

## Research Article

# An Augmented Lagrangian Method for the Optimal $H_\infty$ Model Order Reduction Problem

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This paper treats the computational method of the optimal  $H_\infty$  model order reduction (MOR) problem of linear time-invariant (LTI) systems. Optimal solution of MOR problem of LTI systems can be obtained by solving the LMIs feasibility coupling with a rank inequality constraint, which makes the solutions much harder to be obtained. In this paper, we show that the rank inequality constraint can be formulated as a linear rank function equality constraint. Properties of the linear rank function are discussed. We present an iterative algorithm based on augmented Lagrangian method by replacing the rank inequality with the linear rank function. Convergence analysis of the algorithm is given, which is distinct to the now available heuristic methods. Numerical experiments for the MOR problems of continuous LTI system illustrate the practicality of our method.

## 1. Introduction

Model order reduction (MOR) problem has received considerable attention since it was put up. In physical or engineering system, the mathematical modeling of the system often results in the high-order controllers, while the simulation and physical implementation of the higher order controllers are more difficult to realize due to the high order. The purpose of the MOR is to obtain a stable lower-order system that approximates the higher one according to some given criterion. Many techniques and results on MOR problems have been reported, and there are various approaches such as the aggregation method [1], balanced truncation approach [2], and optimal Hankel norm approximations methods [3, 4]. It is a difficult work to list all the references since we have many references in this field. The  $H_\infty$  MOR problem has been studied by many researchers such as [4–13] and references therein. It mainly consists of finding a lower-order model  $\widehat{G}$  of the original system  $G$  such that the  $H_\infty$  error  $\|G - \widehat{G}\|$  is small. It has received quite a few attentions since the  $H_\infty$  norm of the difference between two systems is one of the most meaningful measures of approximations. Linear matrix inequality (LMI) is a well-known convex feasibility problem which can be solved by the

semidefinite programming (SDP) interior algorithms [14]. LMI is also a powerful tool in control and systems theory and was widely used in output feedback stabilization, reduced-order  $H_\infty$  synthesis,  $\mu$  synthesis with constant scaling [15], and so on. In [6], an LMI approach was followed for the  $H_\infty$  MOR problem, utilizing the Bounded Real Lemma [16], and an iterative two-step algorithm was proposed but with no guaranteed convergence to the global optimum. More recently, [5] studied the deterministic  $H_\infty$  MOR problem for deterministic case, necessarily and sufficient conditions are derived for  $\gamma$ -suboptimal MOR problems in terms of LMIs, numerical techniques based on alternating projections are proposed to solve the MOR problems, and [7] extends the results to the stochastic MOR problem.

The optimal index  $\gamma$  of  $H_\infty$  MOR problem is computed by minimizing the model given in [5], and an algorithm to solve the related LMIs feasibility coupling with a rank constraint is also proposed in that paper. The rank constrained LMI problem can be considered as a generalization of LMI problem, but the lack of convexity makes the rank constrained LMI problem much harder to solve. Now available algorithms for the rank constrained LMI problems are largely heuristic in nature and most of them are not convergent. The existing methods for LMI coupling with rank constrained are those

based on linearization [17], alternating projections [18, 19], trace minimization methods that solve the problem by solving a related convex problem [20], augmented Lagrangian methods (ALF) [21], sequential semidefinite programming [22], and so on. Aside from [21, 22], these methods did not give the proof of convergence and the challenge still remains to find the numerical schemes with verifiable local/global convergence. More recently, nuclear norm minimizations method is a good relaxation approximation method to this problem [23].

The contribution of this paper is as follows: an algorithm is presented using the augmented Lagrangian method by means of the equivalent condition to the rank constraints; ALF method has more advantages compared with the PF method. For PF method, the penalty item becomes more and more possibly ill-conditioned when the penalty parameter become decreasing, which makes the solution more difficult to find. One of the modification methods to solve this problem for PF is the ALF method, which has better performance in the aspect of robustness and convergence rate. So we proposed an ALF method for Problem 10. Convergence of the algorithm is also investigated in this paper. Numerical experiments showed that our algorithm is effective.

