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Research Article

H_∞ Control for Nonlinear Stochastic Systems with Time-Delay and Multiplicative Noise

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This paper studies the infinite horizon H_∞ control problem for a general class of nonlinear stochastic systems with time-delay and multiplicative noise. The exponential/asymptotic mean square H_∞ control design of delayed nonlinear stochastic systems is presented by solving Hamilton-Jacobi inequalities. Two numerical examples are provided to show the effectiveness of the proposed design method.

1. Introduction

It is well known that H_∞ control is one of the most effective approaches to eliminate the effect of the external disturbance [1]. For deterministic linear systems, H_∞ norm is defined by a norm of the transfer function, which cannot be extended to stochastic or nonlinear systems directly. In 1989, Doyle et al. found that, from the view point of time-domain, the norm of a transfer function was the L_2-induced norm of the input-output operator [2], which made it possible to develop the nonlinear or stochastic H_∞ theory [3, 4]. Following along the lines of [4], Zhang and Chen developed infinite and finite horizon nonlinear stochastic H_∞ control designs by means of Hamilton-Jacobi equations [5]. Moreover, the mixed H_2/H_∞ control has also received much attention due to its important significance in practical applications [6].

The phenomena of time-delay are frequently encountered in many engineering applications owing to the finite speed of information processing [7]. Time-delay, nonlinearity, and stochasticity are arguably three of the main sources in reality which result in the complexity of a system. Over the past years, the stability of delayed nonlinear stochastic systems (DNSSs) has gained significant research interests [8–15]. In [8], Mao established the LaSalle-type theorems for the solutions of stochastic differential delay equations, which was applied to establish sufficient criteria for the stochastically asymptotic stability of the delay equations. In [10], the problem of exponential stability for a class of impulsive nonlinear stochastic differential equations with mixed time-delays was investigated, and some interesting results were derived. In [13], the delay-dependent stability conditions for DNSSs were derived based on the convergence theorem for semimartingale inequalities.

Although many results for the stability analysis of DNSSs have been published, the H_∞ control problem of DNSSs has received relatively little attention [16–18]. In [16], the H_∞ analysis problem was studied for a general class of nonlinear stochastic systems with time-delay by using the Razumikhin-type method. In [17], the problem of robust H_∞ output feedback control was studied for a class of uncertain discrete-time DNSSs with missing measurements. In [18], the quantized H_∞ control problem was investigated for delayed nonlinear stochastic network-based systems with data missing. However, most of the above literatures only considered the stochastic systems with state-dependent noise. As pointed in [19], the control input and external disturbance may also be corrupted by noise. Therefore, it is necessary to study the stochastic systems with state, control, and disturbance-dependent noise [20, 21].

Motivated by the preceding discussion, this paper will investigate the infinite horizon H_∞ control for a class of nonlinear stochastic state-delayed systems with multiplicative noise. Compared with [16–18, 22], the considered system in this paper is more general since state, control, and disturbance enter into the diffusion term simultaneously. By means of Hamilton-Jacobi inequalities (HJIs), a sufficient condition is derived for the exponential and asymptotic mean square H_∞ control of DNSSs, respectively. In contrast to the conditions for delay-free H_∞ control [20, 21], the current HJIs depend on more variables owing to the appearance of time-delay. Finally, two numerical examples are given to demonstrate the effectiveness of the obtained results.

Throughout this paper, the following notations will be used. $\mathscr{R}^n$ is n-dimensional Euclidean space. $\mathscr{R}^{n\times m}$ is the set of all $n\times m$ real matrices. A' is the transpose of a matrix A. $A>0$ $(A\geq 0)$: A is a positive definite (positive semidefinite) symmetric matrix. $E[\cdot]$ is the mathematical expectation. $\|x\|$ is the Euclidean norm of a vector x. $L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^l)$ is the space of nonanticipative stochastic processes $y(t)\in\mathscr{R}^l$ with respect to an increasing σ-algebras $\mathscr{F}_t$ $(t\geq 0)$ satisfying $\|y(t)\|_{L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^l)} = (E\int_0^\infty \|y(t)\|^2 dt)^{1/2} < \infty$. $\mathscr{C}^{2,1}(U,T)$ is the class of functions $V(x,t)$ twice continuously differential with respect to $x\in U$ and once continuously differential with respect to $t\in T$, except possibly at the point $x=0$. $\mathscr{C}([-\tau,0],\mathscr{R}^n)$ is a vector space of all continuous $\mathscr{R}^n$-valued functions defined on $[-\tau,0]$. $\text{sym}(M)$: $M+M'$.

2. Definitions and Preliminaries

Consider the following delayed nonlinear stochastic system with multiplicative noise:

$$
\begin{aligned}
dx(t) = {} & [f(x(t),x(t-\tau),t) + g(x(t),x(t-\tau),t)u(t) \\
& + h(x(t),x(t-\tau),t)v(t)]\,dt \\
& + [l(x(t),x(t-\tau),t) + q(x(t),x(t-\tau),t)u(t) \\
& + s(x(t),x(t-\tau),t)v(t)]\,dw(t), \\
z(t) = {} & \operatorname{col}(m(x(t),x(t-\tau),t),u(t)) \\
:= {} & \begin{bmatrix} m(x(t),x(t-\tau),t) \\ u(t) \end{bmatrix}, \quad t\geq 0, \\
x(t) = {} & \phi(t)\in\mathscr{C}^b_{\mathscr{F}_0}([-\tau,0];\mathscr{R}^n), \quad -\tau\leq t\leq 0,
\end{aligned}
\tag{1}
$$

where $x(t)\in\mathscr{R}^n$, $u(t)\in\mathscr{R}^{n_u}$, $v(t)\in\mathscr{R}^{n_v}$, and $z(t)\in\mathscr{R}^{n_z}$ represent the system state, control input, exogenous disturbance, and regulated output, respectively. $w(t)$ is the one-dimensional standard Wiener process defined on a complete filtered space $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in\mathscr{R}^+},P)$, a filtration $\{\mathscr{F}_t\}_{t\in\mathscr{R}^+}$ satisfying usual conditions. $\mathscr{C}^b_{\mathscr{F}_0}([-\tau,0];\mathscr{R}^n)$ denotes all $\mathscr{F}_0$-measurable bounded $\mathscr{C}([-\tau,0];\mathscr{R}^n)$-valued random variable $\xi=\{\xi(\theta): -\tau\leq\theta\leq 0\}$. Assume that f, g, h, l, q, s, and m satisfy the local Lipschitz condition and the linear growth condition, which guarantee system (1) has a unique strong solution [23]. Moreover, suppose that $f(0,0,t)=l(0,0,t)=m(0,0,t)\equiv 0$; hence $x\equiv 0$ is an equilibrium point of (1). For simplicity, we denote $x:=x(t)$ and $x_\tau = x(t-\tau)$.

