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Research Article

Indefinite LQ Optimal Control with Terminal State Constraint for Discrete-Time Uncertain Systems

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Uncertainty theory is a branch of mathematics for modeling human uncertainty based on the normality, duality, subadditivity, and product axioms. This paper studies a discrete-time LQ optimal control with terminal state constraint, whereas the weighting matrices in the cost function are indefinite and the system states are disturbed by uncertain noises. We first transform the uncertain LQ problem into an equivalent deterministic LQ problem. Then, the main result given in this paper is the necessary condition for the constrained indefinite LQ optimal control problem by means of the Lagrangian multiplier method. Moreover, in order to guarantee the well-posedness of the indefinite LQ problem and the existence of an optimal control, a sufficient condition is presented in the paper. Finally, a numerical example is presented at the end of the paper.

1. Introduction

The linear quadratic (LQ) optimal control problem has been pioneered by Kalman [1] for deterministic systems, which is extended to stochastic systems by Wonham [2], and has rapid development in both theory and application [3]. Usually, it is an assumption that the control weighting matrix in the cost is strictly definite. For stochastic LQ optimal control, it is first revealed in [4] that even if the state and control weighting matrices are indefinite the corresponding problem may be still well-posed, which evoked a series of subsequent researches in continuous time [5] and in discrete-time [6]. In fact, some constraints are of considerable importance in many physical systems; the system state and control input are always subject to various constraints, so the constrained stochastic LQ issue has a concrete application background. For that reason, some researchers discussed stochastic LQ optimal problems with indefinite control weights and constraints [7, 8].

As is well known, these stochastic optimal control problems have been well studied by probability theory which is based on a large number of sample sizes. Sometimes, no samples are available to estimate the probability distribution. For such situation, we have to invite some domain experts to evaluate the belief degree that each event will occur. In order to rationally deal with belief degrees, uncertainty theory was established by Liu [9] in 2007 and refined by Liu [10] in 2010. Nowadays, uncertainty theory has become a new branch of mathematics for modeling indeterminate phenomena, which has been well developed and applied in a wide variety of real problems: option pricing problem [11], facility location problem [12], inventory problem [13], assignment problem [14], and production control problem [15].

Based on the uncertainty theory, Zhu [16] proposed an uncertain optimal control model in 2010 and gave an equation of optimality as a counterpart of Hamilton-Jacobi-Bellman equation. After that, some uncertain optimal control problems have been solved. As such, Sheng and Zhu [17] investigated an optimistic value model of uncertain optimal control problem; Yan and Zhu [18] established an uncertain optimal control model for switched systems. Inspired by the preceding work, we will tackle an indefinite LQ optimal control with terminal state constraint for discrete-time uncertain systems, which is a constrained uncertain optimal control problem. The rest of the paper is organized as follows. Section 2 collects some preliminary results. In

Section 3, an indefinite LQ optimal control with terminal state constraint is discussed. We present a general expression for the optimal control set in Section 4. A numerical example is applied in Section 5 to demonstrate the effectiveness of the model. We conclude the paper in Section 6.

For convenience, throughout the paper, we adopt the following notations: $\mathbf{R}^n$ is the real n-dimensional Euclidean space; $\mathbf{R}^{m\times n}$ is the set of all $m\times n$ matrices; M^τ is the transpose of matrix M; and $\operatorname{tr}(M)$ is the trace of a square matrix M. Moreover, $M > 0$ (resp., $M \geq 0$) means that $M = M^\tau$ and M is positive (resp., positive semidefinite) definite.

2. Some Preliminaries

In this section, we introduce some useful definitions about uncertainty theory and Moore-Penrose pseudoinverse of a matrix.

Let Γ be a nonempty set, and let $\mathscr{L}$ be a σ-algebra over Γ. Each element Λ in $\mathscr{L}$ is called an event. An *uncertain measure* was defined by Liu [9] via the following three axioms.

Axiom 1 (normality axiom). $\mathscr{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2 (duality axiom). $\mathscr{M}\{\Lambda\} + \mathscr{M}\{\Lambda^c\} = 1$ for any event Λ.

Axiom 3 (subadditivity axiom). For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathscr{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathscr{M}\{\Lambda_i\}. \tag{1}$$

The triplet $(\Gamma, \mathscr{L}, \mathscr{M})$ is called an *uncertainty space*. Furthermore, Liu [19] defined a product uncertain measure by the product axiom.

Axiom 4 (product axiom). Let $(\Gamma_k, \mathscr{L}_k, \mathscr{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. Then, the product uncertain measure $\mathscr{M}$ on the product σ-algebra satisfies

$$\mathscr{M}\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty}\mathscr{M}_k\{\Lambda_k\}, \tag{2}$$

where Λ_k are arbitrarily chosen events from $\mathscr{L}_k$ for $k = 1, 2, \ldots$, respectively.

An *uncertain variable* is defined by Liu [9] as a function ξ from an uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ to the set of real numbers such that $\{\xi \in B\}$ is an event for any Borel set B. In addition, an *uncertainty distribution* of ξ is defined as

$$\Phi(x) = \mathscr{M}\{\gamma \in \Gamma \mid \xi(\gamma) \leq x\}, \tag{3}$$

for any real number x.

Independence is an important concept in uncertainty theory. The uncertain variables $\xi_1, \xi_2, \ldots, \xi_m$ are said to be *independent* (Liu [19]) if

$$\mathscr{M}\left\{\bigcap_{i=1}^{m}(\xi_i \in B_i)\right\} = \min_{1\leq i\leq m}\mathscr{M}\{\xi_i \in B_i\} \tag{4}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

An uncertain variable ξ is called *linear* (Liu [9]) if it has a linear uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a, \\ \dfrac{(x-a)}{(b-a)}, & \text{if } a \leq x \leq b, \\ 1, & \text{if } x \geq b \end{cases} \tag{5}$$

denoted by $\mathscr{L}(a, b)$, where a and b are real numbers with $a < b$.

