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Research Article

Existence of Almost Periodic Solutions for Impulsive Neutral Functional Differential Equations

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The existence of piecewise almost periodic solutions for impulsive neutral functional differential equations in Banach space is investigated. Our results are based on Krasnoselskii's fixed-point theorem combined with an exponentially stable strongly continuous operator semigroup. An example is given to illustrate the theory.

1. Introduction

In this paper, we study the existence of piecewise almost periodic solutions for a class of abstract impulsive neutral functional differential equations with unbounded delay modeled in the form

$$\begin{gathered}\frac{d}{dt}\left(u(t)+g(t,u_t)\right)=Au(t)+f(t,u_t),\\ t\in R,\quad t\neq t_i,\quad i\in Z,\\ \Delta u(t_i)=I_i\left(u_{t_i}\right),\quad i\in Z,\end{gathered} \tag{1}$$

where A is the infinitesimal generator of an exponentially stable strongly continuous semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on a Banach space $(X,\|\cdot\|)$, the history $u_t : (-\infty,0]\to X$, $u_t(\theta)=u(t+\theta)$ belongs to an abstract phase space $\mathfrak{B}$ defined axiomatically, $f(\cdot), g(\cdot), I_i(\cdot)$ $(i\in Z)$ are appropriate functions, $\{t_i\}_{i\in Z}$ is a discrete set of real numbers such that $t_i<t_j$ when $i<j$, and the symbol $\Delta\xi(t)$ represents the jump of the function $\xi(\cdot)$ at t, which is defined by $\Delta\xi(t)=\xi(t^+)-\xi(t^-)$.

The existence of solutions to impulsive differential equations is one of the most attracting topics in the qualitative theory of impulsive differential equations due to their applications in mechanical, electrical engineering, ecology, biology, and others; see, for instance, [1–6] and the references therein. Some recent contributions on mild solutions to impulsive neutral functional differential equations have been established in [7–14]. On the other hand, the existence of almost periodic solutions for impulsive differential equations has been investigated by many authors; see, for example, [2–5, 15, 16]. However, the existence of almost periodic solutions for the impulsive neutral functional differential equations in the form (1) is an untreated topic in the literature and this fact is the motivation of the present work.

The paper is organized as follows: in Section 2, we recall some notations, concepts, and useful lemmas which are used in this paper. In Section 3, some criteria ensuring the existence of almost periodic solutions for impulsive neutral functional differential equations are obtained. In Section 4, we give an application.

2. Preliminaries

Let $(X,\|\cdot\|)$ be a Banach space. $A : D(A)\to X$ is the infinitesimal generator of a strongly continuous semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on the Banach space X and M_1, δ are positive constants such that $\|T(t)\|\leq M_1e^{-\delta t}$ for $t\geq 0$. Let $0\in\rho(A)$; it is possible to define the fractional power A^α, $0<\alpha<1$, as a closed linear operator with its domain $D(A^\alpha)$. We denote by X_α a Banach space between $D(A)$ and X endowed with the norm $\|x\|_\alpha=\|A^\alpha x\|$, $x\in D(A^\alpha)$; the following properties hold.

Lemma 1 (see [17, 18]). *Let $0 < \alpha < \beta < 1$; then X_β is continuously embedded into X_α with bounded K; that is,*

$$\|x\|_\alpha \leq K\|x\|_\beta. \tag{2}$$

Moreover, the function $s \rightarrow A^\alpha T(s)$ is continuous in the uniform operator topology on $(0, \infty)$ and there exists $M_2 > 0$ such that $\|A^\alpha T(t)\| \leq M_2 e^{-\delta t} t^{-\alpha}$ for every $t > 0$.