For each $V\in\mathscr{C}^{2,1}(\mathscr{R}^n\times\mathscr{R}^+;\mathscr{R}^+)$, an operator $\mathscr{L}V:\mathscr{R}^n\times\mathscr{R}^n\times\mathscr{R}^+\to\mathscr{R}$ associated with (1) is defined as follows [8]:

$$
\begin{aligned}
\mathscr{L}V(x,y,t) = {} & V_t(x,t) + V_x'(x,t) \\
& \times [f(x,y,t) + g(x,y,t)u(t) + h(x,y,t)v(t)] \\
& + \frac{1}{2}[l(x,y,t) + q(x,y,t)u(t) \\
& + s(x,y,t)v(t)]' V_{xx}(x,t) \\
& \times [l(x,y,t) + q(x,y,t)u(t) + s(x,y,t)v(t)],
\end{aligned}
\tag{2}
$$

where $V_t(x,t) = \partial V(x,t)/\partial t$, $V_x(x,t) = (\partial V(x,t)/\partial x_1,\ldots,\partial V(x,t)/\partial x_n)'$, and $V_{xx}(x,t) = (\partial^2 V(x,t)/\partial x_i\partial x_j)_{n\times n}$.

To deal with the infinite horizon H_∞ control of system (1), the following internal stability is needed.

Definition 1 (see [23]). The delayed nonlinear stochastic system,

$$
\begin{aligned}
dx(t) = {} & f(x(t),x(t-\tau),t)\,dt \\
& + l(x(t),x(t-\tau),t)\,dw(t), \quad t\geq 0, \\
x(t) = {} & \phi(t)\in\mathscr{C}^b_{\mathscr{F}_0}([-\tau,0];\mathscr{R}^n), \quad -\tau\leq t\leq 0,
\end{aligned}
\tag{3}
$$

is exponentially mean square stable, if there exist positive constants $\rho>0$ and $\varrho>0$ such that every solution $x(t)$ of (3) satisfies

$$
E\|x(t)\|^2 \leq \rho\|\phi\|^2\exp(-\varrho t), \tag{4}
$$

where $\|\phi\|^2 = E\max_{-\tau\leq t\leq 0}\|\phi(t)\|^2$.

Lemma 2 (see [24]). *System (3) is exponentially mean square stable, if there exist a positive definite Lyapunov function $V(x,t)\in\mathscr{C}^{2,1}(\mathscr{R}^n,\mathscr{R}^+;\mathscr{R}^+)$ and $c_1,c_2,c_3,c_4>0$ with $c_1c_3>c_2c_4$ such that*

(i) $c_1\|x\|^2\leq V(x,t)\leq c_2\|x\|^2$, $\forall(x,t)\in\mathscr{R}^n\times[-\tau,\infty)$,

(ii) $\mathscr{L}V(x,y,t)|_{v=0}\leq -c_3\|x\|^2 + c_4\|y\|^2$, $\forall t>0$.

Definition 3. For given $\gamma>0$, $u(t)=u^*(t)\in L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^{n_u})$ is said to be an exponential mean square H_∞ control of system (1), if

(i) for any nonzero $v(t)\in L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^{n_v})$ and $x(t)\equiv 0$, $t\in[-\tau,0]$, one always has

$$
\|z(t)\|_{L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^{n_z})} \leq \gamma\|v(t)\|_{L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^{n_v})}; \tag{5}
$$

(ii) system (1) with $v(t) = 0$ and $u(t) = u^*(t)$ is internally stable; that is, the system

$$\begin{aligned} dx(t) &= \left[f(x(t), x(t-\tau), t) + g(x(t), x(t-\tau), t)u^*(t)\right]dt \\ &\quad + \left[l(x(t), x(t-\tau), t) + q(x(t), x(t-\tau), t)u^*(t)\right]dw(t) \end{aligned} \tag{6}$$

is exponentially mean square stable.

Equation (5) is equivalent to $\|\mathscr{L}_{zv}\|_\infty \le \gamma$, where the perturbation operator $\mathscr{L}_{zv}$ is defined by $\mathscr{L}_{zv} : L^2_{\mathscr{F}}(\mathscr{R}^+; \mathscr{R}^{n_v}) \mapsto L^2_{\mathscr{F}}(\mathscr{R}^+; \mathscr{R}^{n_z})$ as

$$\begin{aligned} \mathscr{L}_{zv}(v) &= z\left(x\left(t, u^*, v, x_\tau, t\right)\right), \quad t \ge 0, \ v \in L^2_{\mathscr{F}}\left(\mathscr{R}^+; \mathscr{R}^{n_v}\right), \\ \|\mathscr{L}_{zv}\|_\infty &= \sup_{v \in L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^{n_v}), v \ne 0, x(0)=0} \frac{\|z\|_{L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^{n_z})}}{\|v\|_{L^2_{\mathscr{F}}(\mathscr{R}^+;\mathscr{R}^{n_v})}}. \end{aligned} \tag{7}$$

Definition 4. In (ii) of Definition 3, if the equilibrium point of system (6) is asymptotically mean square stable, that is,

$$\lim_{t \to \infty} E\|x(t)\|^2 = 0, \tag{8}$$

and (5) holds, then $u(t) = u^*(t)$ is called an asymptotic mean square H_∞ control.