Let ξ be an uncertain variable. Then, the *expected value* (Liu [9]) of ξ is defined by

$$E[\xi] = \int_0^{+\infty}\mathscr{M}\{\xi \geq r\}\,\mathrm{d}r - \int_{-\infty}^{0}\mathscr{M}\{\xi \leq r\}\,\mathrm{d}r \tag{6}$$

provided that at least one of the two integrals is finite.

Remark 1. For numbers a and b, $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ if ξ and η are independent uncertain variables. Generally speaking, the expected value operator is not necessarily linear if the independence is not assumed.

Remark 2. Let

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1q} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ \xi_{p1} & \xi_{p2} & \cdots & \xi_{pq} \end{pmatrix}, \tag{7}$$

where ξ_{ij} are uncertain variables for $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$. The expected value of $\boldsymbol{\xi}$ is provided by

$$E[\boldsymbol{\xi}] = \begin{pmatrix} E[\xi_{11}] & E[\xi_{12}] & \cdots & E[\xi_{1q}] \\ E[\xi_{21}] & E[\xi_{22}] & \cdots & E[\xi_{2q}] \\ \cdots & \cdots & \cdots & \cdots \\ E[\xi_{p1}] & E[\xi_{p2}] & \cdots & E[\xi_{pq}] \end{pmatrix}. \tag{8}$$

Lemma 3 (Penrose [20]). *Let a matrix $M \in \mathbf{R}^{m\times n}$ be given. Then, there exists a unique matrix $M^+ \in \mathbf{R}^{n\times m}$ such that*

$$\begin{aligned} MM^+M &= M, \\ M^+MM^+ &= M^+, \\ (MM^+)^\tau &= MM^+, \\ (M^+M)^\tau &= M^+M. \end{aligned} \tag{9}$$

The matrix M^+ is called the Moore-Penrose pseudoinverse of M.

Lemma 4 (Penrose [20]). *Let matrices L, M, and N be given with appropriate sizes. Then, the matrix equation $LXM = N$ has a solution X if and only if $LL^+NMM^+ = N$. Moreover, any solution to $LXM = N$ is represented by $X = L^+NM^+ + Y - L^+LYMM^+$, where Y is a matrix with an appropriate size.*

3. Indefinite LQ Optimal Control with Constraints

3.1. Problem Statement. Consider the following indefinite LQ optimal control with terminal state constraint for discrete-time uncertain systems:

$$\begin{aligned} \inf_{\substack{\mathbf{u}_k\\0\le k\le N-1}} \quad & J(\mathbf{x}_0,\mathbf{u}) \\ &= \sum_{k=0}^{N-1} E\left[\mathbf{x}_k^\tau Q_k \mathbf{x}_k + \mathbf{u}_k^\tau R_k \mathbf{u}_k\right] + E\left[\mathbf{x}_N^\tau Q_N \mathbf{x}_N\right] \\ \text{subject to} \quad & \mathbf{x}_{k+1} = A_k\mathbf{x}_k + B_k\mathbf{u}_k + \lambda_k\left(A_k\mathbf{x}_k + B_k\mathbf{u}_k\right)\xi_k, \\ & \qquad k = 0,1,\ldots,N-1,\ \mathbf{x}(0) = \mathbf{x}_0 \\ & E\left[\mathbf{x}_N^\tau \mathbf{x}_N\right] = c, \end{aligned} \tag{10}$$

where $0 \le |\lambda_k| \le 1$, state $\mathbf{x}_k \in \mathbf{R}^n$, control input $\mathbf{u}_k \in \mathbf{R}^m$, $k = 0,1,\ldots,N-1$, and $\mathbf{x}_0 \in \mathbf{R}^n$ is a given crisp vector. Denote $\mathbf{u} = (\mathbf{u}_0,\mathbf{u}_1,\ldots,\mathbf{u}_{N-1})$. Moreover, $Q_0, Q_1,\ldots,Q_N$ and $R_0,R_1,\ldots,R_{N-1}$ are real symmetric matrices with appropriate dimensions. In addition, $c \ge 0$ is a constant; the coefficients $A_0,A_1,\ldots,A_{N-1}$ and $B_0,B_1,\ldots,B_{N-1}$ are crisp matrices having appropriate dimensions determined from context. Besides, the noises $\xi_0,\xi_1,\ldots,\xi_{N-1}$ are independent linear uncertain variables $\mathscr{L}(-1,1)$ with the distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le -1, \\ \dfrac{(x+1)}{2}, & \text{if } -1 \le x \le 1, \\ 1, & \text{if } x \ge 1. \end{cases} \tag{11}$$

In this paper, the weighting matrices in the objective functional are not required to be definite. Therefore, problem (10) is an indefinite LQ optimal control problem. Next, we give the following definitions.

Definition 5. The indefinite LQ problem (10) is called well-posed if

$$V(\mathbf{x}_0) = \inf_{\substack{\mathbf{u}_k\\0\le k\le N-1}} J(\mathbf{x}_0,\mathbf{u}) > -\infty, \quad \forall \mathbf{x}_0 \in \mathbf{R}^n. \tag{12}$$

Definition 6. A well-posed problem is called solvable, if, for $\mathbf{x}_0 \in \mathbf{R}^n$, there is a control sequence $(\mathbf{u}_0^*,\mathbf{u}_1^*,\ldots,\mathbf{u}_{N-1}^*)$ that achieves $V(\mathbf{x}_0)$. In this case, the control $(\mathbf{u}_0^*,\mathbf{u}_1^*,\ldots,\mathbf{u}_{N-1}^*)$ is called an optimal control sequence.

3.2. An Equivalent Problem. Next, we transform the uncertain LQ optimal control problem (10) into an equivalent deterministic LQ optimal control problem which is subject to a matrix difference equation constraint.

Let $X_k = E[\mathbf{x}_k\mathbf{x}_k^\tau]$. Since state $\mathbf{x}_k \in \mathbf{R}^n$, $\mathbf{x}_k\mathbf{x}_k^\tau$ is $n \times n$ matrix whose elements are uncertain variables, and X_k is a symmetric crisp matrix ($k = 0,1,\ldots,N$). Denote $\mathbf{K} = (K_0,K_1,\ldots,K_{N-1})$, where K_i are matrices for $i = 0,1,\ldots,N-1$.

