IJMMS 32:7 (2002) 449-451 PII. S0161171202201015 http://ijmms.hindawi.com © Hindawi Publishing Corp.

## A NOTE ON LOGHARMONIC MAPPINGS

## ZAYID ABDULHADI

Received 5 January 2002

We consider the problem of minimizing the moments of order p for a subclass of logharmonic mappings.

2000 Mathematics Subject Classification: 30C55, 30C62, 49Q05.

**1. Introduction.** Let H(U) be the linear space of all analytic functions defined on the unit disc  $U = \{z = x + iy : |z| < 1\}$ . A logharmonic mapping is a solution of the nonlinear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = (a\overline{f}/f)f_z, \tag{1.1}$$

where the second delatation function a is in H(U) and |a(z)| < 1 for all  $z \in U$ . If f does not vanish on U, then f is of the form

$$f = H \cdot \overline{G},\tag{1.2}$$

where H and G are in H(U). On the other hand, if f vanishes at 0 but has no other zeros in U, then f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z) \overline{g(z)}, \tag{1.3}$$

where

- (a) m is nonnegative integer,
- (b)  $\beta = \overline{a(0)}(1 + a(0))/(1 |a(0)|^2)$  and therefore,  $\Re \beta > -1/2$ ,
- (c) h and g are analytic in U, g(0) = 1, and  $h(0) \neq 0$ .

Univalent logharmonic mappings on the unit disc have been studied extensively. For details see [1, 2, 3, 4, 5, 6, 7, 8]. Suppose that f is a univalent logharmonic mapping defined on the unit disc U. Then, if f(0) = 0, the function  $F(\zeta) = \log(f(e^{\zeta}))$  is univalent and harmonic on the half plane  $\{\zeta : \text{Re } \zeta < 0\}$ . For more details on univalent harmonic mappings defined in the unit disc U, see [9, 10, 11, 12].

In this note, we consider the problem of minimizing the moments of order p over a subclass of the class logharmonic mappings defined over the unit disc U. It is interesting to note that the extremal functions are univalent starlike logharmonic mappings.

## 2. Moments of order p

**THEOREM 2.1.** Let f = zh(z)g(z) be logharmonic mapping defined on the unit disc *U* such that h(0) = g(0) = 1. Let  $M_p(r, f)$  denote the moment of order  $p, p \ge 0$ . Then,

$$M_p(r,f) \ge 2\pi \left(\frac{r^{p+2}}{p+2} - \frac{r^{p+4}}{p+4}\right).$$
 (2.1)

Equality holds if and only if

$$f_1(z) = z \frac{\left(1 + \left((p+2)/(p+4)\right)\overline{z}\right)}{\left(1 + \left((p+2)/(p+4)\right)z\right)}$$
(2.2)

or one of its rotations  $\overline{\eta} f_1(\eta z)$ .

**REMARK 2.2.** If p = 0 in Theorem 2.1, then we have the problem of minimizing the area. Moreover, if p = 2, then we obtain the minimum of the moment of inertia.

**PROOF.** Let  $f = zh(z)\overline{g(z)}$  be logharmonic mapping defined on the unit disc U. Then, f satisfies (1.1) for some  $a \in H(U)$  such that |a(z)| < 1 and a(0) = 0. Hence, using Schwarz's lemma, we have

$$M_{p}(r,f) \geq \int_{0}^{r} \int_{0}^{2\pi} |f|^{p} (|f_{z}|^{2} - |f_{\overline{z}}|^{2}) \rho \, d\theta \, d\rho$$

$$= \int_{0}^{r} \int_{0}^{2\pi} |f|^{p} |f_{z}|^{2} (1 - |a|^{2}) \rho \, d\theta \, d\rho$$

$$\geq \int_{0}^{r} \rho (1 - \rho^{2}) \int_{0}^{2\pi} |f|^{p} |f_{z}|^{2} d\theta \, d\rho.$$
(2.3)

Writing  $(h \cdot g)^{p/2} \cdot (zh)' \cdot g = 1 + \sum_{k=1}^{\infty} c_k z^k$ , we have

$$\int_{0}^{2\pi} |f|^{p} |f_{z}|^{2} d\theta = 2\pi \rho^{p} \left( 1 + \sum_{k=1}^{\infty} |c_{k}|^{2} \rho^{2k} \right)$$
 (2.4)

and therefore,

$$M_p(r,f) \ge 2\pi \int_0^{2\pi} \rho^p (1-\rho^2) d\rho = 2\pi \left(\frac{r^{p+2}}{p+2} - \frac{r^{p+4}}{p+4}\right).$$
 (2.5)