Lemma 5 (see [25]). *For a positive definite symmetric matrix $P > 0 \in \mathscr{R}^{n\times n}$ and any matrices (or vectors) $N_1 \in \mathscr{R}^{n\times m}$ and $N_2 \in \mathscr{R}^{n\times m}$, one has*

$$N_1' P N_2 + N_2' P N_1 \le N_1' P N_1 + N_2' P N_2. \tag{9}$$

Lemma 6 (see [21]). *For any vectors $x, b \in \mathscr{R}^n$ and symmetric matrix $A \in \mathscr{R}^{n\times n}$, A^{-1} exists, and one has*

$$x'Ax + x'b + b'x = \left(x + A^{-1}b\right)' A \left(x + A^{-1}b\right) - b'A^{-1}b. \tag{10}$$

3. Infinite Horizon Stochastic H_∞ Control

In this section, several sufficient conditions are presented for the infinite horizon H_∞ control of system (1) by using inequality technique.

Theorem 7. *Assume that there exist a positive function $V(x,t) \in \mathscr{C}^{2,1}(\mathscr{R}^n \times \mathscr{R}^+; \mathscr{R}^+)$ and $c_1, c_2, c_3, c_4 > 0$ with $c_1c_3 > c_2c_4$ such that*

(i) $c_1\|x\|^2 \le V(x,t) \le c_2\|x\|^2$, $\forall (x,t) \in \mathscr{R}^n \times [-\tau, \infty)$,

(ii) $-\|m(x,y,t)\|^2 \le -c_3\|x\|^2 + c_4\|y\|^2$, $\forall t > 0$.

For given $\gamma > 0$, if $V(x,t)$ solves the Hamilton-Jacobi inequalities (HJIs)

$$\begin{aligned} & V_t(x,t) + V_x'(x,t) f(x,y,t) \\ &\quad + \frac{1}{2} l'(x,y,t) V_{xx}(x,t) l(x,y,t) + m'(x,y,t) m(x,y,t) \\ &\quad + \frac{1}{4}\left[l'(x,y,t) V_{xx}(x,t) s(x,y,t) + V_x'(x,t) h(x,y,t)\right] \\ &\quad \times \left[\gamma^2 I - s'(x,y,t) V_{xx}(x,t) s(x,y,t)\right]^{-1} \\ &\quad \cdot \left[s'(x,y,t) V_{xx}(x,t) l(x,y,t) + h'(x,y,t) V_x(x,t)\right] \\ &\quad - \frac{1}{4}\left[l'(x,y,t) V_{xx}(x,t) q(x,y,t) + V_x'(x,t) g(x,y,t)\right] \\ &\quad \times \left[I + q'(x,y,t) V_{x,x}(x,t) q(x,y,t)\right]^{-1} \\ &\quad \cdot \left[q'(x,y,t) V_{x,x}(x,t) l(x,y,t) + g'(x,y,t) V_x(x,t)\right] \\ & < 0, \end{aligned} \tag{11}$$

$$\gamma^2 I - s'(x,y,t) V_{x,x}(x,t) s(x,y,t) > 0, \tag{12}$$

then

$$\begin{aligned} u^* = -\frac{1}{2}\left[I + q'(x,y,t) V_{x,x}(x,t) q(x,y,t)\right]^{-1} & \\ \times \left[q'(x,y,t) V_{x,x}(x,t) l(x,y,t)\right. & \\ \left. + g'(x,y,t) V_x(x,t)\right] & \end{aligned} \tag{13}$$

is an exponential mean square H_∞ control of (1).

Proof. Applying Itô's formula to $V(x,t)$, we have

$$\begin{aligned} V(x(T), T) &= V(x(0), 0) + \int_0^T \mathscr{L}V(x, x_\tau, t)\, dt \\ &\quad + \int_0^T V_x(x,t) \\ &\quad \times \left[l(x, x_\tau, t) + q(x, x_\tau, t) u \right. \\ &\quad \left. + s(x, x_\tau, t) v\right] dw(t). \end{aligned} \tag{14}$$

Taking mathematical expectation on both sides of (14), we obtain

$$\begin{aligned}
&E\left[V\left(x\left(T\right),T\right)-V\left(x\left(0\right),0\right)\right]\\
&=E\int_0^T\Big\{V_t\left(x,t\right)+V_x'\left(x,t\right)\\
&\qquad\times\left[f\left(x,x_\tau,t\right)+g\left(x,x_\tau,t\right)u+h\left(x,x_\tau,t\right)v\right]\\
&\qquad+\frac{1}{2}\left[l\left(x,x_\tau,t\right)+q\left(x,x_\tau,t\right)u+s\left(x,x_\tau,t\right)v\right]'\\
&\qquad\times V_{xx}\left(x,t\right)\left[l\left(x,x_\tau,t\right)\right.\\
&\qquad\qquad\left.+q\left(x,x_\tau,t\right)u+s\left(x,x_\tau,t\right)v\right]\\
&\qquad+\left\|m(x,x_\tau,t)\right\|^2+\|u\|^2\\
&\qquad-\|z\|^2-\gamma^2\|v\|^2+\gamma^2\|v\|^2\Big\}\,dt\\
&=E\int_0^T\Big\{\Omega_1\left(v,x,x_\tau,t\right)+\Omega_2\left(x,x_\tau,t\right)\\
&\qquad+\Omega_3\left(u,x,x_\tau,t\right)-\|z\|^2+\gamma^2\|v\|^2\\
&\qquad+\frac{1}{2}\mathrm{sym}\left[u'q'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)v\right]\Big\}\,dt,
\end{aligned}\tag{15}$$

where

$$\begin{aligned}
&\Omega_1\left(v,x,x_\tau,t\right)\\
&\quad=v'\left[-\gamma^2I+\frac{1}{2}s'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)\right]v\\
&\qquad+\frac{1}{2}\mathrm{sym}\left[\left(l'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)\right.\right.\\
&\qquad\qquad\left.\left.+V_x'\left(x,t\right)h\left(x,x_\tau,t\right)\right)v\right],\\
&\Omega_2\left(x,x_\tau,t\right)\\
&\quad=V_t\left(x,t\right)+V_x'\left(x,t\right)f\left(x,x_\tau,t\right)\\
&\qquad+\frac{1}{2}l'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)l\left(x,x_\tau,t\right)\\
&\qquad+m'\left(x,x_\tau,t\right)m\left(x,x_\tau,t\right),\\
&\Omega_3\left(u,x,x_\tau,t\right)\\
&\quad=u'\left[I+\frac{1}{2}q'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right)\right]u\\
&\qquad+\frac{1}{2}\mathrm{sym}\left[\left(l'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right)\right.\right.\\
&\qquad\qquad\left.\left.+V_x'\left(x,t\right)g\left(x,x_\tau,t\right)\right)u\right].
\end{aligned}\tag{16}$$