Theorem 7. *If the indefinite LQ problem (10) is solvable by a feedback control*

$$\mathbf{u}_k = K_k\mathbf{x}_k, \tag{13}$$

where K_k are constant crisp matrices, then it is equivalent to the following deterministic optimal control problem:

$$\begin{aligned} \min_{\substack{K_k\\0\le k\le N-1}} \quad & J(X_0,\mathbf{K}) = \sum_{k=0}^{N-1} \operatorname{tr}\left[\left(Q_k + K_k^\tau R_k K_k\right)X_k\right] + \operatorname{tr}\left[Q_N X_N\right] \\ \textit{subject to} \quad & X_{k+1} = \left(1 + \frac{1}{3}\lambda_k^2\right)\left(A_kX_kA_k^\tau + A_kX_kK_k^\tau B_k^\tau + B_kK_kX_kA_k^\tau + B_kK_kX_kK_k^\tau B_k^\tau\right), \\ & X_0 = \mathbf{x}_0\mathbf{x}_0^\tau, \\ & \operatorname{tr}\left[X_N\right] = c, \end{aligned} \tag{14}$$

for $k = 0,1,\ldots,N-1$.

Proof. Assume that the indefinite LQ problem (10) is solvable by a feedback control

$$\mathbf{u}_k = K_k\mathbf{x}_k, \tag{15}$$

for $k = 0,1,\ldots,N-1$. Let $X_k = E[\mathbf{x}_k\mathbf{x}_k^\tau]$ for $k = 0,1,\ldots,N$. Then, we have

$$\begin{aligned} X_{k+1} &= E\left[\mathbf{x}_{k+1}\mathbf{x}_{k+1}^\tau\right] \\ &= E\left\{\left[A_k + B_kK_k + \lambda_k\left(A_k + B_kK_k\right)\xi_k\right]\right. \\ &\quad \left.\cdot \mathbf{x}_k\mathbf{x}_k^\tau\left[A_k^\tau + K_k^\tau B_k^\tau + \lambda_k\left(A_k^\tau + K_k^\tau B_k^\tau\right)\xi_k\right]\right\} \\ &= A_kX_kA_k^\tau + A_kX_kK_k^\tau B_k^\tau + B_kK_kX_kA_k^\tau \\ &\quad + B_kK_kX_kK_k^\tau B_k^\tau + E\left[U_k\xi_k + V_k\xi_k^2\right], \end{aligned} \tag{16}$$

where

$$\begin{aligned} U_k = 2\lambda_k & \left(A_kX_kA_k^\tau + A_kX_kK_k^\tau B_k^\tau + B_kK_kX_kA_k^\tau\right. \\ & \left. + B_kK_kX_kK_k^\tau B_k^\tau\right) \end{aligned}$$

$$V_k = \lambda_k^2 \left(A_k X_k A_k^\tau + A_k X_k K_k^\tau B_k^\tau + B_k K_k X_k A_k^\tau + B_k K_k X_k K_k^\tau B_k^\tau\right). \tag{17}$$

Then, we obtain that $\lambda_k U_k = 2V_k$. Because ξ_k and ξ_k^2 are not independent, we know that

$$E\left[U_k\xi_k + V_k\xi_k^2\right] \neq U_k E\left[\xi_k\right] + V_k E\left[\xi_k^2\right]. \tag{18}$$

We will deal with (18) as follows.

(i) If $V_k = \mathbf{0}$, we obtain

$$E\left[U_k\xi_k + V_k\xi_k^2\right] = E\left[U_k\xi_k\right] = U_k E\left[\xi_k\right] = \mathbf{0}. \tag{19}$$

(ii) If $V_k \neq \mathbf{0}$, we know that $\lambda_k \neq 0$ and $|2/\lambda_k| \geq 2$. According to Example 2 in [21], we have

$$\begin{aligned} E\left[U_k\xi_k + V_k\xi_k^2\right] &= E\left[\frac{2}{\lambda_k}V_k\xi_k + V_k\xi_k^2\right] \\ &= V_k E\left[\frac{2}{\lambda_k}\xi_k + \xi_k^2\right] = \frac{1}{3}V_k. \end{aligned} \tag{20}$$

Therefore, we have

$$E\left[U_k\xi_k + V_k\xi_k^2\right] = \frac{1}{3}V_k. \tag{21}$$

Substituting (21) into (16) produces the following state matrix:

$$\begin{aligned} X_{k+1} = \left(1 + \frac{1}{3}\lambda_k^2\right)&\left(A_k X_k A_k^\tau + A_k X_k K_k^\tau B_k^\tau \right.\\ &\left.+ B_k K_k X_k A_k^\tau + B_k K_k X_k K_k^\tau B_k^\tau\right). \end{aligned} \tag{22}$$

The associated cost function reduces to

$$\begin{aligned} &\min_{\substack{K_k \\ 0\leq k\leq N-1}} J\left(X_0, \mathbf{K}\right) \\ &\quad = \min_{\substack{K_k \\ 0\leq k\leq N-1}} \sum_{k=0}^{N-1} \operatorname{tr}\left[\left(Q_k + K_k^\tau R_k K_k\right) X_k\right] \\ &\qquad + \operatorname{tr}\left[Q_N X_N\right], \end{aligned} \tag{23}$$

and the constraint $E[\mathbf{x}_N^\tau \mathbf{x}_N] = c$ becomes $\operatorname{tr}[X_N] = c$. □

Remark 8. Obviously, if problem (10) has a linear feedback optimal control solution $\mathbf{u}_k^* = K_k^* \mathbf{x}_k$ $(k = 0, 1, \ldots, N-1)$, then K_k^* $(k = 0, 1, \ldots, N-1)$ is the optimal solution of problem (14).

3.3. A Necessary Condition for State Feedback Control. In this subsection, a necessary condition for the optimal linear state feedback control with deterministic gains to the indefinite LQ problem (10) is obtained by applying the deterministic matrix maximum principle [22].

