operators. Finally, in Section 4, we give two examples to demonstrate our results.

2 Preliminaries

In this section, we recall some definitions and facts which will be used in the later analysis. For convenience, we introduce the following notations:

(1) $\lceil\alpha\rceil$: the smallest integer greater than or equal to α;

(2) $[\alpha]$: the integer part of the number α;

(3) $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-t}\,dt$.

Definition 2.1 ([27]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2.2 ([27]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $f : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n-1}}\,dt,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.1 ([23]) *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N},$$

for some $c_i \in R$, $i = 1,2,3,\ldots,N$, where $N = \lceil \alpha \rceil$.

Denote by E a real Banach space. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $\lambda \geq 0 \Rightarrow \lambda x \in P$ and (ii) $x \in P$, $-x \in P \Rightarrow x = \theta$. Suppose that $(E, \|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P \subset E$, *i.e.*, $x \leq y$ if and only if $y - x \in P$. The cone P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, and N is called the normal constant. Putting $P^o = \{x \in P \mid x \text{ is an interior point of } P\}$, the cone P is said to be solid if its interior P^o is non-empty. If $x_1, x_2 \in E$, the set $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$ is called the order interval between x_1 and x_2. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leq y$ implies $Ax \leq Ay$ $(Ax \geq Ay)$.

For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (*i.e.*, $h \geq \theta$ and $h \neq \theta$), let $P_h = \{x \in E \mid x \sim h\}$. If $P_h \subset P^o$, then $P_h = P^o$.

The basic space used in this paper is the space $C[0,1]$, it is a Banach space if it is endowed with the norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$ for any $x \in C[0,1]$. Notice that this space can be equipped with a partial order given by $x, y \in C[0,1]$, $x \leq y \Longleftrightarrow x(t) \leq y(t)$ for $t \in [0,1]$. Let $P \subset C[0,1]$ by $P = \{x \in C[0,1] \mid x(t) \geq 0, t \in [0,1]\}$. Clearly P is a normal cone in $C[0,1]$ and the normality constant is 1.

Lemma 2.2 ([22]) *Given $y \in C[0,1]$ and $1 < \alpha \leq 2$, the unique solution of*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0 \end{cases} \tag{4}$$

is

$$u(t) = \int_0^1 G(t,s) y(s)\,ds,$$

where $G(t,s)$ is the Green function of the BVP (4) *and is given by*

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & s \leq t, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & t \leq s. \end{cases}$$

Lemma 2.3 ([22]) *$G(t,s)$ satisfies the following properties*:

(1) $G(t,s) \in C([0,1] \times [0,1])$;

(2) $G(t,s) > 0$ *for* $t,s \in (0,1)$;

(3) *for all* $t,s \in (0,1)$,

$$\frac{\alpha - 1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-1} s \leq G(t,s) \leq \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}. \tag{5}$$

Lemma 2.4 *Assume that* $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *and* $g : [0,1] \times [0,\infty) \to [0,\infty)$ *are continuous. A function* $u \in P$ *is a solution of the BVP* (1) *if and only if it is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr)\bigr]\,ds. \tag{6}$$

Proof The proof is similar to Lemma 2.2 and omitted. □

Lemma 2.5 ([25]) *Given* $y \in C[0,1]$, *the problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \beta u(\eta) \end{cases} \tag{7}$$

is equivalent to

$$u(t) = \int_0^1 H(t,s) y(s)\,ds,$$

where $H(t,s)$ *is the Green function of the BVP* (7) *and is given by*

$$H(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1} - (t-s)^{\alpha-1}(1-\beta\eta)^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, s \leq \eta, \\ \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}(1-\beta\eta^{\alpha-1})}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \leq \eta \leq s \leq t \leq 1, \\ \frac{[t(1-s)]^{\alpha-1} - \beta t^{\alpha-1}(\eta-s)^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq \eta \leq 1, \\ \frac{[t(1-s)]^{\alpha-1}}{(1-\beta\eta^{\alpha-1})\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \eta \leq s. \end{cases}$$

Lemma 2.6 ([25]) *$H(t,s)$ has the following properties*:

(1) $H(t,s) \in C([0,1] \times [0,1])$;

(2) $H(t,s) > 0$ *for* $t,s \in (0,1)$;

(3) *for all* $t,s \in (0,1)$,

$$\frac{M_0 t^{\alpha-1} s(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})} \leq H(t,s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)(1-\beta\eta^{\alpha-1})}, \tag{8}$$

where $0 < M_0 = \min\{1 - \beta\eta^{\alpha-1}, \beta\eta^{\alpha-1}\} < 1$.