Considering $V_{xx}(x,t)>0$ and Lemma 5, we have

$$\begin{aligned}
&\frac{1}{2}\mathrm{sym}\left[u'q'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)v\right]\\
&\quad\le\frac{1}{2}u'q'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right)u\\
&\qquad+\frac{1}{2}v's'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)v.
\end{aligned}\tag{17}$$

Therefore,

$$\begin{aligned}
&E\left[V\left(x\left(T\right),T\right)-V\left(x\left(0\right),0\right)\right]\\
&\le E\int_0^T\Big\{\Omega_1\left(v,x,x_\tau,t\right)+\Omega_2\left(x,x_\tau,t\right)\\
&\qquad+\Omega_3\left(u,x,x_\tau,t\right)-\|z\|^2+\gamma^2\|v\|^2\\
&\qquad+\frac{1}{2}\left[u'q'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right)u\right.\\
&\qquad\qquad\left.+v's'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)v\right]\Big\}\,dt\\
&=E\int_0^T\left(\widetilde{\Omega}_1\left(v,x,x_\tau,t\right)+\Omega_2\left(x,x_\tau,t\right)\right.\\
&\qquad\left.+\widetilde{\Omega}_3\left(u,x,x_\tau,t\right)-\|z\|^2+\gamma^2\|v\|^2\right)dt,
\end{aligned}\tag{18}$$

where

$$\begin{aligned}
&\widetilde{\Omega}_1\left(v,x,x_\tau,t\right)\\
&\quad=v'\left[-\gamma^2I+s'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)\right]v\\
&\qquad+\frac{1}{2}\mathrm{sym}\left[\left(l'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)\right.\right.\\
&\qquad\qquad\left.\left.+V_x'\left(x,t\right)h\left(x,x_\tau,t\right)\right)v\right],\\
&\widetilde{\Omega}_3\left(u,x,x_\tau,t\right)\\
&\quad=u'\left[I+q'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right)\right]u\\
&\qquad+\frac{1}{2}\mathrm{sym}\left[\left(l'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right)\right.\right.\\
&\qquad\qquad\left.\left.+V_x'\left(x,t\right)g\left(x,x_\tau,t\right)\right)u\right].
\end{aligned}\tag{19}$$

Set

$$\begin{aligned}
\mathbb{A}_1&=-\gamma^2I+s'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right),\\
\mathbb{b}_1'&=\frac{1}{2}\left(l'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)s\left(x,x_\tau,t\right)\right.\\
&\qquad\left.+V_x'\left(x,t\right)h\left(x,x_\tau,t\right)\right),\\
\mathbb{A}_3&=I+q'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right),\\
\mathbb{b}_3'&=\frac{1}{2}\left(l'\left(x,x_\tau,t\right)V_{xx}\left(x,t\right)q\left(x,x_\tau,t\right)\right.\\
&\qquad\left.+V_x'\left(x,t\right)g\left(x,x_\tau,t\right)\right).
\end{aligned}\tag{20}$$

According to Lemma 6, $\widetilde{\Omega}_1(v, x, x_\tau, t)$ and $\widetilde{\Omega}_3(u, x, x_\tau, t)$ can be rewritten as

$$\begin{aligned}\widetilde{\Omega}_1(v, x, x_\tau, t) &= \left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right)' \mathbb{A}_1 \left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right) - \mathbb{b}_1'\mathbb{A}_1\mathbb{b}_1, \\ \widetilde{\Omega}_3(u, x, x_\tau, t) &= \left(u + \mathbb{A}_3^{-1}\mathbb{b}_3\right)' \mathbb{A}_3 \left(u + \mathbb{A}_3^{-1}\mathbb{b}_3\right) - \mathbb{b}_3'\mathbb{A}_3\mathbb{b}_3.\end{aligned} \tag{21}$$

Implementing (21) and $\Omega_2(x, x_\tau, t)$ into (18) yields

$$\begin{aligned}E&[V(x(T), T) - V(x(0), 0)] \\ &= E\int_0^T \Big[\gamma^2\|v\|^2 - \|z\|^2 + \left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right)' \mathbb{A}_1 \left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right) \\ &\qquad + \left(u + \mathbb{A}_3^{-1}\mathbb{b}_3\right)' \mathbb{A}_3 \left(u + \mathbb{A}_3^{-1}\mathbb{b}_3\right) \\ &\qquad + V_t(x, t) + V_x'(x, t) f(x, x_\tau, t) \\ &\qquad + \frac{1}{2} l'(x, x_\tau, t) V_{xx}(x, t) l(x, x_\tau, t) \\ &\qquad + m'(x, x_\tau, t) m(x, x_\tau, t) \\ &\qquad + \mathbb{b}_1'(-\mathbb{A}_1)\mathbb{b}_1 - \mathbb{b}_3'\mathbb{A}_3\mathbb{b}_3\Big] dt.\end{aligned} \tag{22}$$

According to (11), we have

$$\begin{aligned}E&[V(x(T), T) - V(x(0), 0)] \\ &\leq E\int_0^T \Big[\gamma^2\|v\|^2 - \|z\|^2 + \left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right)' \mathbb{A}_1 \left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right) \\ &\qquad + \left(u + \mathbb{A}_3^{-1}\mathbb{b}_3\right)' \mathbb{A}_3 \left(u + \mathbb{A}_3^{-1}\mathbb{b}_3\right)\Big] dt.\end{aligned} \tag{23}$$

Considering (12) and taking $u = u^* = -\mathbb{A}_3^{-1}\mathbb{b}_3$, (23) leads to

$$\begin{aligned}E&\left(\int_0^T \|z\|^2 dt\right) \\ &\leq -E[V(x(T), T)] + \gamma^2 E\left(\int_0^T \|v\|^2 dt\right) \\ &\quad - E\left[\int_0^T \left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right)'(-\mathbb{A}_1)\left(v + \mathbb{A}_1^{-1}\mathbb{b}_1\right) dt\right] \\ &\leq \gamma^2 E\left(\int_0^T \|v\|^2 dt\right).\end{aligned} \tag{24}$$

Let $T \to \infty$, and then (5) of Definition 3 is proved.