Theorem 9. *If the indefinite LQ problem (10) is solvable by a feedback control*

$$\mathbf{u}_k = K_k \mathbf{x}_k, \tag{24}$$

where K_k are constant crisp matrices, then there exist symmetric matrices H_k and a nonnegative $\gamma \in \mathbf{R}^1$ solving the following constrained difference equation:

$$\begin{aligned} H_k &= Q_k + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} A_k - M_k^\tau L_k^+ M_k, \\ L_k L_k^+ M_k - M_k &= 0, \\ L_k &= R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k \geq 0, \\ M_k &= \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} A_k, \\ H_N &= Q_N + \gamma I, \end{aligned} \tag{25}$$

for $k = 0, 1, \ldots, N-1$. Moreover,

$$K_k = -L_k^+ M_k + Y_k - L_k^+ L_k Y_k \tag{26}$$

with $Y_k \in \mathbf{R}^{m\times n}$, $k = 0, 1, \ldots, N-1$, being any given crisp matrices.

Proof. Assume that the indefinite LQ problem (10) is solvable by

$$\mathbf{u}_k = K_k \mathbf{x}_k, \tag{27}$$

where the matrices K_k $(k = 0, 1, \ldots, N-1)$ are viewed as the control to be determined. It is obvious that K_k is also the optimal solution of problem (14) which is deterministic LQ optimal control problem. Hence, we can apply the matrix Lagrangian multiplier method to solve problem (14).

Let matrices H_{k+1} $(k = 0, 1, \ldots, N-1)$ be the Lagrange multipliers of $\mathbf{h}_{k+1}(X_k, K_k)$ $(k = 0, 1, \ldots, N-1)$, and let $\gamma \in \mathbf{R}^1$ be the Lagrange multiplier of $g(X_N) = 0$. Then, the Lagrange function is formed as

$$\begin{aligned} \mathscr{L} = J\left(X_0, \mathbf{K}\right) &+ \sum_{k=0}^{N-1} \operatorname{tr}\left[H_{k+1}\mathbf{h}_{k+1}\left(X_k, K_k\right)\right] \\ &+ \gamma g\left(X_N\right), \end{aligned} \tag{28}$$

where

$$\begin{aligned} J\left(X_0, \mathbf{K}\right) &= \sum_{k=0}^{N-1} \operatorname{tr}\left[\left(Q_k + K_k^\tau R_k K_k\right) X_k\right] + \operatorname{tr}\left[Q_N X_N\right] \\ \mathbf{h}_{k+1}\left(X_k, K_k\right) &= \left(1 + \frac{1}{3}\lambda_k^2\right)\left(A_k X_k A_k^\tau + A_k X_k K_k^\tau B_k^\tau \right.\\ &\quad \left.+ B_k K_k X_k A_k^\tau + B_k K_k X_k K_k^\tau B_k^\tau\right) - X_{k+1}, \\ g\left(X_N\right) &= \operatorname{tr}\left[X_N\right] - c. \end{aligned} \tag{29}$$

According to the first-order necessary conditions for optimality [22], we have

$$\frac{\partial \mathscr{L}}{\partial K_k} = 0 \quad (k = 0, 1, \ldots, N-1), \tag{30}$$

$$H_k = \frac{\partial \mathscr{L}}{\partial X_k} \quad (k = 0, 1, \ldots, N-1), \tag{31}$$

$$H_N = Q_N + \gamma I. \tag{32}$$

Based on the partial rule of gradient matrices [22], (30) can be transformed into

$$\begin{aligned} &\left[R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k\right] K_k \\ &\quad + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} A_k = 0. \end{aligned} \tag{33}$$

Let

$$\begin{aligned} L_k &= R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k, \\ M_k &= \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} A_k. \end{aligned} \tag{34}$$

Then, (33) can be rewritten as $L_k K_k + M_k = 0$. Applying Lemma 4, we have $L_k L_k^+ M_k = M_k$, and

$$K_k = -L_k^+ M_k + Y_k - L_k^+ L_k Y_k, \quad Y_k \in \mathbf{R}^{m\times n}. \tag{35}$$

For (31), according to

$$H_k = \frac{\partial \mathscr{L}}{\partial X_k} \quad (k = 0, 1, \ldots, N-1), \tag{36}$$

we have

$$\begin{aligned} H_k &= Q_k + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} A_k \\ &\quad + K_k^\tau \left[R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k\right] K_k \\ &\quad + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} B_k K_k \\ &\quad + \left(1 + \frac{1}{3}\lambda_k^2\right) K_k^\tau B_k^\tau H_{k+1} A_k. \end{aligned} \tag{37}$$

Substituting (35) into (37), we obtain

$$H_k = Q_k + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} A_k - M_k^\tau L_k^+ M_k. \tag{38}$$

Consider the objective functional

$$\begin{aligned} J(\mathbf{x}_0, \mathbf{u}) &= \sum_{k=0}^{N-1} E\left[\mathbf{x}_k^\tau Q_k \mathbf{x}_k + \mathbf{u}_k^\tau R_k \mathbf{u}_k\right] + E\left[\mathbf{x}_N^\tau Q_N \mathbf{x}_N\right] \\ &= \sum_{k=0}^{N-1} E\left\{\left[\mathbf{x}_k^\tau Q_k \mathbf{x}_k + \mathbf{u}_k^\tau R_k \mathbf{u}_k\right] + E\left[\mathbf{x}_{k+1}^\tau H_{k+1} \mathbf{x}_{k+1}\right]\right. \\ &\quad \left. - E\left[\mathbf{x}_k^\tau H_k \mathbf{x}_k\right]\right\} + E\left[\mathbf{x}_N^\tau Q_N \mathbf{x}_N\right] - E\left[\mathbf{x}_N^\tau H_N \mathbf{x}_N\right] \\ &\quad + \mathbf{x}_0^\tau H_0 \mathbf{x}_0 = \sum_{k=0}^{N-1} \left\{\operatorname{tr}\left[\left(Q_k + K_k^\tau R_k K_k\right) X_k\right]\right. \\ &\quad \left. + \operatorname{tr}\left[H_{k+1} X_{k+1}\right] - \operatorname{tr}\left[H_k X_k\right]\right\} + \operatorname{tr}\left[\left(Q_N - H_N\right)\right. \\ &\quad \left. \cdot X_N\right] + \mathbf{x}_0^\tau H_0 \mathbf{x}_0. \end{aligned} \tag{39}$$