The rest of this paper is structured as follows. Section 2 gives an introduction to the  $H_\infty$  MOR problem and the feasible set of convex LMI constraints coupling with rank constraint. Section 3 presents an equivalent condition of the rank constraints and reformulates the MOR problem as an NLP problem using a cost function which combines  $H_\infty$  gains index  $\gamma$  and a penalty term accounting for the rank constraint. An augmented Lagrangian method solving the NLP problem is proposed in Section 4, as well as convergence results. Section 5 gives some computational details about the algorithms. Numerical experiments are given in Section 6 for continuous LTI MOR problems. Section 7 concludes this paper.

The standard notation used throughout this paper is as follows.  $S^n$  denotes the set of all symmetric matrix in  $R^n$ , and  $I_n$  denotes the  $n$  identity matrix. For a matrix  $A$ , its transpose, rank, inverse, and trace are denoted by  $A^T$ ,  $\text{rank}(A)$ ,  $A^{-1}$ , and  $\text{trace}(A)$ , respectively. If  $A$  is a symmetric matrix,  $A > (\geq) 0$  means that  $A$  is positive (semidefinite) definite. Symbols  $\nabla$  and  $\nabla^2$  mean the gradient and Hessian matrix of a function,  $\sigma(A)$  denotes the maximum singular value of a matrix,  $\|\cdot\|$  denotes the  $H_\infty$  norm of a rational transfer function, and  $\langle A, B \rangle$  denotes the inner product of two matrices.

## 2. Problem Formulation

Consider an asymptotically stable  $n$ th-order linear, time-invariant continuous system  $G$  with a state space representation

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \end{aligned} \quad (1)$$

where  $A \in R^{n \times n}$ ,  $B \in R^{n \times m}$ ,  $C \in R^{p \times n}$ , and  $D \in R^{p \times m}$ . The optimal  $H_\infty$  MOR problem is to find a stable  $\hat{n}$ th order system  $\widehat{G}(\hat{n} \ll n)$  with a state space representation

$$\begin{aligned} \dot{\hat{x}} &= \widehat{A}\hat{x} + \widehat{B}\hat{u}, \\ \hat{y} &= \widehat{C}\hat{x} + \widehat{D}\hat{u}, \end{aligned} \quad (2)$$

where  $\widehat{A} \in R^{\hat{n} \times \hat{n}}$ ,  $\widehat{B} \in R^{\hat{n} \times m}$ ,  $\widehat{C} \in R^{p \times \hat{n}}$ , and  $D \in R^{p \times m}$ , such that the  $H_\infty$  gain index of  $\|G - \widehat{G}\|_\infty$  with respect to  $\widehat{G}$  is minimized. The balanced truncation method for MOR problem [2] and the optimal Hankel norm MOR method [3] are the most popular among the now available methods. Methods based on the LMIs are the ones which can reformulate the MOR problem as LMIs by means of the Bounded real lemma. The optimal MOR solution can be obtained by solving a feasible point of LMIs feasible set coupling with a rank constraint. The method is proposed in [6] in terms of time domain response of the error system and [5] extends the result to the  $H_\infty$  MOR problem in the state space. Reference [7] extends the results to the stochastic case. More recently, basing on LMIs method, [24] presents a sufficient and necessary condition for positivity-preserving  $H_\infty$  MOR of LTI positive system.

**Theorem 1** (see [5]). *There exists a stable system  $\widehat{G}$  such that the  $H_\infty$  MOR problem of the  $n$ th-order continuous-time system in the form of (1) is solvable for a given  $\gamma > 0$  if and only if there exists positive definite matrix  $X > 0$ ,  $Y > 0$  satisfying*

$$\min_{\gamma, X, Y} \gamma \quad (3)$$

$$AX + XA^T + BB^T < 0, \quad (4)$$

$$YA + A^T Y + C^T C < 0, \quad (5)$$

$$\begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \geq 0, \quad (6)$$

$$\text{rank} \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \leq n + \hat{n}.$$