Lemma 2.7 *Assume that* $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *and* $g : [0,1] \times [0,\infty) \to [0,\infty)$ *are continuous. A function* $u \in P$ *is a solution of the BVP* (2) *if and only if it is a solution of the integral equation*

$$u(t) = \int_0^1 H(t,s)\bigl[f\bigl(s,u(s),u(s)\bigr) + g\bigl(s,u(s)\bigr)\bigr]\,ds. \tag{9}$$

Proof The proof is similar to Lemma 2.5 and omitted. □

Definition 2.3 ([28]) $A : P \times P \to P$ is said to be a mixed monotone operator if $A(u, v)$ is increasing in u and decreasing in v, *i.e.*, u_i, v_i $(i = 1, 2) \in P$, $u_1 \leq u_2$, $v_1 \geq v_2$ imply $A(u_1, v_1) \leq A(u_2, v_2)$. The element $x \in P$ is called a fixed point of A if $A(x, x) = x$.

Definition 2.4 ([29]) An operator $A : E \to E$ is said to be positive homogeneous if it satisfies $A(tx) = tAx$, $\forall t > 0, x \in E$. An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies

$$A(tx) \geq tAx, \quad \forall t \in (0, 1), x \in P. \tag{10}$$

Definition 2.5 ([29]) Let $D = P$ and r be a real number with $0 \leq r < 1$. An operator $A : P \to P$ is said to be r-concave if it satisfies

$$A(tx) \geq t^r Ax, \quad \forall t \in (0, 1), x \in D. \tag{11}$$

Lemma 2.8 ([28]) *Let $h > \theta$ and $\beta \in (0, 1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies*

$$A\big(tx, t^{-1}y\big) \geq t^{\beta} A(x, y), \quad \forall t \in (0, 1), x, y \in P. \tag{12}$$

$B : P \to P$ is an increasing sub-homogeneous operator. Assume that

(i) *there is a $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*

(ii) *there exists a constant $\delta_0 > 0$ such that $A(x, y) \geq \delta_0 Bx$, $\forall x, y \in P$.*

Then

(1) *$A : P_h \times P_h \to P_h$, $B : P_h \to P_h$;*

(2) *there exist $u_0, v_0 \in P_h$ and $\gamma \in (0, 1)$ such that $\gamma v_0 \leq u_0 < v_0$, $u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0$;*

(3) *the operator equation $A(x, x) + Bx = x$ has a unique solution x^* in P_h;*

(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively sequences $x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}$, $y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}$, $n = 1, 2, \ldots$, we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.*

Lemma 2.9 ([28]) *Let $h > \theta$ and $\alpha \in (0, 1)$. $A : P \times P \to P$ is a mixed monotone operator and satisfies*

$$A\big(tx, t^{-1}y\big) \geq tA(x, y), \quad \forall t \in (0, 1), x, y \in P. \tag{13}$$

$B : P \to P$ is an increasing α-concave operator. Assume that

(i) *there is a $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*

(ii) *there exists a constant $\delta_0 > 0$ such that $A(x, y) \leq \delta_0 Bx$, $\forall x, y \in P$.*

Then

(1) *$A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$;*

(2) *there exist $u_0, v_0 \in P_h$ and $\gamma \in (0, 1)$ such that $\gamma v_0 \leq u_0 < v_0$, $u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0$;*

(3) *the operator equation $A(x, x) + Bx = x$ has a unique solution x^* in P_h;*

(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences* $x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}$, $y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}$, $n = 1, 2, \ldots$, *we have* $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

Lemma 2.10 ([28]) *Let* $\alpha \in (0,1)$ *and* $A : P \times P \to P$ *be a mixed monotone operator. Assume that* (12) *holds and there is* $h_0 > \theta$ *such that* $A(h_0, h_0) \in P_h$.

