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FUNCTIONAL EQUATION OF A SPECIAL DIRICHLET SERIES

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ABSTRACT. In this paper we study the special Dirichlet series

$$L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin(\frac{2\pi n}{3})n^{-s} \quad , \ s \in C$$

This series converges uniformly in the half-plane Re(s) > 1 and thus represents a holomorphic function there. We show that the function L can be extended to a holomorphic function in the whole complex-plane. The values of the function L at the points 0,±1,-2,±3,-4,±5,... are obtained. The values at the positive integers 1,3,5,... are determined by means of a functional equation satisfied by L.

KEY WORDS AND PHRASES. Dirichlet Series, Analytic Continuation, Functional Equation, Γ-Function.
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1. INTRODUCTION.

By a Dirichlet series we mean a series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s}$$

where the coefficients a_n are any given numbers, and s is a complex variable [1], [2].

In this paper we study the special Dirichlet series

$$L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin(\frac{2\pi n}{3})n^{-s} \quad , \ s \in C$$

which converges uniformly in the half-plane Re(s) > 1 and thus represents an analytic function there. In section 1 we study the analytic behaviour of the function L beyond the half-plane Re(s) > 1, and prove that the function L can be extended to a holomorphic function in the whole complex-plane. Moreover values of L at the points -m (m=0,1,2,3,...) are obtained at the end of this section. The values of L at the positive integers 1,3,5,... are determined by means of the functional equation

$$L(s) = \frac{2}{\sqrt{3}}(\frac{2\pi}{3})^{s-1}\Gamma(1-s)\cos(\frac{1}{2}\pi s)L(1-s) \quad , s \in C$$

satisfied by the function L, which we prove in section 2.

2. ANALYTIC CONTINUATION OF L.

$$L(s) = \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{3}\right) n^{-s} \quad , (s \in \mathbb{C}) \tag{2.1}$$

is uniformly convergent in the half-plane Re(s) > 1 and so it represents an analytic function there. The aim of this section is to extend L to the whole complex plane and to prove that L is holomorphic in C.

LEMMA 2.1. For all values of s in the half-plane Re(s) > 1

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} G(t) t^{s-1} dt \quad \text{,where}$$

$$G(t) = \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{3}\right) e^{-nt} \quad \text{,Re(t)>o}$$

$$= \frac{1}{e^{t}+e^{-t}+1}$$

PROOF. Consider the Euler's integral

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt$$

Substitution of nt,n $\in$ N, for t in the above integral yields

$$n^{-s}\,\Gamma(s) = \int_0^{\infty} e^{-nt} t^{s-1} dt \quad \text{,Re(s)>o}$$

Thus for Re(s) > 1, we get

$$\Gamma(s)L(s) = \frac{2}{\sqrt{3}} \sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{3}\right) \int_0^{\infty} e^{-nt} t^{s-1} dt$$

i.e.

$$\Gamma(s)L(s) = \frac{2}{\sqrt{3}} \int_0^{\infty} \sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{3}\right) e^{-nt} t^{s-1} dt \quad ,$$

Thus

$$\Gamma(s)L(s) = \int_0^{\infty} G(t) t^{s-1} dt$$

Now

$$G(t) = \frac{1}{i\sqrt{3}} \sum_{n=1}^{\infty} ((\varepsilon)^n - (\bar{\varepsilon})^n) e^{-nt} \quad \text{,where } \varepsilon = e^{2\pi i/3}.$$

i.e.

$$G(t) = \frac{1}{i\sqrt{3}} \left(\sum_{n=1}^{\infty} (\varepsilon)^n e^{-nt} - \sum_{n=1}^{\infty} (\bar{\varepsilon})^n e^{-nt} \right) \quad \text{,Re(t)>0} .$$

Thus

$$G(t) = \frac{1}{i\sqrt{3}} \left(\frac{1}{(1-\varepsilon e^{-t})} - \frac{1}{(1-\bar{\varepsilon} e^{-t})} \right).$$

By using the identities $\varepsilon - \bar{\varepsilon} = i\sqrt{3}$, $\varepsilon + \bar{\varepsilon} + 1 = 0$ and $\varepsilon \bar{\varepsilon} = 1$, we get

$$G(t) = \frac{1}{e^{t} + e^{-t} + 1}$$

The function $G(t) = (e^t + e^{-t} + 1)^{-1}$ is analytic near $t=0$; therefore it can be expanded as a power series in t. So we have

LEMMA 2.2. $G(t)$ has the Taylor series expantion

$$G(t) = \sum_{n=0}^{\infty} a_n t^{2n} \quad , \ |t| < 2\pi/3$$

where the coefficients a_n satisfy the recursion formula

$$a_0 = 1/3 \ , \ 3a_n + 2\sum_{k=1}^{n} \frac{1}{(2k)!} a_{n-k} = 0 \ , n \geq 1 \tag{2.2}$$