Since $X_{k+1} = (1 + (1/3)\lambda_k^2)(A_k X_k A_k^\tau + A_k X_k K_k^\tau B_k^\tau + B_k K_k X_k A_k^\tau + B_k K_k X_k K_k^\tau B_k^\tau)$, the objective functional can be rewritten as

$$\begin{aligned} J(X_0, \mathbf{K}) &= \sum_{k=0}^{N-1} \left\{\operatorname{tr}\left[\left(Q_k + K_k^\tau R_k K_k\right) + \left(1 + \frac{1}{3}\lambda_k^2\right)\right.\right. \\ &\quad \cdot \left(A_k^\tau H_{k+1} A_k + K_k^\tau B_k^\tau H_{k+1} A_k + A_k^\tau H_{k+1} B_k K_k\right. \\ &\quad \left.\left.\left. + K_k^\tau B_k^\tau H_{k+1} B_k K_k\right) - H_k\right] X_k\right\} + \operatorname{tr}\left[\left(Q_N - H_N\right)\right. \\ &\quad \left.\cdot X_N\right] + \mathbf{x}_0^\tau \mathrm{H}_0 \mathbf{x}_0 = \sum_{k=0}^{N-1} \operatorname{tr}\left\{\left[Q_k\right.\right. \\ &\quad \left. + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} A_k - H_k\right] + \left(1 + \frac{1}{3}\lambda_k^2\right) \\ &\quad \cdot K_k^\tau B_k^\tau H_{k+1} A_k + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} B_k K_k \\ &\quad \left. + K_k^\tau \left[R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k\right] K_k\right\} X_k \\ &\quad + \operatorname{tr}\left[\left(Q_N - H_N\right) X_N\right] + \mathbf{x}_0^\tau H_0 \mathbf{x}_0 \\ &= \sum_{k=0}^{N-1} \operatorname{tr}\left[M_k^\tau L_k^+ M_k + K_k^\tau M_k + M_k^\tau K_k + K_k^\tau L_k K_k\right] \\ &\quad \cdot X_k + \operatorname{tr}\left[\left(Q_N - H_N\right) X_N\right] + \mathbf{x}_0^\tau H_0 \mathbf{x}_0. \end{aligned} \tag{40}$$

By applying (32) and Lemma 3, a completion of square implies

$$\begin{aligned} &J(X_0, \mathbf{K}) \\ &\quad = \sum_{k=0}^{N-1} \operatorname{tr}\left[\left(K_k + L_k^+ M_k\right)^\tau L_k \left(K_k + L_k^+ M_k\right) X_k\right] - c\gamma \\ &\qquad + \mathbf{x}_0^\tau H_0 \mathbf{x}_0. \end{aligned} \tag{41}$$

We assert that L_k $(k = 0, 1, \ldots, N-1)$ must satisfy

$$L_k = R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k \geq 0. \tag{42}$$

If it is not so, there is an L_p for $p \in \{0,1,\ldots,N-1\}$ with a negative eigenvalue λ. Denote the unitary eigenvector with respect to λ as $\mathbf{v}_\lambda$ (i.e., $\mathbf{v}_\lambda^\tau \mathbf{v}_\lambda = 1$ and $L_p \mathbf{v}_\lambda = \lambda \mathbf{v}_\lambda$). Let $\delta \neq 0$ be an arbitrary scalar and construct a control sequence $\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2, \ldots, \tilde{\mathbf{u}}_{N-1})$ as follows:

$$\tilde{\mathbf{u}}_k = \begin{cases} -L_k^{+} M_k \mathbf{x}_k, & k \neq p, \\ \delta \left|\lambda\right|^{-1/2} \mathbf{v}_\lambda - L_k^{+} M_k \mathbf{x}_k, & k = p. \end{cases} \tag{43}$$

The associated cost functional becomes

$$\begin{aligned} J\left(\mathbf{x}_0, \tilde{\mathbf{u}}\right) &= \sum_{k=0}^{N-1} \operatorname{tr}\left[\left(\widetilde{K}_k + L_k^{+} M_k\right)^\tau L_k \left(\widetilde{K}_k + L_k^{+} M_k\right) X_k\right] \\ &\quad - c\gamma + \mathbf{x}_0^\tau H_0 \mathbf{x}_0 \\ &= \sum_{k=0}^{N-1} E\left[\left(\tilde{\mathbf{u}}_k + L_k^{+} M_k \mathbf{x}_k\right)^\tau L_k \left(\tilde{\mathbf{u}}_k + L_k^{+} M_k \mathbf{x}_k\right)\right] \\ &\quad - c\gamma + \mathbf{x}_0^\tau H_0 \mathbf{x}_0 \\ &= \left[\frac{\delta}{\left|\lambda\right|^{1/2}} \mathbf{v}_\lambda\right]^\tau L_p \left[\frac{\delta}{\left|\lambda\right|^{1/2}} \mathbf{v}_\lambda\right] - c\gamma + \mathbf{x}_0^\tau H_0 \mathbf{x}_0 \\ &= -\delta^2 - c\gamma + \mathbf{x}_0^\tau H_0 \mathbf{x}_0. \end{aligned} \tag{44}$$

Let $\delta \to \infty$. Then, $J(\mathbf{x}_0, \tilde{\mathbf{u}}) \to -\infty$, which contradicts the well-posedness of problem (10). □

3.4. Special Cases. We have obtained that $L_k \geq 0$ in the constrained difference equation (25) of Theorem 9. The following corollaries are special cases of the above result if we have $L_k > 0$ and $L_k = 0$.