Then

(1) $A : P_h \times P_h \to P_h$;

(2) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \leq u_0 \leq v_0$, $u_0 \leq A(u_0, v_0) \leq A(v_0, u_0) \leq v_0$;

(3) *the operator equation* $A(x, x) = x$ *has a unique solution* x^* *in* P_h;

(4) *for any initial values* $x_0, y_0 \in P_h$, *constructing successively the sequences* $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, *we have* $x_n \to x^*$ *and* $y_n \to x^*$ *as* $n \to \infty$.

3 Main results

In this section, we establish the existence and uniqueness of positive solutions results for the problems (1) and (2), respectively.

First, we give the existence and uniqueness of positive solutions to the problem (1).

Theorem 3.1 *Assume that*

(H_1) $f(t,x,y) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing in* $x \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $y \in [0,+\infty)$ *decreasing in* $y \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $x \in [0,+\infty)$;

(H_2) $g(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing in* $x \in [0,+\infty)$ *for fixed* $t \in [0,1]$, $g(t,0) \not\equiv 0$;

(H_3) *there exists a constant* $\delta_0 > 0$ *such that* $f(t,x,y) \geq \delta_0 g(t,x)$, $t \in [0,1]$, $x, y \geq 0$;

(H_4) $g(t,\lambda x) \geq \lambda g(t,x)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u \in [0,+\infty)$ *and there exists a constant* $\xi \in (0,1)$ *such that* $f(t,\lambda x,\lambda^{-1}y) \geq \lambda^{\xi} f(t,x,y)$, $\forall t \in [0,1]$, $x, y \in [0,+\infty)$.

Then

(1) *there exist* $x_0, y_0 \in P_h$ *and* $\gamma \in (0,1)$ *such that* $\gamma y_0 \leq x_0 < y_0$ *and*

$$x_0(t) \leq \int_0^1 G(t,s)\bigl[f\bigl(t, x_0(s), y_0(s)\bigr) + g\bigl(s, x_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1],$$

$$y_0(t) \geq \int_0^1 G(t,s)\bigl[f\bigl(s, x_0(s), y_0(s)\bigr) + g\bigl(s, x_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\alpha-1}(1-t)$, $t \in [0,1]$;

(2) *the BVP* (1) *has a unique positive solution* x^* *in* P_h. *For any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s, x_n(s), y_n(s)\bigr) + g\bigl(s, x_n(s)\bigr)\bigr]\,ds, \quad n = 0, 1, 2, \ldots,$$

$$y_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s, y_n(s), x_n(s)\bigr) + g\bigl(s, y_n(s)\bigr)\bigr]\,ds, \quad n = 0, 1, 2, \ldots,$$

we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof Firstly, according to Lemma 2.4, the BVP (1) is equivalent to the integral formulation given by

$$x(t) = \int_0^1 G(t,s)\big[f\big(s,x(s),x(s)\big) + g\big(s,x(s)\big)\big]\,ds.$$

Let $A_1 : P \times P \to E$ be the operator defined by

$$A_1(x,y)(t) = \int_0^1 G(t,s)f\big(s,x(s),y(s)\big)\,ds,$$

and $B_1 : P \to E$ be the operator defined by

$$(B_1x)(t) = \int_0^1 G(t,s)g\big(s,x(s)\big)\,ds.$$

It is simple to show that x is the solution of the BVP (1) if and only if x solves the operator equation $x = A_1(x,x) + B_1x$. From (H_1) and (H_2) we know that $A_1 : P \times P \to P$ and $B_1 : P \to P$.