PROOF. Since G is an even function, the expantion of G can be expressed as

$$G(t) = \sum_{n=0}^{\infty} a_n t^{2n}$$

which is valid near zero (in fact valid in the disk $|t| < \frac{2}{3}\pi$ which extends to the nearest singularities $t = \pm\frac{2\pi}{3}$ of G(t)). The relation $G(t)(e^t + e^{-t} + 1) = 1$ gives

$$\left(\sum_{n=0}^{\infty} a_n t^{2n}\right)\left(1 + 2\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\right) = 1$$

i.e.

$$\sum_{n=0}^{\infty} a_n t^{2n} + 2\left(\sum_{n=0}^{\infty} a_n t^{2n}\right)\left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\right) = 1$$

i.e.

$$\sum_{n=0}^{\infty} a_n t^{2n} + 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{(2k)!} a_{n-k}\right) t^{2n} = 1$$

Thus for the coefficients a_n we have the recursion formula

$$a_0 = 1/3 \ , \ 3a_n + 2\sum_{k=1}^{n} \frac{1}{(2k)!} a_{n-k} = 0 \ , n \geq 1 \quad .$$

This completes the proof of the lemma.

The coefficient a_n can be determined successively by (2.2). The first few are easily determined to be

$$a_0 = \frac{1}{3} \ , \qquad a_1 = -\frac{1}{9}$$

$$a_2 = \frac{1}{36} \ , \qquad a_3 = -\frac{7}{1080}$$

THEOREM 2.1. The function L defined by

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} G(t) t^{s-1} dt \quad , \mathrm{Re}(s) > 1$$

can be extended to a holomorphic function in the whole complex plane.

PROOF. Let us define P and Q for Re(s) > 1 by

$$P(s) = \int_0^1 G(t) t^{s-1} dt$$

$$Q(s) = \int_1^{\infty} G(t) t^{s-1} dt$$

The integral

$$\int_1^\infty G(t)t^{s-1}dt$$

exists and converges uniformly in any finite region of the s-plane, since the function

$$(e^{-t}\ t^{Re(s)+1})/(e^{-t}+e^{-2t}+1)$$

is bounded for all values of Re(s), and we can compare the integral with that of $1/t^2$. Thus Q is an entire function. Recall from Lemma 2.2 that

$$G(t)=\sum_{n=0}^{\infty} a_n t^{2n} \qquad ,t\in[0,1]$$

the convergence being uniform on [0,1]. We deduce for Re(s) > 1 that

$$P(s)=\sum_{n=0}^{\infty}\int_0^1 a_n t^{2n+s-1}dt$$

$$=\sum_{n=0}^{\infty}\frac{1}{2n+s}a_n$$

Thus P is a meromorphic function on C with simple poles at 0,-2,-4,-6,... . Since $1/\Gamma$ is an entire function we may now extend L to the whole of C by

$$L(s)=\frac{P(s)}{\Gamma(s)}+\frac{Q(s)}{\Gamma(s)} \tag{2.3}$$

Since Q and $1/\Gamma$ are entire functions, the singularities of L can only be those of P/Γ. We have seen that P has simple poles at 0,-2,-4,-6,... . Since $1/\Gamma$ has simple zeros at 0,-2,-4,... it follows that L is regular for all values of s in the complex plane. This completes the proof of the theorem.

LEMMA 2.3. (i) L has zeros at -1,-3,-5,...

(ii) The values of L at 0,-2,-4,-6,... are given by

$$L(-2m)=(2m)!a_m \qquad ,m=0,1,2,3,4,....$$

PROOF. (i) This follows immediately from the fact that $1/\Gamma$ has zeros at 0,-1, -2,-3,..., and thus

$$L(1-2m)=\frac{P(1-2m)}{\Gamma(1-2m)}+\frac{Q(1-2m)}{\Gamma(1-2m)}=0 \qquad ,m\in N .$$

(ii) As in (i) we use the partial fraction (2.3) of L to get

$$L(-2m)=\lim_{s\to-2m}\frac{P(s)}{\Gamma(s)}+\frac{Q(s)}{\Gamma(s)}$$

$$=\lim_{s\to-2m}\frac{P(s)}{\Gamma(s)}=\lim_{s\to-2m}\frac{1}{\Gamma(s)}\sum_{n=0}^{\infty}\frac{1}{2n+s}a_n$$

i.e.

$$L(-2m)=\lim_{s\to-2m}\frac{1}{\Gamma(s)}\cdot\frac{1}{2m+s}a_m.$$

Since Γ has simple poles at the points $-m$ $(m=0,1,2,3,\dots)$ with residues $(-1)^m/m!$, we get

$$\lim_{s\to -2m} (2m+s)\ \Gamma(s) = \mathrm{Res}(\Gamma, -2m) = \frac{1}{(2m)!}$$

Thus

$$L(-2m) = (2m)!\, a_m \qquad , m = 0,1,2,3,\dots$$

where a_m can be determined successively by (2.2).