The optimal  $H_\infty$  error and the corresponding feasible solution for  $X, Y$  can be obtained by solving the following optimization Problem 2, and all  $\gamma$ -suboptimal of  $\hat{n}$ th-order models that correspond to a feasible matrix pair  $(X, Y)$  are given by

$$\begin{pmatrix} \widehat{D} & \widehat{C} \\ \widehat{A} & \widehat{B} \end{pmatrix} = \widehat{G}_1 + \widehat{G}_2 L \widehat{G}_3, \quad (7)$$

where  $L \in R^{(p+\hat{n}) \times (m+\hat{n})}$  is any matrix such that  $\|L\| < 1$ , and

$$\widehat{G}_1 = (M_1 - Q_{12} Q_{22}^{-1} M_2^T) (M_2 Q_{22}^{-1} M_2^T)^{-1}, \quad (8)$$

$$\widehat{G}_2 = (-Q_{11} + Q_{12} Q_{22}^{-1} Q_{12}^T - G_1 G_3^2 G_1^T)^{1/2}, \quad (9)$$

$$\widehat{G}_3 = -(M_2 Q_{22}^{-1} M_2^T)^{1/2}, \quad (10)$$

where

$$\begin{aligned}
 M_1 &= \begin{pmatrix} 0 & 0 \\ 0 & R_x^2 \end{pmatrix}, \\
 M_2 &= \begin{pmatrix} 0 & I \\ R_x L_x^T & 0 \end{pmatrix}, \\
 Q_{11} &= \begin{pmatrix} -\gamma^2 I & CL_x R_x \\ R_x L_x^T C^T & 0 \end{pmatrix}, \\
 Q_{12} &= \begin{pmatrix} CX & D \\ R_x L_x^T A^T & 0 \end{pmatrix}, \\
 Q_{22} &= \begin{pmatrix} AX + XA^T & B \\ B^T & I \end{pmatrix}
 \end{aligned} \tag{11}$$

and  $R_x \in \mathbb{R}^{\hat{n} \times \hat{n}}$  is an arbitrary positive definite matrix,  $L_x \in \mathbb{R}^{n \times \hat{n}}$  is an arbitrary matrix factor such that  $L_x L_x^T = X - \gamma^2 Y^{-1}$ .

**Problem 2.** According to Theorem 1, the solution of the optimal  $H_\infty$  MOR problem can be obtained by solving the following LMIs constrained minimization problem coupling with rank constraints.

$$\min_{\gamma, X, Y} \gamma \tag{12}$$

$$AX + XA^T + BB^T < 0, \tag{13}$$

$$YA + A^T Y + C^T C < 0, \tag{14}$$

$$\begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \geq 0, \tag{15}$$

$$\text{rank} \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \leq n + \hat{n}.$$

**Assumption 3.** Throughout the paper, we assume that Problem 2 has at least one feasible solution.

**Theorem 4** (see [12]). *The rank of a symmetric positive semidefinite matrix  $W$  satisfies  $\text{rank}(W) \leq r$  if and only if there exist matrices  $V \in \mathbb{R}^{n \times (n-r)}$  with  $V^T V = I_{n-r}$  such that  $\text{tr}(V^T W V) = 0$ .*

Theorem 4 is a lemma in [12]; we give a detailed different proof here.

*Proof.* Given a symmetric positive semidefinite matrix  $W \geq 0$ ,  $\text{rank}(W) \leq r$ ,  $r > 0$ , there exists an orthogonal matrix  $U$  such that

$$U^T W U = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \tag{16}$$

where the eigenvalues are ordered  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , since the rank of matrix  $W$  is less than or equal to  $r$ ; there must be at most  $r$  eigenvalues that are not 0. The columns of matrix  $U$  are the corresponding eigenvectors of the related eigenvalues; take  $V = (u_{r+1}, \dots, u_n) \in \mathbb{R}^{n \times (n-r)}$ ; the columns of  $V$  consist of the eigenvectors corresponding to the  $n - r$  smallest eigenvalues of  $W$ , and  $V^T V = I_{n-r}$ ; this ends the necessity part.