Corollary 10. *The indefinite LQ problem (10) is uniquely solvable if and only if $L_k > 0$ for $k = 0,1,\ldots,N-1$. Moreover, the unique optimal control is given by*

$$\mathbf{u}_k = -L_k^{-1} M_k \mathbf{x}_k, \quad k = 0,1,\ldots,N-1. \tag{45}$$

Proof. By using Theorem 9, we immediately obtain the corollary. □

Corollary 11. *If $L_k = 0$ for $k = 0,1,\ldots,N-1$, then any admissible control of the indefinite LQ problem (10) is optimal and the constrained difference equation (25) reduces to the following linear system:*

$$\begin{aligned} H_k &= Q_k + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} A_k, \\ R_k &+ \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k = 0, \\ B_k^\tau H_{k+1} A_k &= 0, \\ H_N &= Q_N + \gamma I, \end{aligned} \tag{46}$$

for $k = 0,1,\ldots,N-1$.

Proof. Letting $L_k = 0$ in (25), it is easy to obtain the linear system (46). Letting $L_k = 0$ in (41), (41) is simplified as

$$J\left(\mathbf{x}_0, \mathbf{u}\right) = -c\gamma + \mathbf{x}_0^\tau H_0 \mathbf{x}_0, \tag{47}$$

which implies that $V(\mathbf{x}_0) = -c\gamma + \mathbf{x}_0^\tau H_0 \mathbf{x}_0$ for any admissible control. Then, any admissible control of the indefinite LQ problem (10) is optimal. □

3.5. Well-Posedness of the Indefinite LQ Problem. In the following, it is shown that the solvability of the constrained difference equation (25) is sufficient for the well-posedness of the indefinite LQ problem and the existence of an optimal control. Moreover, any optimal control can be represented explicitly as a linear state feedback by the solution of (25).

Theorem 12. *The indefinite LQ problem (10) is well-posed if there exist symmetric matrices H_k and $\gamma \in \mathbf{R}^1$ satisfying the constrained difference equation (25). Moreover, the optimal control is given by*

$$\begin{aligned} \mathbf{u}_k = &-\left[R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k\right]^{+} \\ &\cdot \left[\left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} A_k\right] \mathbf{x}_k, \\ &k = 0,1,\ldots,N-1. \end{aligned} \tag{48}$$

Furthermore, the optimal cost of the indefinite LQ problem (10) is

$$V\left(\mathbf{x}_0\right) = \mathbf{x}_0^\tau H_0 \mathbf{x}_0 - c\gamma. \tag{49}$$

Proof. Let H_k and $\gamma \in \mathbf{R}^1$ satisfy (25). Then,

$$\begin{aligned} J\left(\mathbf{x}_0, \mathbf{u}\right) &= \sum_{k=0}^{N-1} E\left[\mathbf{x}_k^\tau Q_k \mathbf{x}_k + \mathbf{u}_k^\tau R_k \mathbf{u}_k\right] + E\left[\mathbf{x}_N^\tau Q_N \mathbf{x}_N\right] \\ &= \sum_{k=0}^{N-1} \{E\left[\mathbf{x}_k^\tau Q_k \mathbf{x}_k + \mathbf{u}_k^\tau R_k \mathbf{u}_k\right] + E\left[\mathbf{x}_{k+1}^\tau H_{k+1} \mathbf{x}_{k+1}\right] \\ &\quad - E\left[\mathbf{x}_k^\tau H_k \mathbf{x}_k\right]\} + E\left[\mathbf{x}_N^\tau Q_N \mathbf{x}_N\right] - E\left[\mathbf{x}_N^\tau H_N \mathbf{x}_N\right] \\ &\quad + \mathbf{x}_0^\tau H_0 \mathbf{x}_0 = \sum_{k=0}^{N-1} \{\operatorname{tr}\left[\left(Q_k + K_k^\tau R_k K_k\right) X_k\right] \\ &\quad + \operatorname{tr}\left[H_{k+1} X_{k+1}\right] - \operatorname{tr}\left[H_k X_k\right]\} + \operatorname{tr}\left[\left(Q_N - H_N\right) \right.\\ &\quad \left.\cdot X_N\right] + \mathbf{x}_0^\tau H_0 \mathbf{x}_0 \\ &= \sum_{k=0}^{N-1} \operatorname{tr}\left\{\left[Q_k + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} A_k - H_k\right]\right. \end{aligned}$$

$$+\left(1+\frac{1}{3}\lambda_k^2\right)K_k^\tau B_k^\tau H_{k+1}A_k$$
$$+\left(1+\frac{1}{3}\lambda_k^2\right)A_k^\tau H_{k+1}B_kK_k$$
$$+K_k^\tau\left[R_k+\left(1+\frac{1}{3}\lambda_k^2\right)B_k^\tau H_{k+1}B_k\right]K_k\Bigg\}X_k$$
$$+\operatorname{tr}\left[\left(Q_N-H_N\right)X_N\right]+\mathbf{x}_0^\tau H_0\mathbf{x}_0$$
$$=\sum_{k=0}^{N-1}\operatorname{tr}\left[M_k^\tau L_k^+M_k+K_k^\tau M_k+M_k^\tau K_k+K_k^\tau L_kK_k\right]$$
$$\cdot X_k+\operatorname{tr}\left[\left(Q_N-H_N\right)X_N\right]+\mathbf{x}_0^\tau H_0\mathbf{x}_0. \tag{50}$$

By applying Lemma 3, a completion of square implies

$$J\left(X_0,\mathbf{K}\right)=\sum_{k=0}^{N-1}\operatorname{tr}\left[\left(K_k+L_k^+M_k\right)^\tau L_k\left(K_k+L_k^+M_k\right)X_k\right]+\operatorname{tr}\left[\left(Q_N-H_N\right)X_N\right]+\mathbf{x}_0^\tau H_0\mathbf{x}_0. \tag{51}$$

Since $L_k\geq 0$, from (51), we can easily deduce that the cost function of problem (10) is bounded from below by

$$V\left(\mathbf{x}_0\right)=\operatorname{tr}\left[\left(Q_N-H_N\right)X_N\right]+\mathbf{x}_0^\tau H_0\mathbf{x}_0>-\infty,\quad \forall\mathbf{x}_0\in\mathbf{R}^n. \tag{52}$$

Hence, the indefinite LQ problem (10) is well-posed. It is clear that it is solvable by the feedback control

$$\mathbf{u}_k=-K_k\mathbf{x}_k=-L_k^+M_k\mathbf{x}_k,\quad k=0,1,\ldots,N-1. \tag{53}$$

Furthermore, by using $\operatorname{tr}[X_N]=c$ and $H_N=Q_N+\gamma I$ which we have obtained in Theorems 7 and 9, (52) indicates that the optimal value of problem (10) equals

$$V\left(\mathbf{x}_0\right)=\mathbf{x}_0^\tau H_0\mathbf{x}_0-c\gamma. \tag{54}$$

□

4. General Expression for the Optimal Control Set

In this part, we will present a general expression for the optimal control set based on the solution to (25).