Secondly, we show that A_1 and B_1 satisfy all the assumptions of Lemma 2.8. To begin with, we prove that A_1 is a mixed monotones operator. In fact, for $x_i, y_i \in P$, $i = 1,2$ with $x_1 \geq x_2$, $y_1 \leq y_2$, we know that $x_1(t) \geq x_2(t)$ and $y_1(t) \leq y_2(t)$ for all $t \in [0,1]$. From (H_1) and Lemma 2.3 we know that $A_1(x_1,y_1)(t) = \int_0^1 G(t,s)f(s,x_1(s),y_1(s))\,ds \geq \int_0^1 G(t,s)f(s,x_2(s),y_2(s))\,ds = A_1(x_2,y_2)(t)$. That is, $A_1(x_1,y_1) \geq A_1(x_2,y_2)$. Furthermore, it follows from (H_2) and Lemma 2.3 that B_1 is increasing. Next we prove that A_1 satisfies (12). For any $\lambda \in (0,1)$ and $x, y \in P$, together with (H_4), we have

$$\begin{aligned}
A_1\big(\lambda x, \lambda^{-1}y\big)(t) &= \int_0^1 G(t,s)f\big(s,\lambda x(s),\lambda^{-1}y(s)\big)\,ds \\
&\geq \lambda^{\xi}\int_0^1 G(t,s)f\big(s,x(s),y(s)\big)\,ds \\
&= \lambda^{\xi}A_1(x,y)(t).
\end{aligned}$$

That is, $A_1(\lambda x, \lambda^{-1}y) \geq \lambda^{\xi}A_1(x,y)$ for $\lambda \in (0,1)$, $x, y \in P$. Hence the operator A_1 satisfies (12). Also, for any $\lambda \in (0,1)$, $x \in P$, using (H_4) we have

$$B_1(\lambda x)(t) = \int_0^1 G(t,s)g\big(s,\lambda x(s)\big)\,ds \geq \lambda\int_0^1 G(t,s)g\big(s,x(s)\big)\,ds = \lambda B_1x(t).$$

Thus, $B_1(\lambda x) \geq \lambda B_1(x)$ for any $\lambda \in (0,1)$, $x \in P$. Hence the operator B_1 is sub-homogeneous. Now we prove that $A_1(h,h) \in P_h$ and $B_1h \in P_h$. We only need to verify the following conclusions:

(a) $\exists a_1 > 0$ and $a_2 > 0$, such that, for all $t \in [0,1]$, $a_2h(t) \leq A_1(h,h)(t) \leq a_1h(t)$;

(b) $\exists b_1 > 0$ and $b_2 > 0$, such that, for all $t \in [0,1]$, $b_2h(t) \leq B_1h(t) \leq b_1h(t)$.

Let

$$a_1 := \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}f(s,1,0)\,ds,$$

$$a_2 := \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2} s f(s,0,1)\,ds,$$
$$b_1 := \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2} g(s,1)\,ds,$$
$$b_2 := \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2} s g(s,0)\,ds.$$

It follows from (H_1), (H_2), and Lemma 2.3 that, for any $t \in [0,1]$,

$$\begin{aligned} A_1(h,h)(t) &= \int_0^1 G(t,s) f\bigl(s,h(s),h(s)\bigr)\,ds \\ &\le \frac{1}{\Gamma(\alpha)} h(t) \int_0^1 (1-s)^{\alpha-2} f(s,1,0)\,ds \\ &= a_1 h(t) \end{aligned}$$

and

$$\begin{aligned} A_1(h,h)(t) &= \int_0^1 G(t,s) f\bigl(s,h(s),h(s)\bigr)\,ds \\ &\ge \frac{\alpha-1}{\Gamma(\alpha)} h(t) \int_0^1 (1-s)^{\alpha-1} f(s,0,1)\,ds \\ &= a_2 h(t). \end{aligned}$$

According to (H_1), (H_2), and (H_3), we get

$$f(s,1,0) \ge f(s,0,1) \ge \delta_0 g(s,0) \ge 0.$$

Since $g(t,0) \not\equiv 0$, we obtain

$$\int_0^1 f(s,1,0)\,ds \ge \int_0^1 f(s,0,1)\,ds \ge \delta_0 \int_0^1 g(s,0)\,ds > 0$$

and, in consequence, $a_1 > 0$ and $a_2 > 0$. Thus, $a_2 h(t) \le A_1(h,h)(t) \le a_1 h(t)$, $t \in [0,1]$, and hence we get (a). In the same way, we get

$$B_1 h(t) \le \frac{1}{\Gamma(\alpha)} h(t) \int_0^1 (1-s)^{\alpha-2} g(s,1)\,ds = b_1 h(t),$$
$$B_1 h(t) \ge \frac{\alpha-1}{\Gamma(\alpha)} h(t) \int_0^1 (1-s)^{\alpha-2} s g(s,0)\,ds = b_2 h(t).$$