If there exists a matrix  $V \in \mathbb{R}^{n \times (n-r)}$ , with  $V^T V = I_{n-r}$  satisfying  $\text{tr}(V^T W V) = 0$ , suppose the eigenvalues of the symmetric matrix  $V^T W V$  are  $\lambda_1, \lambda_2, \dots, \lambda_{n-r}$ . They are also the eigenvalues of the matrix  $W$  since matrix  $V$  is full rank which means  $\lambda_1, \lambda_2, \dots, \lambda_{n-r} \geq 0$  because  $W$  is a symmetric positive semidefinite matrix. We have

$$\lambda_1 + \lambda_2 + \dots + \lambda_{n-r} = \text{tr}(V^T W V) = 0, \tag{17}$$

from the nonnegativity of the eigenvalues  $\lambda_i, i = 1, 2, \dots, n-r$ , we have  $\lambda_i = 0, i = 1, 2, \dots, n-r$ , which means  $\text{rank}(W) \leq r$ ; this ends the proof of sufficiency part.  $\square$

For simplification, we note the closure of the semipositive feasible constraints set in Problem 2 as  $\Omega = \{(\gamma, X, Y) \mid \gamma \geq 0, \exists X, Y > 0, \text{ such that } AX + XA^T + BB^T \leq 0, YA + A^T Y + C^T C \leq 0, -\begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \leq 0\}$ .

**Proposition 5.** *Set  $\Omega$  as a convex in variable  $\gamma$ .*

*The set is the intersection of some polynomial inequalities constrained set. The proof of Proposition 5 can be done by the positive definiteness (positive semidefiniteness) of matrix.*

**Proposition 6.** *Denote  $h(\gamma; V) = \text{tr}(V^T W V)$ , where  $W = \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix}$ ; then  $h(\gamma)$  is a linear function in matrix variable  $X, Y$  for a given fixed matrix  $V$ .*

*Proof.* From Theorem 1, matrices  $X, Y$  are functions in variable  $\gamma$ ; according to the definition of function  $h(\gamma)$  and the inner product of matrix, we have

$$h(\gamma) = \text{tr}(V^T W V) = \langle V, W V \rangle, \quad W = \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix}. \tag{18}$$

Its obvious that  $h(\gamma)$  is a linear function in  $X, Y$  for a given  $V$  and  $W = \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix}$  but keep in mind that  $X, Y$  are related to variable  $\gamma$ , and  $V$  varies with matrix  $W$ . Matrix  $V$  can be obtained by the singular decomposition of the specific matrix  $W$ .  $\square$

**Proposition 7.** *For a given matrix  $W$  with a fixed rank  $\hat{n}$ , function  $h(\gamma; V)$  is continuous differentiable in variable  $\gamma$  for fixed  $X, Y$ . Suppose  $\gamma^*$  is the solution of equation  $h(\gamma) = 0$ ; then when  $\gamma \rightarrow \gamma^*, h(\gamma) \rightarrow 0$ .*

*Proof.* From Theorem 4, matrix  $V$  is a submatrix of the orthogonal transformation matrix, so  $V^T W V$  has the same eigenvalues as matrix  $W$ . The value of  $h(\gamma; V)$  is the addition of some eigenvalues of matrix  $W$ . From the definition of eigenvalues of a matrix and Ostrowski Theorem, the eigenvalues of matrix  $W$  are a continuous and differentiable function

in the entries of the matrix  $W$  (while matrix  $V$  is not the case). The proof of this lemma is obvious. Conclusions can be drawn according to the continuity of  $h(\gamma)$  and Proposition 6.  $\square$

Proposition 7 suggests that  $|h(\gamma; V)| \leq \varepsilon$  can be used as the iteration termination criterion, although the rank condition for matrix  $W$  in Theorem 1 does not hold.