Theorem 13. *Assume that H_k $(k=0,1,\ldots,N-1)$ and $\gamma\geq 0\in\mathbf{R}^1$ solves the constrained difference equation (25). A sufficient and necessary condition that $\mathbf{u}_k$ is in the set of all optimal feedback controls for indefinite LQ problem (10) is that*

$$\mathbf{u}_k=-\left(L_k^+M_k+Y_k-L_k^+L_kY_k\right)\mathbf{x}_k+Z_k-L_k^+M_kZ_k,\quad k=0,1,\ldots,N-1, \tag{55}$$

where $Y_k\in\mathbf{R}^{m\times n}$ and $Z_k\in\mathbf{R}^m$ are arbitrary variables with appropriate size.

Proof.

Sufficiency. According to the same calculation as in Theorem 9, we have

$$J\left(\mathbf{x}_0,\mathbf{u}\right)=\sum_{k=0}^{N-1}E\left[\mathbf{x}_k^\tau Q_k\mathbf{x}_k+\mathbf{u}_k^\tau R_k\mathbf{u}_k\right]+E\left[\mathbf{x}_N^\tau Q_N\mathbf{x}_N\right]$$
$$=\sum_{k=0}^{N-1}\operatorname{tr}\left\{\left[Q_k+\left(1+\frac{1}{3}\lambda_k^2\right)A_k^\tau H_{k+1}A_k-H_k\right]\right.$$
$$+\left(1+\frac{1}{3}\lambda_k^2\right)K_k^\tau B_k^\tau H_{k+1}A_k+\left(1+\frac{1}{3}\lambda_k^2\right)$$
$$\cdot A_k^\tau H_{k+1}B_kK_k+K_k^\tau\left[R_k+\left(1+\frac{1}{3}\lambda_k^2\right)B_k^\tau H_{k+1}B_k\right] \tag{56}$$
$$\left.\cdot K_k\right\}X_k-c\gamma+\mathbf{x}_0^\tau H_0\mathbf{x}_0=\sum_{k=0}^{N-1}E\left[\mathbf{x}_k^\tau\left(M_k^\tau L_k^+M_k\right.\right.$$
$$\left.\left.+K_k^\tau M_k+M_k^\tau K_k+K_k^\tau L_kK_k\right)\mathbf{x}_k\right]-c\gamma+\mathbf{x}_0^\tau H_0\mathbf{x}_0$$
$$=\sum_{k=0}^{N-1}E\left[\mathbf{x}_k^\tau M_k^\tau L_k^+M_k\mathbf{x}_k+2\mathbf{x}_k^\tau M_k^\tau\mathbf{u}_k+\mathbf{u}_k^\tau L_k\mathbf{u}_k\right]$$
$$-c\gamma+\mathbf{x}_0^\tau H_0\mathbf{x}_0.$$

By denoting $T_k^1=-(Y_k-L_k^+L_kY_k)$ and $T_k^2=-(Z_k-L_k^+L_kZ_k)$, we obtain

$$L_kT_k^1=0,\qquad L_kT_k^2=0. \tag{57}$$

According to (56) and (57), we obtain

$$J\left(\mathbf{x}_0,\mathbf{u}\right)=\sum_{k=0}^{N-1}E\left[\mathbf{u}_k+\left(L_k^+M_k+T_k^1\right)\mathbf{x}_k+T_k^2\right]^\tau\cdot L_k\left[\mathbf{u}_k+\left(L_k^+M_k+T_k^1\right)\mathbf{x}_k+T_k^2\right]-c\gamma+\mathbf{x}_0^\tau H_0\mathbf{x}_0. \tag{58}$$

As $L_k\geq 0$, we know that the control $\mathbf{u}_k=-[(L_k^+M_k+T_k^1)\mathbf{x}_k+T_k^2]$ minimizes $J(\mathbf{x}_0,\mathbf{u})$ with the optimal value $-c\gamma+\mathbf{x}_0^\tau H_0\mathbf{x}_0$ for $k=0,1,\ldots,N-1$.

Necessity. If any control sequence $\tilde{\mathbf{u}}=(\tilde{\mathbf{u}}_1,\tilde{\mathbf{u}}_2,\ldots,\tilde{\mathbf{u}}_{N-1})$ which minimizes the cost function $J(\mathbf{x}_0,\mathbf{u})$, then we have

$$J\left(\mathbf{x}_0,\tilde{\mathbf{u}}\right)=\sum_{k=0}^{N-1}E\left[\left(\tilde{\mathbf{u}}_k+L_k^+M_k\mathbf{x}_k\right)^\tau L_k\left(\tilde{\mathbf{u}}_k+L_k^+M_k\mathbf{x}_k\right)\right]-c\gamma+\mathbf{x}_0^\tau H_0\mathbf{x}_0, \tag{59}$$

for $k=0,1,\ldots,N-1$. The above equality implies that

$$\sum_{k=0}^{N-1}E\left[\left(\tilde{\mathbf{u}}_k+L_k^+M_k\mathbf{x}_k\right)^\tau L_k\left(\tilde{\mathbf{u}}_k+L_k^+M_k\mathbf{x}_k\right)\right]=0,\quad k=0,1,\ldots,N-1. \tag{60}$$

Since $L_k \geq 0$, we get the following equivalent condition:

$$L_k \left(\tilde{\mathbf{u}}_k + L_k^+ M_k \mathbf{x}_k\right) = 0, \quad k = 0, 1, \ldots, N-1. \tag{61}$$