Next, we will prove system (6) to be exponentially mean square stable. Let $\mathscr{L}_{u^*}$ be the infinitesimal generator of the system (6), and then

$$\begin{aligned}\mathscr{L}_{u^*}V(x, y, t) &= V_t(x, t) \\ &\quad + V_x'(x, t)[f(x, y, t) + g(x, y, t)u^*] \\ &\quad + \frac{1}{2}[l(x, y, t) + q(x, y, t)u^*]' \\ &\quad \times V_{xx}(x, t)[l(x, y, t) + q(x, y, t)u^*] \\ &= V_t(x, t) + V_x'(x, t) f(x, y, t) \\ &\quad + \frac{1}{2} l'(x, y, t) V_{xx}(x, t) l(x, y, t) \\ &\quad + V_x'(x, t) g(x, y, t) u^* \\ &\quad + \frac{1}{2}\text{sym}\left[l'(x, y, t) V_{xx}(x, t) q(x, y, t) u^*\right] \\ &\quad + \frac{1}{2} u^{*\prime} q(x, y, t)' V_{xx}(x, t) q(x, y, t) u^*.\end{aligned} \tag{25}$$

Setting

$$\begin{aligned}\Sigma_1 &= V_x'(x, t) g(x, y, t) u^* \\ &\quad + \frac{1}{2}\text{sym}\left[l'(x, y, t) V_{xx}(x, t) q(x, y, t) u^*\right],\end{aligned} \tag{26}$$

$$\Sigma_2 = \frac{1}{2} u^{*\prime} q(x, y, t)' V_{xx}(x, t) q(x, y, t) u^*$$

and implementing

$$\begin{aligned}u^* &= -\frac{1}{2}\left[I + q'(x, y, t) V_{x,x}(x, t) q(x, y, t)\right]^{-1} \\ &\quad \times \left[q'(x, y, t) V_{x,x}(x, t) l(x, y, t) \right. \\ &\qquad \left. + g'(x, y, t) V_x(x, t)\right]\end{aligned} \tag{27}$$

into Σ_1 and Σ_2, it yields

$$\begin{aligned}\Sigma_1 &= -\frac{1}{2} V_x'(x, t) g(x, y, t) \\ &\quad \times \left[I + q'(x, y, t) V_{x,x}(x, t) q(x, y, t)\right]^{-1} \\ &\quad \cdot \left[q'(x, y, t) V_{x,x}(x, t) l(x, y, t) \right. \\ &\qquad \left. + g'(x, y, t) V_x(x, t)\right] \\ &\quad - \frac{1}{4}\text{sym}\Big\{\left[l'(x, y, t) V_{xx}(x, t) q(x, y, t)\right] \\ &\qquad \times \left[I + q'(x, y, t) V_{x,x}(x, t) q(x, y, t)\right]^{-1} \\ &\qquad \cdot \left[q'(x, y, t) V_{x,x}(x, t) l(x, y, t) \right. \\ &\qquad\quad \left. + g'(x, y, t) V_x(x, t)\right]\Big\},\end{aligned}$$

$$= -\frac{1}{2}\left[l'(x,y,t)V_{xx}(x,t)q(x,y,t) + V_x'(x,t)g(x,y,t)\right] \times \left[I + q'(x,y,t)V_{x,x}(x,t)q(x,y,t)\right]^{-1} \cdot \left[q'(x,y,t)V_{x,x}(x,t)l(x,y,t) + g'(x,y,t)V_x(x,t)\right],$$

$$\Sigma_2 = \frac{1}{8}\left[l'(x,y,t)V_{xx}(x,t)q(x,y,t) + V_x'(x,t)g(x,y,t)\right] \times \left[I + q'(x,y,t)V_{x,x}(x,t)q(x,y,t)\right]^{-1} \cdot q'(x,y,t)V_{xx}(x,t)q(x,y,t) \times \left[I + q'(x,y,t)V_{x,x}(x,t)q(x,y,t)\right]^{-1} \cdot \left[q'(x,y,t)V_{x,x}(x,t)l(x,y,t) + g'(x,y,t)V_x(x,t)\right]$$
$$\leq \frac{1}{8}\left[l'(x,y,t)V_{xx}(x,t)q(x,y,t) + V_x'(x,t)g(x,y,t)\right] \times \left[I + q'(x,y,t)V_{x,x}(x,t)q(x,y,t)\right]^{-1} \cdot \left[q'(x,y,t)V_{x,x}(x,t)l(x,y,t) + g'(x,y,t)V_x(x,t)\right]. \quad (28)$$

Substituting (28) into (25) and considering conditions (i), (ii), and (11) in Theorem 7, it follows that