3. DERIVATION OF THE FUNCTIONAL EQUATION OF L.

In this section we derive the equation

$$L(s) = \frac{2}{\sqrt{3}}\left(\frac{2}{3}\pi\right)^{s-1} \Gamma(1-s)\cos\left(\frac{1}{2}\pi s\right)L(1-s) \qquad , s \in C.$$

where L is the Dirichlet series (2.1)

$$L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{3}\right)n^{-s} \qquad , \quad s \in C$$

Finally we determine the values of L at 1,3,5,..., by the use of the functional equation obtained above.

LEMMA 3.1. There exists an integral function I such that

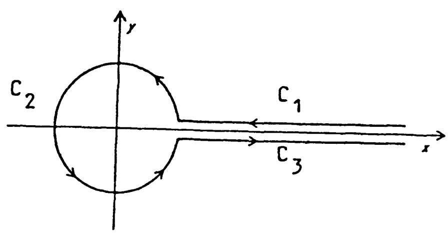
$$L(s) = -\Gamma(1-s)I(s) \qquad , s \in C .$$

PROOF. Let $0 < r < 1$, and let C_r be the contour consisting of the paths C_1, C_2 and C_3, where

$C_1 = (\infty, r]$

$C_2 = \partial_+ D_r(0)$ is a circle of radius r and the center at the origin oriented in the positive direction.

$C_3 = [r, \infty)$.



Define the function I_r by

$$I_r(s) = \frac{1}{2\pi i}\int_{C_r} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt$$

We prove now that I_r is independent of r. We have

$$I_r(s) - I_{r'}(s) = \frac{1}{2\pi i}\int_{C_0} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1} \qquad ,$$

where C_0 is the contour shown in figure (a). Now

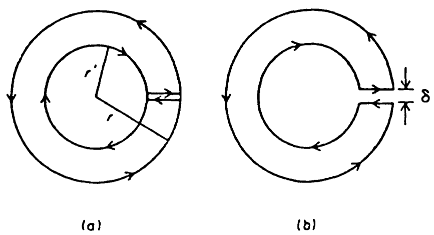
$$\int_{C_0} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt = \lim_{\delta \to 0} \int_C \frac{(-t)^{s-1}}{e^t + e^{-t} + 1} dt ,$$

where C is the contour in figure (b).

According to Cauchy's theorem, the integral around C is zero. Thus

$$\int_{C_0} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt = 0$$

It follows that I_r is independent of r.



(a) (b)

Now,

$$I_r(s) = \frac{1}{2\pi i} \int_\infty^r \frac{e^{(\log t - \pi i)(s-1)}}{e^t + e^{-t} + 1}\, dt +$$

$$\frac{1}{2\pi i} \int_{C_2} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1} dt + \frac{1}{2\pi i} \int_r^\infty \frac{e^{(\log t + \pi i)(s-1)}}{e^t + e^{-t} + 1}\, dt .$$

The middle term approaches zero as $r \to 0$ provided $\text{Re}(s) > 0$, since

$$\left| \int_{C_2} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt \right| < M \int_0^{2\pi} r^{\text{Re}(s)-1} e^{-(\pi+\theta)\text{Im}(s)} r\, d\theta$$

$$< M' r^{\text{Re}(s)} \quad .$$

Hence

$$\lim_{r \to 0} I_r(s) : \frac{-e^{-\pi i(s-1)} + e^{\pi i(s-1)}}{2\pi i} \int_0^\infty \frac{t^{s-1}}{e^t + e^{-t} + 1} dt.$$

Define the function I by

$$I(s) = \lim_{r \to 0} I_r(s)$$

Thus we have

$$I(s) = -\frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{t^{s-1}}{e^t + e^{-t} + 1} dt.$$

We have seen in the proof of theorem 2.1 that the function defined by the integral

$$\int_0^\infty \frac{t^{s-1}}{e^t + e^{-t} + 1}$$

is a meromorphic function with simple poles at the points $0,-2,-4,\ldots$. Since the function $\sin(\pi s)$ has simple zeros at $0,-2,-4,\ldots$ it follows that I is regular for

all values of s in the complex plane.
Moreover we have

$$I(s) = -\frac{\Gamma(s)\sin(\pi s)}{\pi} L(s)$$

Thus

$$I(s)\,\Gamma(1-s) = -\,L(s)$$

THEOREM 3.1. The function L satisfies the functional equation

$$L(s) = \frac{2}{\sqrt{3}}\left(\frac{2}{3}\pi\right)^{s-1} \Gamma(1-s)\cos\left(\frac{1}{2}\pi s\right)L(1-s)$$