Equality holds if and only if

$$(h)^{p/2} \cdot (g)^{(p+2)/2} \equiv 1 \tag{2.6}$$

and  $a(z) = \eta z$ ,  $|\eta| = 1$ . This implies that

$$(h)^{(p+2)/2} \cdot (g)^{p/2} g' = \eta \tag{2.7}$$

and then,

$$z \cdot \frac{\partial (h \cdot g)^{(p+2)/2}}{\partial z} = \frac{(p+2)(1 - (h \cdot g)^{(p+2)/2} + \eta z)}{2}.$$
 (2.8)

The solution of the differential equation

$$z \cdot u(z)' + \frac{(p+2) \cdot u(z)}{2} = \frac{(p+2)(1+\eta z)}{2}; \quad u(0) = 1$$
 (2.9)

is  $u(z) = (h(z)g(z))^{(p+2)/2} = 1 + ((p+2)/(p+4))\eta z$ . Together with (2.6), we get

$$\frac{g(z)'}{g(z)} = \frac{\eta}{(1 + ((p+2)/(p+4))\eta z)}$$
(2.10)

and therefore,

$$g(z) = \left(1 + \frac{p+2}{p+4} \eta z\right)^{(p+4)/(p+2)},$$

$$zh(z) = \frac{z}{\left(1 + \left((p+2)/(p+4)\right) \eta z\right)},$$
(2.11)

which leads to the solution  $\overline{\eta} f_1(\eta z)$ . Since

$$\phi(z) = \frac{zh(z)}{g(z)} = \frac{z}{(1 + ((p+2)/(p+4))\eta z)^{(2p+6)/(p+2)}}$$
(2.12)

is a starlike univalent analytic, it follows from [4, Theorem 2.1] that  $f_1$  is a starlike univalent logharmonic mapping.

## REFERENCES

- Z. Abdulhadi, Close-to-starlike logharmonic mappings, Int. J. Math. Math. Sci. 19 (1996), no. 3, 563-574.
- [2] \_\_\_\_\_\_, Typically real logharmonic mappings, Int. J. Math. Math. Sci. 31 (2002), no. 1, 1-9.
- [3] Z. Abdulhadi and D. Bshouty, *Univalent functions in*  $H \cdot \bar{H}(D)$ , Trans. Amer. Math. Soc. **305** (1988), no. 2, 841–849.
- [4] Z. Abdulhadi and W. Hengartner, *Spirallike logharmonic mappings*, Complex Variables Theory Appl. **9** (1987), no. 2-3, 121–130.
- [5] \_\_\_\_\_\_, Univalent harmonic mappings on the left half-plane with periodic dilatations, Univalent Functions, Fractional Calculus, and Their Applications (Kōriyama, 1988), Ellis Horwood Series: Mathematics and Its Applications, Horwood, Chichester, 1989, pp. 13-28.
- [6] \_\_\_\_\_, Univalent logharmonic extensions onto the unit disk or onto an annulus, Current Topics in Analytic Function Theory, World Scientific Publishing, New Jersey, 1992, pp. 1-12.
- [7] \_\_\_\_\_\_, One pointed univalent logharmonic mappings, J. Math. Anal. Appl. 203 (1996), no. 2, 333-351.
- [8] \_\_\_\_\_, Polynomials in  $H\bar{H}$ , Complex Variables Theory Appl. 46 (2001), no. 2, 89–107.
- [9] J. Clunie and T. Sheil-Small, Harmonic univalent functions, Ann. Acad. Sci. Fenn. Ser. A I Math. 9 (1984), 3–25.
- [10] W. Hengartner and G. Schober, *Harmonic mappings with given dilatation*, J. London Math. Soc. (2) **33** (1986), no. 3, 473–483.
- [11] \_\_\_\_\_\_, *On the boundary behavior of orientation-preserving harmonic mappings*, Complex Variables Theory Appl. 5 (1986), no. 2-4, 197-208.
- [12] \_\_\_\_\_\_, Univalent harmonic functions, Trans. Amer. Math. Soc. 299 (1987), no. 1, 1-31.

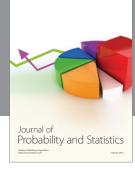
ZAYID ABDULHADI: DEPARTMENT OF COMPUTER SCIENCE, MATHEMATICS AND STATISTICS, AMERICAN UNIVERSITY OF SHARJAH, P.O. BOX 26666, SHARJAH, UNITED ARAB EMIRATES *E-mail address*: zahadi@aus.ac.ae











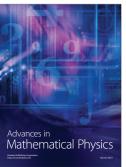


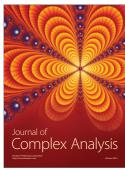




Submit your manuscripts at http://www.hindawi.com











Journal of Discrete Mathematics

