

A REMARK ON CERTAIN p-VALENT FUNCTIONS

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ABSTRACT. The object of the present paper is to prove an interesting result for certain analytic and p -valent functions in the unit disc $U = \{z : |z| < 1\}$.

KEY WORDS AND PHRASES. Analytic, p -valent, Ruscheweyh derivative.
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1. INTRODUCTION.Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N} = \{1, 2, \dots\}) \quad (1.1)$$

which are analytic and p -valent in the unit disc $U = \{z : |z| < 1\}$. For functions $f_j(z)$ ($j=1, 2$) defined by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \quad (1.2)$$

we define the convolution product $f_1 * f_2(z)$ of functions $f_1(z)$ and $f_2(z)$ by,

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k. \quad (1.3)$$

Using the above convolution product, we define

$$D^{n+p-1} f(z) = \left(\frac{z^p}{(1-z)^{n+p}} \right) * f(z) \quad (f(z) \in A(p)), \quad (1.4)$$

where n is any integer greater than $-p$. We note that

$$D^{n+p-1} f(z) = \frac{z^p (z^{n-1} f(z))^{(n+p-1)}}{(n+p-1)!}. \quad (1.5)$$

The symbol D^{n+p-1} when $p = 1$ was introduced by Ruscheweyh [8], and the symbol D^{n+p-1} was introduced by Goel and Sohi [5]. This symbol was named the Ruscheweyh derivative of $(n+p-1)$ -th order by Chen and Owa [4].

It follows from (1.5) that

$$z(D^{n+p-1} f(z))' = (n+p) D^{n+p} f(z) - n D^{n+p-1} f(z) \quad (\text{cf. [4] and [5]}). \quad (1.6)$$

Recently, Chen and Lan ([1], [2]), Chen, Lee and Owa [3], Chen and Owa [4] and Srivastava, Owa and Pashkouleva [9] have been proved some interesting results of analytic functions involving Ruscheweyh derivatives. In the present paper, we prove an interesting result for functions $f(z) \in A(p)$ satisfying

$$\operatorname{Re} \left\{ \frac{D^{n+p+1}f(z)}{z^p} \right\} > \alpha, \quad 0 \leq \alpha < 1 \text{ and } n \in N_0 = N \cup \{0\}.$$

2. MAIN RESULT .

In order to prove our main result , we recall here the following lemma:

LEMMA (Miller [6]; Miller and Mocanu [7]).

Let $\varphi(u, v)$ be a complex - valued function ,

$\varphi: D \rightarrow C$, $D \subset C \times C$ (C is the complex plane),

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\varphi(u, v)$ satisfies the following conditions:

- (i) $\varphi(u, v)$ is continuous in D ;
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\varphi(1, 0)\} > 0$;

$$(iii) \operatorname{Re}\{\varphi(iu_2, v_1)\} \leq 0 \text{ for all } (iu_2, v_1) \in D \text{ and such that } v_1 \leq -\frac{(1+u_2^2)}{2}.$$

Let $q(z) = 1 + q_1 z + q_2 z^2 + \dots$ be regular in the unit disc U such that $(q(z), zq'(z)) \in D$ for all $z \in U$. If

$$\operatorname{Re}\{\varphi(q(z), zq'(z))\} > 0 \quad (z \in U),$$

then

$$\operatorname{Re}\{q(z)\} > 0 \quad (z \in U).$$

Applying the above Lemma , we derive the following:

THEOREM. Let the function $f(z)$ be in the class $A(p)$ satisfy

$$\operatorname{Re} \left\{ \frac{D^{n+p+1}f(z)}{z^p} \right\} > \alpha, \quad (z \in U) \tag{2.1}$$

for $0 \leq \alpha < 1$ and $n \in N_0$. Then

$$\operatorname{Re} \left\{ \sqrt{\frac{D^{n+p}f(z)}{z^p}} \right\} > \beta, \quad (z \in U), \tag{2.2}$$

where

$$\beta = \frac{1 + \sqrt{1 + 4\alpha(n+p+1)(n+p+2)}}{2(n+p+2)}. \tag{2.3}$$

PROOF. For $f(z)$ in $A(p)$, we define the function $q(z)$ by

$$\sqrt{\frac{D^{n+p}f(z)}{z^p}} = \beta + (1-\beta)q(z), \tag{2.4}$$

where β is given by (2.3). Then $q(z)$ is regular in U and $q(z) = 1 + q_1 z + q_2 z^2 + \dots$. Taking the derivatives of both sides in (2.4) , we have

$$\frac{z(D^{n+p}f(z))' - p(D^{n+p}f(z))}{z^p} = 2(1-\beta)[\beta + (1-\beta)q(z)]zq'(z). \tag{2.5}$$

Since the identity (1.5) implies

$$z(D^{n+p}f(z))' = (n+p+1)D^{n+p+1}f(z) - (n+1)D^{n+p}f(z), \quad (2.6)$$

(2.5) becomes

$$\frac{D^{n+p+1}f(z)}{z^p} = [\beta + (1-\beta)q(z)]^2 + \frac{2(1-\beta)[\beta + (1-\beta)q(z)]zq'(z)}{(n+p+1)}, \quad (2.7)$$

or

$$\operatorname{Re}\left\{\frac{D^{n+p+1}f(z)}{z^p} - \alpha\right\} = \operatorname{Re}\left\{[\beta + (1-\beta)q(z)]^2 + \frac{2(1-\beta)[\beta + (1-\beta)q(z)]zq'(z)}{(n+p+1)} - \alpha\right\} > 0. \quad (2.8)$$

Taking $q(z) = u = u_1 + iu_2$ and $zq'(z) = v = v_1 + iv_2$, we define the function $\varphi(u, v)$ by

$$\varphi(u, v) = [\beta + (1-\beta)u]^2 + \frac{2(1-\beta)[\beta + (1-\beta)u]v}{(n+p+1)} - \alpha. \quad (2.9)$$

Then it follows from (2.9) that

- (i) $\varphi(u, v)$ is continuous in $D = C \times C$,
- (ii) $(1, 0) \in D$ and $\operatorname{Re}\{\varphi(1, 0)\} = 1 - \alpha > 0$,

$$(iii) \text{ for all } (iu_2, v_1) \in D \text{ such that } v_1 \leq -\frac{(1+u_2^2)}{2},$$

$$\begin{aligned} \operatorname{Re}\{\varphi(iu_2, v_1)\} &= \beta^2 - (1-\beta)^2 u_2^2 + \frac{2\beta(1-\beta)v_1}{(n+p+1)} - \alpha \\ &\leq \beta^2 - (1-\beta)^2 u_2^2 - \frac{\beta(1-\beta)(1+u_2^2)}{(n+p+1)} - \alpha \\ &< 0 \end{aligned}$$

for $0 \leq \alpha < 1$, $n \in N_0$, $n > -p$ and β is given by (2.3). Therefore, the function $\varphi(u, v)$ satisfies the conditions in the lemma. Thus we have $\operatorname{Re}\{q(z)\} > 0$ ($z \in U$), that is,

$$\operatorname{Re}\left\{\sqrt{\frac{D^{n+p}f(z)}{z^p}}\right\} > \beta = \frac{1 + \sqrt{1 + 4\alpha(n+p+1)(n+p+2)}}{2(n+p+2)} \quad (2.10)$$

which completes the proof of the Theorem.