$$\begin{aligned}\mathscr{L}_{u^*}V(x,y,t) &\leq V_t(x,t) + V_x'(x,t)f(x,y,t) + \frac{1}{2}l'(x,y,t)V_{xx}(x,t)l(x,y,t) \\ &\quad - \frac{3}{8}\left[l'(x,y,t)V_{xx}(x,t)q(x,y,t) + V_x'(x,t)g(x,y,t)\right] \\ &\quad \times \left[I + q'(x,y,t)V_{x,x}(x,t)q(x,y,t)\right]^{-1} \cdot \left[q'(x,y,t)V_{x,x}(x,t)l(x,y,t) + g'(x,y,t)V_x(x,t)\right] \\ &\leq V_t(x,t) + V_x'(x,t)f(x,y,t) + \frac{1}{2}l'(x,y,t)V_{xx}(x,t)l(x,y,t) \\ &\quad - \frac{1}{4}\left[l'(x,y,t)V_{xx}(x,t)q(x,y,t) + V_x'(x,t)g(x,y,t)\right] \\ &\quad \times \left[I + q'(x,y,t)V_{x,x}(x,t)q(x,y,t)\right]^{-1} \cdot \left[q'(x,y,t)V_{x,x}(x,t)l(x,y,t) + g'(x,y,t)V_x(x,t)\right] \\ &< -\frac{1}{4}\left[l'(x,y,t)V_{xx}(x,t)s(x,y,t) + V_x'(x,t)h(x,y,t)\right] \\ &\quad \times \left[\gamma^2 I - s'(x,y,t)V_{xx}(x,t)s(x,y,t)\right]^{-1} \cdot \left[s'(x,y,t)V_{xx}(x,t)l(x,y,t) + h'(x,y,t)V_x(x,t)\right] \\ &\quad - m'(x,y,t)m(x,y,t) \\ &\leq -\|m(x,y,t)\|^2 \\ &\leq -c_3\|x\|^2 + c_4\|y\|^2.\end{aligned} \quad (29)$$

From Lemma 2, system (6) is exponentially mean square stable. This theorem is proved. □

The following theorem is derived for the asymptotic mean square H_∞ control, which is weaker than the exponential mean square H_∞ control.

Theorem 8. *Assume that* $V(x,t) \in \mathscr{C}^{2,1}(\mathscr{R}^n \times \mathscr{R}^+; \mathscr{R}^+)$ *has an infinitesimal upper limit; that is,* $\lim_{\|x\|\to\infty}\inf_{t>0}V(x,t) = \infty$ *and* $V(x,t) > c\|x\|^2$ *for some* $c > 0$. *If* $V(x,t)$ *solves HJIs (11)-(12), then (13) is an asymptotic mean square* H_∞ *control of (1).*

Proof. It only needs to prove that system (6) is asymptotically mean square stable when $v = 0$. We know that $\mathscr{L}_{u^*}V(x,y,t) < 0$ from (29), which implies that system (6) is globally asymptotically stable in probability 1 [26]. According to Itô formula and the property of stochastic integration, we obtain

$$\begin{aligned}EV(x(t),t) &= EV(x(0),0) + E\int_0^t \mathscr{L}_{u^*}V(x(s),s)\big|_{v=0}\,ds \\ &\quad + E\int_0^t V_x(x(s),s) \times \left[l(x,x_\tau,t) + q(x,x_\tau,t)u^*\right]\big|_{v=0}\,dw(s)\end{aligned}$$

$$= EV(x(0),0) + E\int_0^t \mathscr{L}_{u^*}V(x(s),s)\big|_{v=0}\,ds$$
$$\leq EV(x(0),0) - E\int_0^t \|m(x(s),x(s-\tau),s)\|^2\,ds$$
$$\leq EV(x(0),0) < \infty. \quad (30)$$

Let $\widetilde{\mathscr{F}}_t = \mathscr{F}_t \cup \sigma(y(s), 0 \leq s \leq t)$, and then (30) leads to

$$E\left[V(x(t),t) \mid \widetilde{\mathscr{F}}_s\right] \leq V(x(s),s) \quad \text{a.s.}, \quad (31)$$

which means that $\{V(x(t),t), \widetilde{\mathscr{F}}_t, 0 \leq s \leq t\}$ is a nonnegative supermartingale with respect to $\{\widetilde{\mathscr{F}}_t\}_{t\geq 0}$. According to Doob's convergence theorem [27] and $\lim_{t\to\infty}x(t) = 0$ a.s., we have $V(x(\infty),\infty) = \lim_{t\to\infty}V(x(t),t) = 0$ a.s. Furthermore, $\lim_{t\to\infty}EV(x(t),t) = EV(x(\infty),\infty) = EV(0,\infty) = 0$. Since $V(x,t) > c\|x\|^2$ for some $c > 0$, it yields that $\lim_{t\to\infty}E\|x(t)\|^2 = 0$. The proof is completed. □

Remark 9. In [24], Zhang et al. studied the robust H_∞ filtering problem of nonlinear stochastic systems with time delay. However, the H_∞ control problem was not tackled in [24], mainly due to mathematical difficulties in dealing with the case that state, control, and disturbance enter into the diffusion term simultaneously. In this paper, Lemma 6 is applied to solve this problem, and two sufficient conditions for H_∞ control of delayed nonlinear stochastic systems are obtained in Theorems 7 and 8.

Remark 10. A further development of the present issue is twofold. On the one hand, in order to avoid solving HJIs (11) and (12), the global linearization approach [25] or fuzzy approach based on Takagi-Sugeno model [28] can be used to design H_∞ control for delayed nonlinear stochastic systems. On the other hand, Lévy noise is more versatile and interesting with a wider range of applications in comparison to the standard Gaussian noise [29, 30]. Therefore, the H_∞ control of stochastic differential equations with Lévy noise is another valuable research topic.

4. Numerical Examples

In this section, two numerical examples are given to illustrate the proposed H_∞ control design.

Example 1. Consider the following one-dimensional nonlinear stochastic state-delayed system:

$$\begin{aligned} dx(t) = {} & \left[-2x(t) + x(t)x^2(t-\tau) + 4x(t-\tau)u(t) + x(t-\tau)v(t)\right]dt \\ & + \left[x(t)x(t-\tau) + u(t) + v(t)\right]dw(t), \\ z(t) = {} & \begin{bmatrix} 2x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \\ x(t) = {} & \phi(t) \in \mathscr{C}^b_{\mathscr{F}_0}([-\tau,0];\mathscr{R}^n), \quad -\tau \leq t \leq 0. \end{aligned} \quad (32)$$

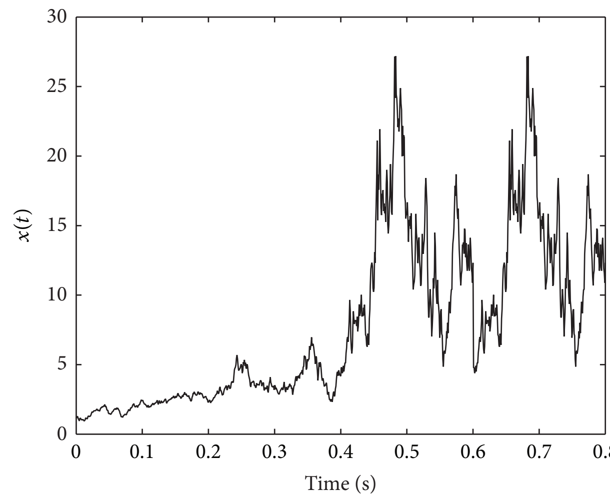

Figure 1: The state $x(t)$ of the unforced system in Example 1.