From $g(t,0) \not\equiv 0$ we have $b_1 > 0$ and $b_2 > 0$. Therefore, $b_2 h(t) \le B_1 h(t) \le b_1 h(t)$, $t \in [0,1]$. Thus, (b) is satisfied. Hence the condition (i) of Lemma 2.8 is proved. For $x, y \in P$, and for any $t \in [0,1]$, from (H_3) we have

$$A_1(x,y)(t) = \int_0^1 G(t,s) f\bigl(s,x(s),y(s)\bigr)\,ds \ge \delta_0 \int_0^1 G(t,s) g\bigl(s,x(s)\bigr)\,ds = \delta_0 B_1 x(t),$$

that is, $A_1(x,y) \geq \delta_0 B_1 x$, $\forall x, y \in P$. It follows from the above inequality that the condition (ii) of Lemma 2.8 is satisfied. Applying Lemma 2.8, we can get the conclusion of Theorem 3.1. □

Combining the proof of Theorem 3.1 with Lemma 2.10, we can prove the following conclusion.

Corollary 3.1 *Assume that*

$(H_1)'$ $f(t,x,y):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ *is continuous and increasing in* $x \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $y \in [0,+\infty)$ *decreasing in* $y \in [0,+\infty)$ *for fixed* $t \in [0,1]$ *and* $x \in [0,+\infty)$, $f(t,0,1) \not\equiv 0$;

$(H_2)'$ *there exists a constant* $\xi \in (0,1)$ *such that* $f(t,\lambda x,\lambda^{-1}y) \geq \lambda^{\xi} f(t,x,y)$, $t \in [0,1]$, $\lambda \in (0,1)$, $x, y \in [0,+\infty)$.

Then

(1) *there exist* $x_0, y_0 \in P_h$ *and* $\gamma \in (0,1)$ *such that* $\gamma y_0 \leq x_0 < y_0$ *and*

$$x_0(t) \leq \int_0^1 G(t,s) f\bigl(t, x_0(s), y_0(s)\bigr)\,ds, \quad t \in [0,1],$$
$$y_0(t) \geq \int_0^1 G(t,s) f\bigl(s, y_0(s), x_0(s)\bigr)\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\alpha-1}(1-t)$, $t \in [0,1]$;

(2) *the problem*

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t, x(t), x(t)), & 0 < t < 1, \\ x(0) = x(1) = 0 \end{cases}$$

has a unique positive solution x^* *in* P_h *and for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s) f\bigl(s, x_n(s), y_n(s)\bigr)\,ds, \quad n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_0^1 G(t,s) f\bigl(s, y_n(s), x_n(s)\bigr)\,ds, \quad n = 0,1,2,\ldots,$$

we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

By using Lemma 2.9 we can also prove the following theorem.

Theorem 3.2 *Assume that* (H_1), (H_2), *and*

(H_5) *there exists a constant* $\delta_0 > 0$ *such that* $f(t,x,y) \leq \delta_0 g(t,x)$, $t \in [0,1]$, $x, y \geq 0$;

(H_6) *there exists a constant* $\xi \in (0,1)$ *such that* $g(t,\lambda x) \geq \lambda^{\xi} g(t,x)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,+\infty)$, *and* $f(t,\lambda x,\lambda^{-1}y) \geq \lambda f(t,x,y)$, $\forall t \in [0,1]$, $\lambda \in (0,1)$, $x, y \in [0,+\infty)$.