Letting $\alpha = 0$, the theorem gives:

COROLLARY 1. Let the function $f(z)$ be in the class $A(p)$ satisfy

$$\operatorname{Re}\left\{\frac{D^{n+p+1}f(z)}{z^p}\right\} > 0 \quad (z \in U) \quad (2.11)$$

for $n \in N_0$ and $n > -p$. Then

$$\operatorname{Re}\left\{\sqrt{\frac{D^{n+p}f(z)}{z^p}}\right\} > \frac{1}{(n+p+2)} \quad (z \in U). \quad (2.12)$$

Taking $n = 1-p$ in the above theorem, we have

COROLLARY 2. Let the function $f(z)$ be in the class $A(p)$ satisfy

$$\operatorname{Re} \left\{ \frac{D^2 f(z)}{z^p} \right\} > \alpha \quad (z \in U) \quad (2.13)$$

for $0 \leq \alpha < 1$ and $p \in N$. Then

$$\operatorname{Re} \left\{ \sqrt{\frac{zf'(z) + (1-p)f(z)}{z^p}} \right\} > \frac{1 + \sqrt{1 + 24\alpha}}{6} \quad (2.14)$$

REMARK 1. Putting $p = 1$ in the above results, we get the results obtained by Chen, Lee and Owa [3].

REMARK 2. Using the same technique as in the theorem (or putting $\frac{zf'(z)}{p}$ instead of $f(z)$ in the theorem), we have the following result :

COROLLARY 3. Let the function $f(z)$ be in the class $A(p)$ satisfy

$$\operatorname{Re} \left\{ \frac{(D^{n+p+1}f(z))'}{pz^{p-1}} \right\} > \alpha \quad (z \in U)$$

for $0 \leq \alpha < 1$ and $n \in N_0$. Then

$$\operatorname{Re} \left\{ \sqrt{\frac{(D^{n+p}f(z))'}{pz^{p-1}}} \right\} > \beta$$

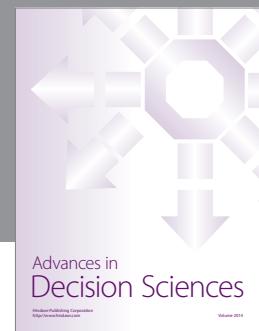
where β is given by (2.3)

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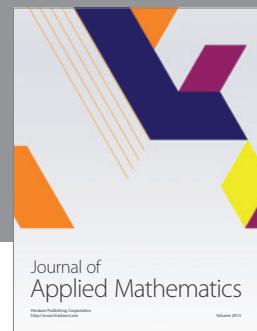
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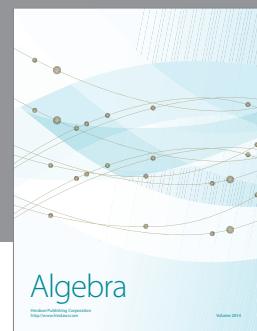
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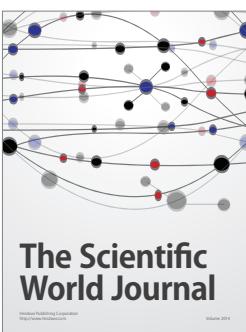
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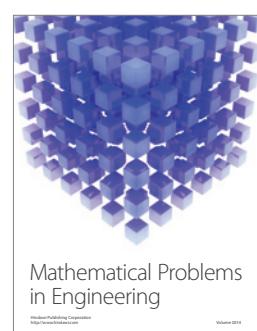
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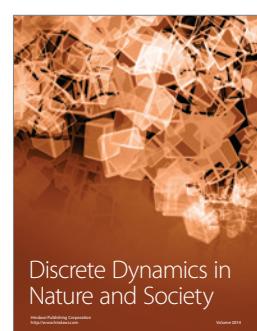
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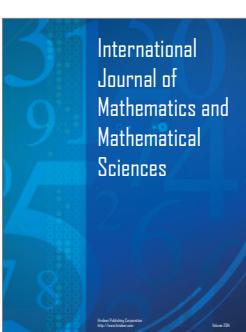
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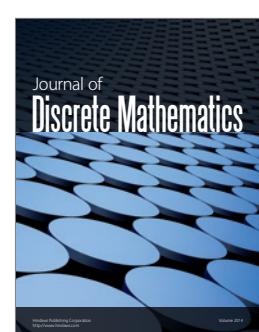
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