Set $V(x) = px^2$, $p > 0$ to be determined, and then HJIs (11)-(12) become

$$\begin{aligned} & 2px \cdot \left(-2x + xx_\tau^2\right) \\ & + \frac{1}{2}xx_\tau \cdot 2p \cdot xx_\tau + 2x \cdot 2x \\ & + \frac{1}{4}\left(xx_\tau \cdot 2p + 2px \cdot x_\tau\right)^2\left(\gamma^2 - 2p\right)^{-1} \\ & - \frac{1}{4}\left(xx_\tau \cdot 2p + 2px \cdot 4x_\tau\right)^2\left(1 + 2p\right)^{-1} < 0, \\ & \gamma^2 - 2p > 0. \end{aligned} \quad (33)$$

Given $\gamma = \sqrt{3}$, the above inequalities have a solution $p = 1$. From Theorem 7, the H_∞ control of system (32) is $u^* = -(5/3)xx_\tau$.

The initial condition is chosen as $\phi(t) = 1.2$ for any $t \in [-\tau, 0]$ with $\tau = 0.2$ and $v(t) = e^{-t}$. Applying the Euler-Maruyama method [31], the state responses of the unforced system ($u = 0$) and the controlled system ($u = u^*$) and the control input are shown in Figures 1, 2, and 3, respectively. It is found that the controlled system can achieve stability and attenuation performance by using the proposed H_∞ control.

Example 2. Consider a two-dimensional system (1) with the following parameters:

$$\begin{aligned} & f(x) = \begin{bmatrix} x_2(t) \\ -x_2^3(t) - x_2(t) - x_1(t) \end{bmatrix}, \\ & g(x) = \begin{bmatrix} 0 \\ 2x_2(t-\tau) \end{bmatrix}, \qquad m(x) = \sqrt{2}x_2(t), \\ & h(x) = \begin{bmatrix} 0 \\ x_2(t-\tau) \end{bmatrix}, \qquad l(x) = \begin{bmatrix} 0 \\ x_2(t)x_2(t-\tau) \end{bmatrix}, \\ & q(x) = 0, \qquad s(x) = 0. \end{aligned} \quad (34)$$

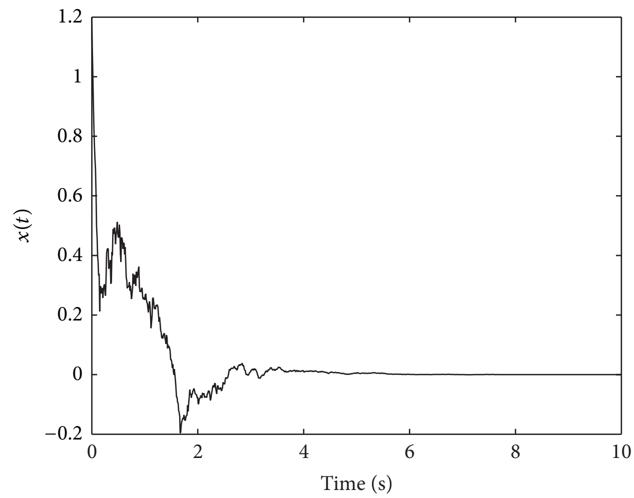

Figure 2: The state $x(t)$ of the controlled system in Example 1.

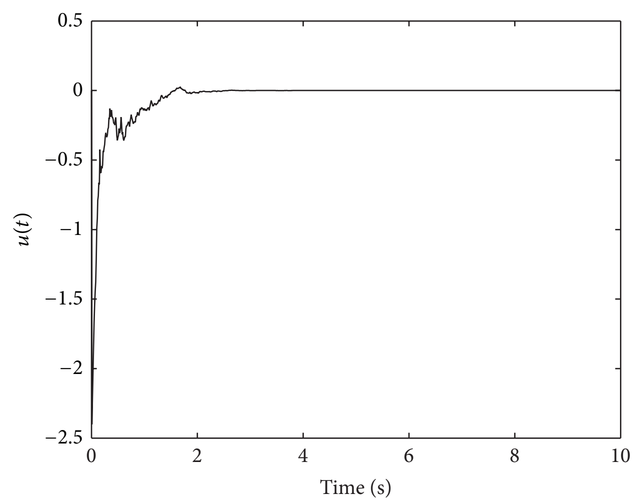

Figure 3: The control input $u(t)$ in Example 1.

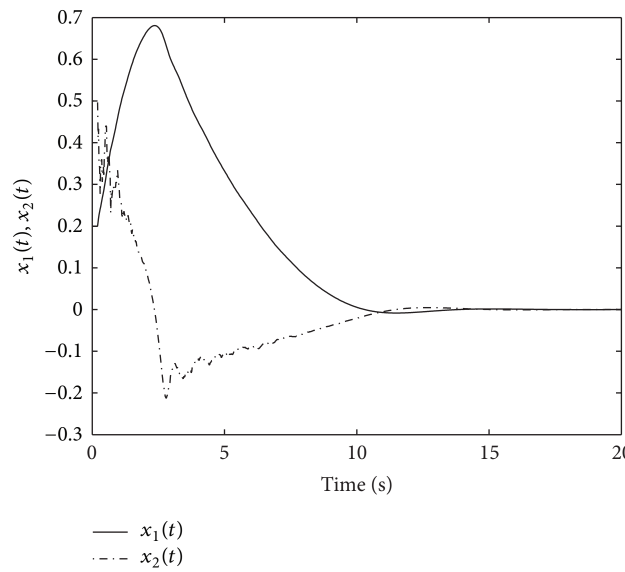

Figure 4: The states $x_1(t)$ and $x_2(t)$ of the controlled system in Example 2.