**Proposition 8.** *If the rank of matrix  $W$  is full, function  $h(\gamma; V) > 0$  for any  $\gamma \geq 0$  and  $X, Y$  satisfies the LMIs; if the rank of matrix  $W$  is less than or equal to  $n + \hat{n}$  ( $\hat{n} \ll n$ ), there must exist a  $\bar{\gamma} > 0$  such that  $h(\bar{\gamma}; V) = 0$ .*

*Proof.*  $W = \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix}$ , it is obvious that  $\text{rank}(W) \geq n$  since  $W$  has a block matrix  $X$  which is positive definite. If  $\text{rank}(W) \leq n + \hat{n}$ , then

$$\text{rank}(W) = \text{rank} \begin{pmatrix} X & 0 \\ 0 & \gamma^2 I - XY \end{pmatrix} \quad (19)$$

$$= \text{rank}(X) + \text{rank}(\gamma^2 I - XY) \leq n + \hat{n}.$$

Since  $\text{rank}(X) = n$ , we have  $\text{rank}(\gamma^2 I - XY) \leq \hat{n}$ .

Matrix  $Y = S^2$ , where  $S$  is also a symmetric positive definite matrix for the reason that  $Y$  is a symmetric positive definite matrix; then we have

$$\text{rank}(\gamma^2 I - XY) = \text{rank}\{S^{-1}(\gamma^2 I - SXS)S\}; \quad (20)$$

there must exist an orthogonal matrix  $Q$  such that

$$Q^T SXSQ = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}, \quad (21)$$

so we have

$$\begin{aligned} \text{rank}(\gamma^2 I - XY) &= \text{rank}\{S^{-1}(\gamma^2 I - SXS)S\} \\ &= \text{rank} \left\{ S^{-1}Q \begin{pmatrix} \gamma^2 - \mu_1 & & \\ & \ddots & \\ & & \gamma^2 - \mu_n \end{pmatrix} Q^T S \right\}, \quad (22) \end{aligned}$$

so there must exist at least  $n - \hat{n}$  eigenvalues which are zero; that is,  $\bar{\gamma}^2 = \mu_j$ ,  $j = 1, 2, \dots, n - \hat{n}$ . The property  $h(0) > 0$  comes from the positive definite of matrix  $X, Y$ . Propositions 5–8 show that the rank constraint in (15) can be replaced by the equality constraint  $h(\gamma; V) = 0$ .  $\square$

### 3. Reformulation of Problem 2

By means of the function  $h(\gamma, X, Y)$  and Theorem 1, Problem 2 can be reformulated as follows.

*Problem 9.*

$$\begin{aligned} \min_{\gamma, X, Y} \quad & \gamma \\ & AX + XA^T + BB^T < 0, \\ & YA + A^T Y + C^T C < 0, \\ & \begin{pmatrix} X & \gamma I \\ \gamma I & Y \end{pmatrix} \geq 0, \\ & h(\gamma; V) = 0. \end{aligned} \quad (23)$$

The above minimization Problem 9 can be written into the following standard optimization Problem 10, but keep in mind that  $X, Y$  is related to  $\gamma$ . We make a notation  $x = (\gamma, X, Y)$  (we omit the dimension of matrix here), which is the decision variable.

*Problem 10.*

$$\min_{\gamma, X, Y} f(x) \quad (24)$$

$$h(x; V) = 0, \quad x = (\gamma, X, Y) \in \Omega. \quad (25)$$

Problem 10 is an optimization problem with equality constraint and closed feasible set; standard optimization techniques and algorithms can be used to solve this problem. Recently, [12] presented a penalty function (PF) algorithm by means of the rank function given in this paper. Numerical experiments showed that the algorithm is effective while there is no convergence analysis given in that paper. Also there are some disadvantages such as ill conditions for penalty function methods. This motivates us to present a convergent algorithm with good numerical performance for Problem 10 (also Problem 2) by means of the function  $h(\gamma; V)$ .