We see that $\tilde{\mathbf{u}}_k$ solves the following equation:

$$L_k \tilde{\mathbf{u}}_k + L_k L_k^+ M_k \mathbf{x}_k = 0, \quad k = 0, 1, \ldots, N-1. \tag{62}$$

By using Lemma 3 with $L = L_k$, $M = I$, $N = -L_k L_k^+ M_k \mathbf{x}_k$, it is easy to verify that

$$LL^+ NMM^+ = N. \tag{63}$$

Then, we obtain the solution of (62) with

$$\begin{aligned}\tilde{\mathbf{u}}_k &= -L_k^+ M_k \mathbf{x}_k + Z_k - L_k^+ L_k Z_k, \\ &\qquad Z_k \in \mathbf{R}^m, \ k = 0, 1, \ldots, N-1.\end{aligned} \tag{64}$$

As in (35), the optimal control can be represented by

$$\begin{aligned}\mathbf{u}_k &= -\left(L_k^+ M_k + Y_k - L_k^+ L_k Y_k\right) \mathbf{x}_k + Z_k - L_k^+ M_k Z_k, \\ &\qquad k = 0, 1, \ldots, N-1.\end{aligned} \tag{65}$$

□

5. Numerical Example

In this section, application of Theorem 9 to solve constraint optimal control problem is illustrated. We present a two-dimensional indefinite LQ problem with terminal state constraint for discrete-time uncertain systems. A set of specific parameters of the coefficients are given as follows:

$$\begin{aligned}\mathbf{x}_0 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ c &= 2.0408, \\ N &= 2, \\ A_0 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ A_1 &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \\ B_0 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \\ B_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \lambda_0 &= -\frac{\sqrt{3}}{2}, \\ \lambda_1 &= \frac{\sqrt{3}}{2}.\end{aligned} \tag{66}$$

The state weights and the control weights are as follows:

$$\begin{aligned}Q_0 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\ Q_1 &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\ Q_2 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ R_0 &= -1, \\ R_1 &= -2.\end{aligned} \tag{67}$$

Note that, in this example, the state weight Q_0 is negative semidefinite, Q_1 is negative definite, and Q_2 is positive semidefinite and the control weights R_0 and R_1 are negative definite.

In order to find the optimal controls and optimal cost value of this example, we have to solve the following equations:

$$\begin{aligned}H_k &= Q_k + \left(1 + \frac{1}{3}\lambda_k^2\right) A_k^\tau H_{k+1} A_k - M_k^\tau L_k^+ M_k, \\ L_k L_k^+ M_k &- M_k = 0, \\ L_k &= R_k + \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} B_k \geq 0, \\ M_k &= \left(1 + \frac{1}{3}\lambda_k^2\right) B_k^\tau H_{k+1} A_k, \quad k = 0, 1, \\ H_2 &= Q_2 + \gamma I, \\ X_{k+1} &= \left(1 + \frac{1}{3}\lambda_k^2\right) \left(A_k X_k A_k^\tau + A_k X_k K_k^\tau B_k^\tau \right. \\ &\quad \left. + B_k K_k X_k A_k^\tau + B_k K_k X_k K_k^\tau B_k^\tau\right), \\ &\qquad k = 0, 1, \ X_0 = \mathbf{x}_0 \mathbf{x}_0^\tau.\end{aligned} \tag{68}$$

Firstly, we have

$$X_0 = \mathbf{x}_0 \mathbf{x}_0^\tau = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{69}$$

Then, we get $\gamma = 2$ by solving (68), and we obtain

$$H_2 = Q_2 + \gamma I = \gamma \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}. \tag{70}$$

Secondly, by applying Theorem 9, we obtain the optimal feedback control and optimal cost value as follows.

For $k = 1$, we obtain

$$L_1 = R_1 + \left(1 + \frac{1}{3}\lambda_1^2\right) B_1^\tau H_2 B_1 = 0.5 > 0,$$
$$M_1 = \left(1 + \frac{1}{3}\lambda_1^2\right) B_1^\tau H_2 A_1 = (5, 0),$$
$$H_1 = Q_1 + \left(1 + \frac{1}{3}\lambda_1^2\right) A_1^\tau H_2 A_1 - M_1^\tau L_1^+ M_1 = \begin{pmatrix} -41 & 0 \\ 0 & 1.5 \end{pmatrix}. \quad (71)$$

The optimal feedback control is $\mathbf{u}_1 = K_1 \mathbf{x}_1$, where

$$K_1 = -L_1^+ M_1 = (-10, 0). \quad (72)$$

For $k = 0$, we obtain

$$L_0 = R_0 + \left(1 + \frac{1}{3}\lambda_0^2\right) B_0^\tau H_1 B_0 = 0.875 > 0,$$
$$M_0 = \left(1 + \frac{1}{3}\lambda_0^2\right) B_0^\tau H_1 A_0 = (1.875, 1.875),$$
$$H_0 = Q_0 + \left(1 + \frac{1}{3}\lambda_0^2\right) A_0^\tau H_1 A_0 - M_0^\tau L_0^+ M_0 = \begin{pmatrix} -54.3929 & -2.1429 \\ -2.1429 & -1.1429 \end{pmatrix}. \quad (73)$$

The optimal feedback control is $\mathbf{u}_0 = K_0 \mathbf{x}_0$, where

$$K_0 = -L_0^+ M_0 = (-2.1429, -2.1429). \quad (74)$$

Finally, the optimal cost value is

$$V(\mathbf{x}_0) = \mathbf{x}_0^\tau H_0 \mathbf{x}_0 - c\gamma = -5.2245. \quad (75)$$

6. Conclusion

We have considered the indefinite LQ optimal control with terminal state constraint involving state and control dependent uncertain noises. We first transform the uncertain LQ optimal control problem into a deterministic LQ optimal control problem. By means of the matrix maximum principle, we have presented a necessary condition for the existence of optimal linear state feedback control. Besides, we have proved the well-posedness of the indefinite LQ constraint problem by applying the technique of completing squares. For further work, we will consider discrete-time indefinite LQ optimal control model with inequality constraint.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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