Then

(1) *there exist* $x_0, y_0 \in P_h$ *and* $\gamma \in (0,1)$ *such that* $\gamma y_0 \leq x_0 < y_0$ *and*

$$x_0(t) \leq \int_0^1 G(t,s)\big[f\big(t, x_0(s), y_0(s)\big) + g\big(s, x_0(s)\big)\big]\,ds, \quad t \in [0,1],$$

$$y_0(t) \geq \int_0^1 G(t,s)\big[f\big(s, y_0(s), x_0(s)\big) + g\big(s, x_0(s)\big)\big]\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\alpha-1}(1-t)$, $t \in [0,1]$;

(2) *the BVP* (1) *has a unique positive solution* x^* *in* P_h *and for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s, x_n(s), y_n(s)\big) + g\big(s, x_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$

$$y_{n+1}(t) = \int_0^1 G(t,s)\big[f\big(s, y_n(s), x_n(s)\big) + g\big(s, y_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$

we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof The proof is similar to that given for Theorem 3.1. We omit it. □

Next, we present the existence and uniqueness of a positive solution to the problem (2).

Theorem 3.3 *Assume that* (H_1)-(H_4) *hold. Then*

(1) *there exist* $x_0, y_0 \in P_h$ *and* $\gamma \in (0,1)$ *such that* $\gamma y_0 \leq x_0 < y_0$ *and*

$$x_0(t) \leq \int_0^1 H(t,s)\big[f\big(t, x_0(s), y_0(s)\big) + g\big(s, x_0(s)\big)\big]\,ds, \quad t \in [0,1],$$

$$y_0(t) \geq \int_0^1 H(t,s)\big[f\big(s, y_0(s), x_0(s)\big) + g\big(s, x_0(s)\big)\big]\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\alpha-1}$, $t \in [0,1]$;

(2) *the BVP* (2) *has a unique positive solution* x^* *in* P_h *and for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 H(t,s)\big[f\big(s, x_n(s), y_n(s)\big) + g\big(s, x_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$

$$y_{n+1}(t) = \int_0^1 H(t,s)\big[f\big(s, y_n(s), x_n(s)\big) + g\big(s, y_n(s)\big)\big]\,ds, \quad n = 0,1,2,\ldots,$$

we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof Firstly, according to Lemma 2.7, the BVP (2) is equivalent to the integral formulation given by

$$x(t) = \int_0^1 H(t,s)\big[f\big(s, x(s), x(s)\big) + g\big(s, x(s)\big)\big]\,ds.$$

Let $A_2 : P \times P \to E$ be the operator defined by

$$A_2(x,y)(t) = \int_0^1 H(t,s)f\bigl(s,x(s),y(s)\bigr)\,ds,$$

and $B_2 : P \to E$ be the operator defined by

$$(B_2x)(t) = \int_0^1 H(t,s)g\bigl(s,x(s)\bigr)\,ds.$$

It is easy to prove that x is the solution of the BVP (2) if and only if x solves the operator equation $x = A_2(x,x) + B_2x$. Similar to the proof of Theorem 3.1, we can prove that $A_2 : P \times P \to P$ is a mixed monotone operator and satisfies $A_2(tx, t^{-1}y) \geq t^{\xi} A_2(x,y)$, $B_2 : P \to P$ is an increasing sub-homogeneous operator from (H_1), (H_2), and Lemma 2.6. Secondly, we only need to prove that (i) and (ii) in Lemma 2.8. Next, we show that $A_2(h,h) \in P_h$ and $B_2h \in P_h$. We only need to verify the following conclusions:

(a) $\exists a_1' > 0$ and $a_2' > 0$, such that, for all $t \in [0,1]$, $a_2'h(t) \leq A_2(h,h)(t) \leq a_1'h(t)$;

(b) $\exists b_1' > 0$ and $b_2' > 0$, such that, for all $t \in [0,1]$, $b_2'h(t) \leq B_2h(t) \leq b_1'h(t)$.