## 4. Augmented Lagrangian Function Algorithm

*4.1. ALF Algorithm.* One of the effective convergent algorithms for Problem 10 with equality constraints is the augmented Lagrangian function (ALF) method. ALF method has more advantages compared with PF method. For PF method, the penalty term becomes more and more possibly ill-conditioned when the penalty parameter is decreasing, which makes the solution more difficult to find. One of the modification methods to overcome this problem for PF is the ALF method, which has better performance in the aspect of robustness and convergence rate. So we proposed an ALF method for Problem 10 in this paper. For more detailed information about the ALF methods, refer to [25–28].

Problem 10 can be considered as a standard equality constrained nonlinear programming problem (NLP). Augmented Lagrangian function method is an effective method to solve an optimization with equality constraints. We reformulate Problem 10 as the form of augmented Lagrangian function optimization Problem 11, where  $\lambda$  is the multiplier and  $c$  is the penalty parameter.

*Problem 11.*

$$\min_{\gamma, X, Y} \Phi_c = f(x) + \lambda h(x; V) + \frac{c}{2} h^2(x; V) \quad (26)$$

$$x = (\gamma, X, Y) \in \Omega, \quad (27)$$

$$\lambda \in \Lambda.$$

Suppose the  $k$ th iteration solution of Problem 11 is  $x^k = (\gamma^k, X^k, Y^k)$ . The rationale for the ALF method is based on the fact that when  $\lambda^k$  is bounded and  $c^k \rightarrow \infty$ , then the term  $\lambda h(x; V) + (c/2)h^2(x; V)$  tends to infinity if  $h(x; V) \neq 0$  and is equal to 0 if  $h(x; V) = 0$ . The minimum of Problem 10 can be obtained by solving the unconstrained Problem 11 under some conditions.

*ALF Algorithm for NLP Problem 10*

*Step 1* (initialization). Initialize the algorithm by determining a feasible of the LMI constraints  $X^0, Y^0$ , and  $\gamma^0$ . Then initialize the Lagrangian multiplier  $\lambda^0$  and penalty parameter  $c^0 > 0$ ,  $k := 0$ ,

*Step 2* (inner optimization). For  $k = 0, 1, 2, \dots$ , solving the NLP problem for fixed  $c^k, \lambda^k$

$$\min_{\gamma, X, Y} \Phi_{c^k} = f(x) + \lambda^k h(x; V) + \frac{c^k}{2} h^2(x; V) \quad (28)$$

$$x = (\gamma, X, Y) \in \Omega, \quad (29)$$

$$\lambda \in \Lambda.$$

Using algorithm solving LMI feasibility (such as SeDuMi [29] or YALMIP [29]) to solve the subproblem, an eigenvalue decomposition of matrix  $W$  needs to be done at  $k$ th iteration, and let

$$x^k = (\gamma^k, X^k, Y^k) \quad (30)$$

be the solution of  $x^k = \arg \min \Phi_{c^k} f(x) + \lambda^k h(x; V) + (c^k/2)h^2(x; V)$ . If  $x^k$  satisfies the termination criterion in Step 4, stop!

*Step 3* (update penalty and multiplier).

$$\lambda^{k+1} = \lambda^k + c^k h(x^{k+1}; V),$$

$$c^{k+1} = \begin{cases} \rho c^k, & \text{if } |h(x^{k+1})| > \mu |h(x^k; V)|; \\ c^k, & \text{if } |h(x^{k+1}; V)| \leq \mu |h(x^k; V)| \end{cases} \quad (31)$$

for given  $\rho > 1, 0 < \mu < 1$ . To ensure that  $\lambda^k \rightarrow \lambda^*$  during the iteration, we use the above update of the multiplier. This method is discussed in [25, 26]. Let  $k = k + 1$ ; go to Step 2.

*Step 4* (terminating phase). Due to the linearity of the constraints function  $h(x)$ , the terminate criterion of the algorithm can be chosen with  $(\gamma, X, Y) \in \Omega$  and  $|h(x)| \leq \varepsilon$ .