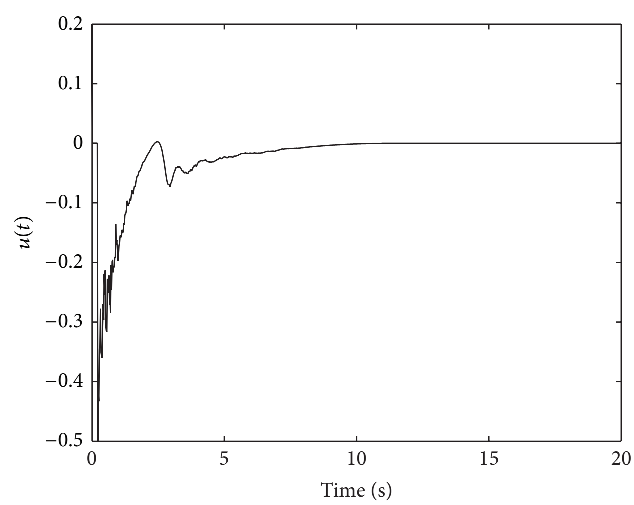

Figure 5: The control input $u(t)$ in Example 2.

Take $V(x) = x'Px$ with $P = \text{diag}\{p_1, p_2\} < 0$ to be determined. For a given disturbance attenuation level $\gamma = 1$, $P = \text{diag}\{1, 1\}$ is a solution to (11)-(12). According to Theorem 7, $\overline{u}_\infty = -2x_2x_{2\tau}$ is an H_∞ control of system (1). The initial condition is chosen as $\phi(t) = [0.2 \;\; 0.5]'$ for any $t \in [-0.2, 0]$, and take $v(t) = e^{-t}$. By using a similar method in Example 1, the states of the controlled system and the control input are shown in Figures 4-5, which show the effectiveness of the designed controller.

5. Conclusions

For general delayed nonlinear stochastic systems with state, control, and disturbance-dependent noise, this paper has presented a sufficient condition for exponential/asymptotic mean square H_∞ control problem in terms of HJIs. There still remain many interesting topics, for example, how to derive delay-dependent conditions or how to design H_2/H_∞ control for delayed nonlinear stochastic systems. These issues deserve further research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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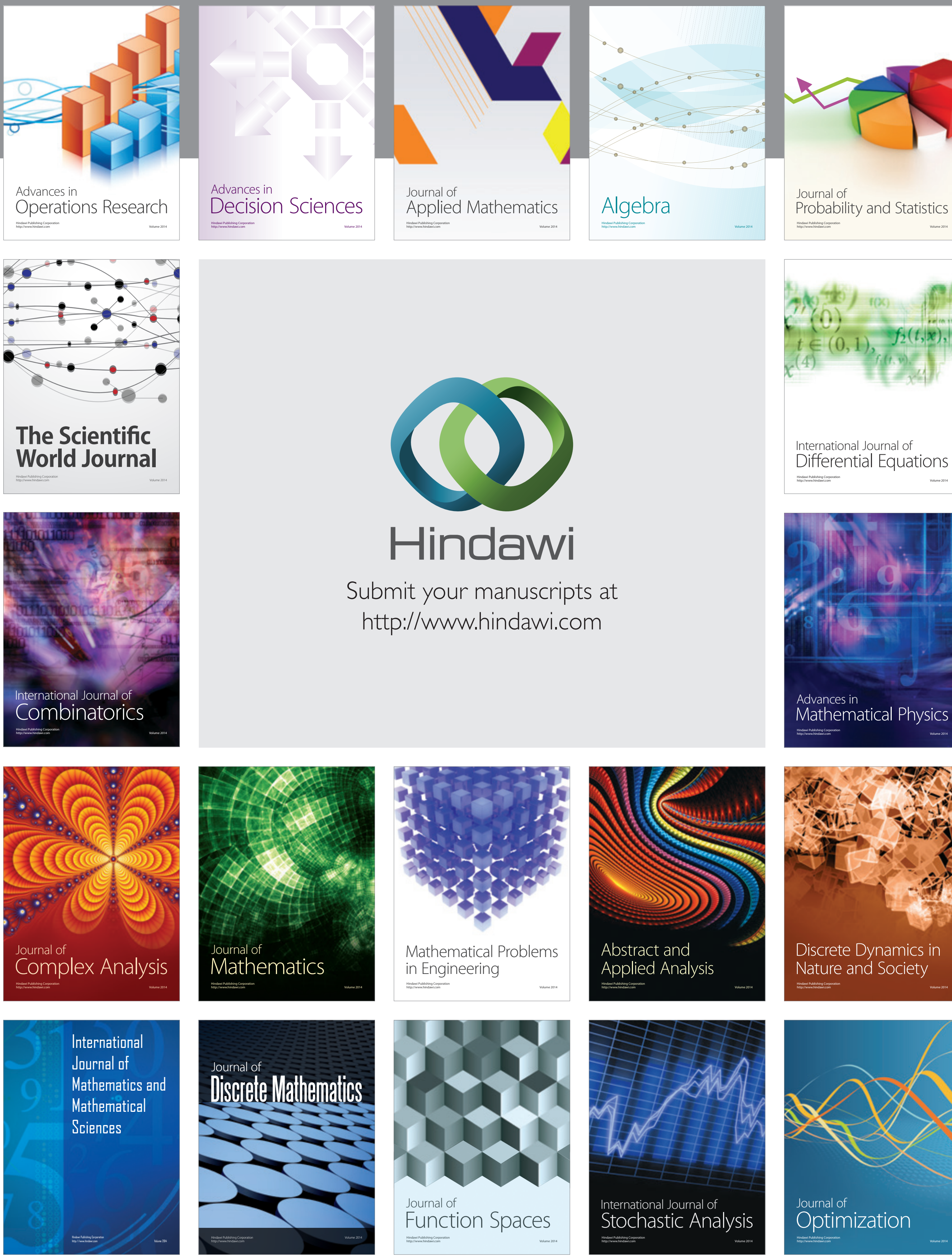
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