Let

$$\begin{aligned}
a_1' &:= \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} f(s,1,0)\,ds, \\
a_2' &:= \frac{M_0}{\Gamma(\alpha)} \int_0^1 \frac{s(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} f(s,0,1)\,ds, \\
b_1' &:= \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} g(s,1)\,ds, \\
b_2' &:= \frac{M_0}{\Gamma(\alpha)} \int_0^1 \frac{s(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} g(s,0)\,ds.
\end{aligned}$$

It follows from (H_1), (H_2), and Lemma 2.6 that

$$\begin{aligned}
A_2(h,h)(t) &= \int_0^1 H(t,s)f\bigl(s,h(s),h(s)\bigr)\,ds \\
&\leq \frac{1}{\Gamma(\alpha)} h(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} f(s,1,0)\,ds \\
&= a_1'h(t), \quad \forall t \in [0,1],
\end{aligned}$$

and

$$\begin{aligned}
A_2(h,h)(t) &= \int_0^1 H(t,s)f\bigl(s,h(s),h(s)\bigr)\,ds \\
&\geq \frac{M_0}{\Gamma(\alpha)} h(t) \int_0^1 \frac{s(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} f(s,0,1)\,ds \\
&= a_2'h(t), \quad \forall t \in [0,1].
\end{aligned}$$

According to (H_1), (H_2), and (H_3), we have

$$f(s,1,0) \geq f(s,0,1) \geq \delta_0 g(s,0) \geq 0.$$

Since $g(t,0) \not\equiv 0$, we obtain

$$\int_0^1 f(s,1,0)\,ds \ge \int_0^1 f(s,0,1)\,ds \ge \delta_0 \int_0^1 g(s,0)\,ds > 0$$

and, in consequence, $a_1' > 0$ and $a_2' > 0$. Therefore, $a_2' h(t) \le A_2(h,h)(t) \le a_1' h(t)$, $t \in [0,1]$, and hence we get (a). In the same way, we get

$$B_2 h(t) \le \frac{1}{\Gamma(\alpha)} h(t) \int_0^1 \frac{(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} g(s,1)\,ds = b_1' h(t)$$

and

$$B_2 h(t) \ge \frac{M_0}{\Gamma(\alpha)} h(t) \int_0^1 \frac{s(1-s)^{\alpha-1}}{1-\beta\eta^{\alpha-1}} g(s,0)\,ds = b_2' h(t).$$

From $g(t,0) \not\equiv 0$, we have $b_1' > 0$ and $b_2' > 0$. Therefore, $b_2' h(t) \le B_2 h(t) \le b_1' h(t)$, $t \in [0,1]$, and (b) is satisfied. Hence the condition (i) of Lemma 2.8 is proved. Next, we show that the condition (ii) of Lemma 2.8 is satisfied. For $x, y \in P$ and for any $t \in [0,1]$, according to (H_3), we have

$$A_2(x,y)(t) = \int_0^1 H(t,s) f\bigl(s, x(s), y(s)\bigr)\,ds \ge \delta_0 \int_0^1 H(t,s) g\bigl(s, u(s)\bigr)\,ds = \delta_0 B_2 u(t).$$

Then we get $A_2(x,y) \ge \delta_0 B_2 x$, $x, y \in P$. Thus, the conclusions of Theorem 3.3 follow from Lemma 2.8. □

From Lemma 2.9 we also prove the following theorem.

Theorem 3.4 *Assume that* (H_1), (H_2), (H_5), *and* (H_6) *hold. Then*

(1) *there exist* $x_0, y_0 \in P_h$ *and* $\gamma \in (0,1)$ *such that* $\gamma y_0 \le x_0 < y_0$ *and*

$$x_0(t) \le \int_0^1 H(t,s)\bigl[f\bigl(t, x_0(s), y_0(s)\bigr) + g\bigl(s, x_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1],$$
$$y_0(t) \ge \int_0^1 H(t,s)\bigl[f\bigl(s, y_0(s), x_0(s)\bigr) + g\bigl(s, x_0(s)\bigr)\bigr]\,ds, \quad t \in [0,1],$$

where $h(t) = t^{\alpha-1}$, $t \in [0,1]$;

(2) *the BVP* (2) *has a unique positive solution* x^* *in* P_h *and for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 H(t,s)\bigl[f\bigl(s, x_n(s), y_n(s)\bigr) + g\bigl(s, x_n(s)\bigr)\bigr]\,ds, \quad n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_0^1 H(t,s)\bigl[f\bigl(s, y_n(s), x_n(s)\bigr) + g\bigl(s, y_n(s)\bigr)\bigr]\,ds, \quad n = 0,1,2,\ldots,$$

we have $\|x_n - x^*\| \to 0$ *and* $\|y_n - x^*\| \to 0$ *as* $n \to \infty$.