**Proposition 12.** *ALF algorithm is well defined.*

*Proof.* From Assumption 3, the feasible set of Problem 11 is nonempty; Propositions 5 and 6 showed that there must exist  $\bar{x}$  such that  $h(\bar{x}) = 0$ ; it is obvious that the algorithm is well defined.  $\square$

*4.2. The Optimality Conditions for Problem 11.* We note the Lagrangian part of  $\Phi_c(x, \lambda)$  as  $\Phi(x, \lambda)$ ; that is,

$$\Phi_c(x, \lambda) = \Phi(x, \lambda) + \frac{c}{2} h^2(x; V). \quad (32)$$

Let  $x^*$  be a local minimizer of Problem 10; suppose there exists a Lagrangian multiplier  $\lambda^*$  such that the first- and second-order optimality conditions for Problem 10 are satisfied [25]. Then there exists a large enough  $\bar{c} > 0$  for any  $c > \bar{c}$ , at the point  $(x^*, \lambda^*)$ ; we have

$$\Phi_c(x^*, \lambda^*) = \Phi(x^*, \lambda^*),$$

$$\nabla \Phi_c(x^*, \lambda^*) = \nabla \Phi(x^*, \lambda^*), \quad (33)$$

$$\nabla^2 \Phi_c(x^*, \lambda^*) \leq \nabla^2 \Phi(x^*, \lambda^*)$$

so we can obtain the optimality conditions for Problem 11,

$$\nabla \Phi_c(x^*, \lambda^*)^T (x - x^*) \geq 0,$$

for any  $x \in \Omega$ , (First order condition) (34)

$$(x - x^*)^T \nabla^2 \Phi_c(x^*, \lambda^*) (x - x^*) \geq 0,$$

for any  $x \neq x^* \geq 0$ . (Second order optimality condition).

Then from the Taylor expansion and the optimality conditions, we have

$$\Phi_c(x, \lambda^*) \geq \Phi_c(x^*, \lambda^*) + \frac{\eta}{2} \|x - x^*\|^2, \quad (35)$$

for any  $x \in \Omega, \|x - x^*\| < \varepsilon$ , and certain  $\eta, \varepsilon > 0, c > \bar{c}$ .

**Proposition 13.** *From the above inequality (35), we conclude that there must exist a local minimum of  $\Phi_c(x, \lambda)$  that is close to  $x^*$  for every  $c > \bar{c}$  when  $\lambda$  is close to  $\lambda^*$ .*

*4.3. The Convergence of ALF Algorithm*

**Theorem 14** (local convergence of the algorithm). *Suppose the feasible set  $\{x \mid x = (\gamma, X, Y) \in \Omega\}$  is nonempty. For  $k = 1, 2, \dots$ , let  $x^k$  be global minima of the objective function in Problem 10 over the feasible set. If the Lagrangian multiplier sequence  $\lambda^k$  is bounded, for all  $k, 0 < c^k < c^{k+1}$ , and  $c^k \rightarrow 0$ , then every convergent point of the sequence  $x^k$  is a global optimal solution of Problem 10.*



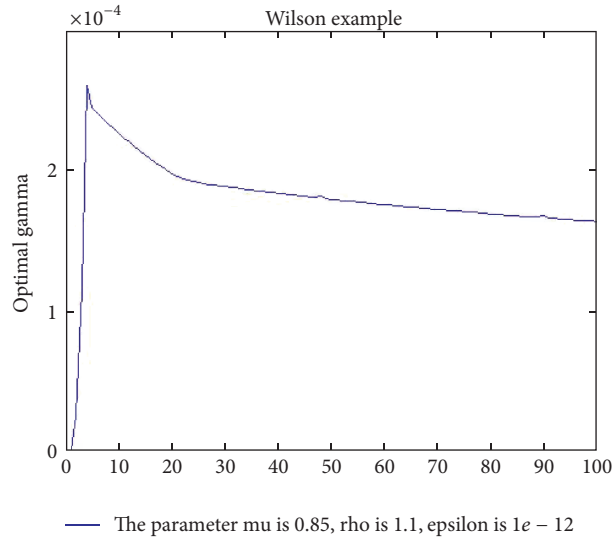


FIGURE 1: Gamma convergence.