PROOF. Let $R_n = n+\frac{1}{2}$, $n = 1,2,3,\ldots$, and let $C_{n,r}$ $(0<r<1)$ be the contour consisting of the positive real axis from R_n to r, a circle radius r and center at the origin oriented in the positive direction, the positive real axis from r to R_n, and finally a circle of radius R_n with center at the origin oriented in the negative direction.
i.e.

$$C_{n,r} = \left[R_n, r\right] + \underset{+}{\partial} D_r(o) + \left[r, R_n\right] + \underset{-}{\partial} D_{R_n}(o) \quad .$$

To deduce the functional equation of L we evaluate the integral

$$\frac{1}{2\pi i} \int_{C_{r,n}} \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1} dt$$

If we assume $s = x$ is a negative real number, then we have

$$(-t)^{x-1} = e^{(x-1)\log(-t)}$$

It follows that

$$|(-t)|^{x-1} = |t|^{x-1}$$

Since the function $(e^{t}+e^{-t}+1)^{-1}$ is bounded on the circle $\underset{-}{\partial} D_{R_n}(0)$,

$$\left| \int_{\underset{-}{\partial} D_{R_n}(o)} \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1} dt \right| < 2\, M^{*} R_n^{x} \quad ,$$

which goes to zero as n goes to infinity.
Thus we have

$$I(s) = \lim_{n\to\infty} \left(\frac{1}{2\pi i} \int_{C_{n,r}} \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1} dt \right) .$$

Now between $\partial D_{R_n}(0)$ and $D_r(0)$ the integrand has poles at the points

$$\pm \frac{2\pi i}{3}\ ,\ \pm \frac{2\pi i}{3}(3m+1) \text{ and } \pm \frac{2\pi i}{3}(3m-1),\ m=1,2,3,\ldots.$$

Denote

$$H(t) = \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1}$$

Thus we have

$$\mathrm{Res}(H, \tfrac{2\pi i}{3}) = \frac{1}{\sqrt{3}}(\tfrac{2\pi}{3})^{s-1} e^{-\pi is/2} .$$

$$\mathrm{Res}(H, -\tfrac{2\pi i}{3}) = \frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{\pi is/2} .$$

$$\mathrm{Res}(H, \tfrac{2\pi i}{3}(3m+1)) = \frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{-\pi is/2}(3m+1)^{s-1} .$$

$$\mathrm{Res}(H, -\tfrac{2\pi i}{3}(3m+1)) = \frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{\pi is/2}(3m+1)^{s-1} .$$

$$\mathrm{Res}(H, \tfrac{2\pi i}{3}(3m-1)) = -\frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{-\pi is/2}(3m-1)^{s-1} .$$

$$\mathrm{Res}(H, -\tfrac{2\pi i}{3}(3m-1) = -\frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{\pi is/2}(3m-1)^{s-1} .$$

The sum of the residues between $\partial D_{R_n}(0)$ and $\partial D_r(0)$ equals

$$\frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)\Big(1+\sum_{m=1}^{n}[(3m+1)^{s-1}-(3m-1)^{s-1}]\Big)$$

One can easily verify the identity

$$1+\sum_{m=1}^{n}[(3m+1)^{s-1}-(3m-1)^{s-1}] = \frac{2}{\sqrt{3}}\sum_{m=1}^{3n+1} \sin(\tfrac{2}{3}\pi m)m^{s-1}.$$

Thus the sum of the residues is

$$\frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)\Big(\frac{2}{\sqrt{3}}\sum_{m=1}^{3n+1}\sin(\tfrac{2}{3}\pi m)m^{s-1}\Big) .$$

It follows that

$$\begin{aligned} -I(s) &= \frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)\Big(\frac{2}{\sqrt{3}}\sum_{m=1}^{\infty}\sin(\tfrac{2}{3}\pi m)m^{s-1}\Big). \\ &= \frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)L(1-s) \qquad (3.1) \end{aligned}$$

We have seen that $-I(s)\Gamma(1-s) = L(s)$ for all $s \in C$, so by the identity theorem the formula (3.1) is true for all $s \in C$. Thus we have proved the functional equation

$$L(s) = \frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)\,\Gamma(1-s)L(1-s) .$$

LEMMA 3.2. The values of L at the points $s=2m+1, m=0,1,2,3,...$ are given by the formula

$$L(1+2m) = (-1)^m \frac{\sqrt{3}}{2}(\tfrac{2}{3}\pi)^{2m+1} a_m .$$

where a_m's are determined by (2.2).

PROOF. For $s = -2m$ the functional equation and the identity

$$L(-2m) = (2m)!\, a_m \quad , \; m = 0,1,2,\ldots$$

of the previous section give the proof of the lemma.

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