Proof The proof is similar to that given for Theorem 3.3. We omit it. □

4 Example

In this section, we give two examples to illustrate our results.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{\frac{5}{3}}x(t) = x^{\frac{1}{3}} + \arctan x + y^{-\frac{1}{3}} + t^2 + t^3 + \frac{\pi}{2}, & 0 < t < 1, \\ x(0) = x(1) = 0. \end{cases} \tag{14}$$

In this case, $\alpha = \frac{5}{3}$. Problem (14) can be regarded as a boundary value problem of the form (1) with $f(t,x,y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2}$ and $g(t,x) = \arctan x + t^3$. Now we verify that conditions (H_1)-(H_4) are satisfied.

Firstly, it is easy to see (H_1) and (H_2) are satisfied and $g(t,0) = t^3 \not\equiv 0$. Secondly, take $\delta_0 \in (0,1]$, we obtain

$$f(t,x,y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2} \geq t^2 + \frac{\pi}{2} \geq t^3 + \arctan x \geq \delta_0\bigl(t^3 + \arctan x\bigr) = \delta_0 g(t,x).$$

Thus, (H_3) is satisfied. Moreover, for any $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,\infty)$, $y \in [0,\infty)$, we get $\arctan(\lambda x) \geq \lambda \arctan x$. Therefore

$$g(t,\lambda x) \geq \lambda g(t,x),$$

$$f\bigl(t,\lambda x,\lambda^{-1}y\bigr) = \lambda^{\frac{1}{3}}x^{\frac{1}{3}} + \lambda^{\frac{1}{3}}y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2} \geq \lambda^{\frac{1}{3}}\left(x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2}\right) = \lambda^{\gamma} f(t,x,y),$$

where $\gamma = \frac{1}{3}$. We conclude that condition (H_4) is satisfied. Therefore, Theorem 3.1 ensures that the BVP (14) has a unique positive solution in P_h with $h(t) = t^{\frac{1}{3}}(1-t)$.

Example 4.2 Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{\frac{3}{2}}x(t) = 2x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2 + t^3, & 0 < t < 1, \\ x(0) = 0, \qquad x(1) = \frac{1}{2}x(\frac{1}{2}). \end{cases} \tag{15}$$

In this case, $\alpha = \frac{3}{2}$. Problem (15) can be regard as a boundary value problem of form (2) with $f(t,x,y) = x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2$ and $g(t,x) = x^{\frac{1}{2}} + t^3$. Now we verify that conditions (H_1)-(H_4) are satisfied.

Firstly, it is easy to see (H_1) and (H_2) are satisfied and $g(t,0) = t^3 \not\equiv 0$. Secondly, take $\delta_0 \in (0,1]$, we obtain

$$f(t,x,y) = x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2 \geq x^{\frac{1}{2}} + t^3 \geq \delta_0\bigl(x^{\frac{1}{2}} + t^3\bigr) = \delta_0 g(t,x).$$

Thus, (H_3) is satisfied. Moreover, for any $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,\infty)$, $y \in [0,\infty)$, we have

$$g(t,\lambda x) = \lambda^{\frac{1}{2}}x^{\frac{1}{2}} + t^3 \geq \lambda^{\frac{1}{2}}\bigl(x^{\frac{1}{2}} + t^3\bigr) \geq \lambda g(t,x),$$

$$f\bigl(t,\lambda x,\lambda^{-1}y\bigr) = \lambda^{\frac{1}{2}}x^{\frac{1}{2}} + \lambda^{\frac{1}{2}}y^{-\frac{1}{2}} + t^2 \geq \lambda^{\frac{1}{2}}\bigl(x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2\bigr) = \lambda^{\gamma} f(t,x,y),$$

where $\gamma = \frac{1}{2}$. We conclude that condition (H_4) is satisfied. Therefore, Theorem 3.3 ensures that the BVP (15) has a unique positive solution in P_h with $h(t) = t^{\frac{1}{2}}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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