$$\begin{aligned}
 & + \begin{pmatrix} -4.336 \\ -3.691 \\ 10.141 \\ -1.612 \\ 16.629 \\ -242.476 \\ -14.261 \\ 13.672 \\ 82.187 \end{pmatrix} u(t), \\
 y(t) &= (-0.422 \quad -0.736 \quad -0.0416 \quad 0.232 \quad -0.816 \quad -0.715 \quad 0.546 \quad -0.235 \quad -0.080) x(t).
 \end{aligned} \tag{39}$$

Tables 1 and 2 list the optimal indexes obtained by our algorithm. Figures 1 and 2 indicate the error and the convergence by the ALF algorithm. The computation parameters used in the ALF algorithms were  $\epsilon = 1e - 12$ ,  $\mu = 0.85$ ,  $\rho = 1.1$ ,  $c^0 = 1000$ , and  $\lambda^0 = 1000$ . Our experience showed that this algorithm works quite well for both problems compared with the now available algorithms [31, 32] under the frame of feasibility coupled with a rank constraint. The iterations were both less than 100 times when the termination were satisfied; this showed that our algorithm has good convergence performance and also the error is less than the results obtained by other methods, especially for Boiler System; our algorithm performed quite better than that in [32]. The suboptimal errors can be obtained as well as the corresponding positive matrices  $X, Y$  by means of our algorithm. The reduced state space can be evaluated according to (6)–(8) (Theorem 1 in [5]), and the obtained reduced system is stable.

## 7. Conclusion

We presented an ALF algorithm for optimal  $H_\infty$  MOR problem of the LTI system by means of an augmented Lagrangian method. First, we give a rank function which can reformulate the rank constraint of the optimal  $H_\infty$  MOR problem as a linear function equality constraint; then we reformulate the original problem as an optimization problem with LMIs constraints and linear equality constraints; an ALF algorithm is presented in the following. Compared with the now available algorithms, the goodness of our algorithm is the better convergence, which is opposite to the heuristic algorithm; our algorithm can also overcome the ill condition of PF algorithm in the process iteration. The algorithm is applicable to both continuous system and discrete system. Numerical experiments showed that the algorithm is effective for deterministic  $H_\infty$  MOR problem. MOR is an interesting and practical topic; not only is it applied in physics or

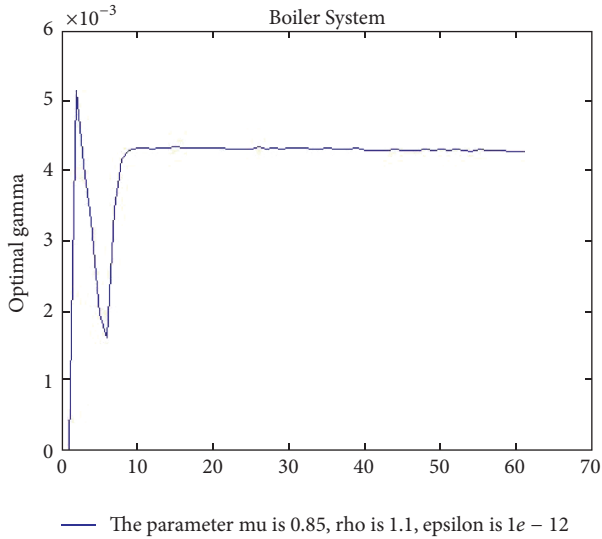


FIGURE 2: Gamma convergence.

TABLE 1: Wilson's example.

Method	$\gamma$
Wilson's model reduction	$2.525e - 4$
$\ H\ _{\infty}$ model reduction	$2.729e - 4$
ALF model reduction	$1.638e - 4$

TABLE 2: Boiler system.

Method	$\gamma$
$\ H\ _{\infty}$ model reduction	$1.127e - 1$
ALF model reduction	$0.043e - 1$

engineering systems, but also it can be used in some models in biological systems [33–36].

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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