Throughout this paper, let $\mathbb{T}$ be the set consisting of all real sequences $\{t_i\}_{i\in Z}$ such that $\gamma = \inf_{i\in Z}(t_{i+1} - t_i) > 0$. For $\{t_i\}_{i\in Z} \in \mathbb{T}$, let $PC(R, X_\alpha)$ be the space formed by all piecewise continuous functions $\phi : R \rightarrow X_\alpha$ such that $\phi(\cdot)$ is continuous at t for any $t \notin \{t_i\}_{i\in Z}$ and $\phi(t_i) = \phi(t_i^-)$ for all $i \in Z$; let $PC(R\times X_\alpha, X_\alpha)$ be the space formed by all piecewise continuous functions $\phi : R \times X_\alpha \rightarrow X_\alpha$ such that, for any $x \in X_\alpha$, $\phi(\cdot, x)$ is continuous at t for any $t \notin \{t_i\}_{i\in Z}$ and $\phi(t_i, x) = \phi(t_i^-, x)$ for all $i \in Z$ and for any $t \in R$, $\phi(t, \cdot)$ is continuous at $x \in X_\alpha$.

Definition 2. A number $\tau \in R$ is called an ϵ-translation number of the function $\phi \in PC(R, X_\alpha)$ if

$$\|\phi(t + \tau) - \phi(t)\|_\alpha < \epsilon, \tag{3}$$

for all $t \in R$ which satisfies $|t - t_i| > \epsilon$, for all $i \in Z$. Denote by $T(\phi, \epsilon)$ the set of all ϵ-translation numbers of ϕ.

Definition 3 (see [1]). A function $\phi \in PC(R, X_\alpha)$ is said to be piecewise almost periodic if the following conditions are fulfilled.

(1) $\{t_i^j = t_{i+j} - t_i\}$, $i \in Z$, $j = 0, \pm1, \pm2, \ldots$, are equipotentially almost periodic; that is, for any $\epsilon > 0$, there exists a relatively dense set of ϵ-almost periods that are common to all the sequences $\{t_i^j\}$.

(2) For any $\epsilon > 0$, there exists a positive number $\delta = \delta(\epsilon)$ such that if the points t' and t'' belong to a same interval of continuity of ϕ and $|t' - t''| < \delta$, then $\|\phi(t') - \phi(t'')\|_\alpha < \epsilon$.

(3) For every $\epsilon > 0$, $T(\phi, \epsilon)$ is a relatively dense set in R.

We denote by $AP_T(R, X_\alpha)$ the space of all piecewise almost periodic functions. $AP_T(R, X_\alpha)$ endowed with the uniform convergence topology is a Banach space.

Definition 4. $f \in PC(R \times X_\alpha, X_\alpha)$ is said to be piecewise almost periodic in t uniformly in $x \in X_\alpha$ if for each compact set $K \subseteq X_\alpha$, $\{f(\cdot, x) : x \in K\}$ is uniformly bounded and, given $\epsilon > 0$, there exists a relatively dense set $\Omega(\epsilon)$ such that

$$\|f(t + \tau, x) - f(t, x)\|_\alpha < \epsilon, \tag{4}$$

for all $x \in K$, $\tau \in \Omega(\epsilon)$, and $t \in R$, $|t - t_i| > \epsilon$ for all $i \in Z$. Denote by $AP_T(R \times X_\alpha, X_\alpha)$ the set of all such functions.

Lemma 5 (see [15]). *Let $\phi \in AP_T(R, X_\alpha)$; then the range of ϕ, $R(\phi)$, is a relatively compact subset of X_α.*

Lemma 6. *Suppose that $f(t, x) \in AP_T(R \times X_\alpha, X_\alpha)$ and $f(t, \cdot)$ is uniformly continuous on each compact subset $K \subseteq X_\alpha$ uniformly for $t \in R$; that is, for every $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in K$ and $\|x - y\|_\alpha < \delta$ implies that $\|f(t, x) - f(t, y)\|_\alpha < \epsilon$ for all $t \in R$. Then $f(\cdot, x(\cdot)) \in AP_T(R, X_\alpha)$ for any $x \in AP_T(R, X_\alpha)$.*

Proof. Since $x \in AP(R, X_\alpha)$, by Lemma 5, $R(x)$ is a relatively compact subset of X_α. Because $f(t, \cdot)$ is uniformly continuous on each compact subset $K \subseteq X_\alpha$ uniformly for $t \in R$, then for any $\epsilon > 0$, there exist a number $\delta : 0 < \delta \leq \epsilon/2$, such that

$$\|f(t, x_1) - f(t, x_2)\|_\alpha < \frac{\epsilon}{2}, \tag{5}$$

where $x_1, x_2 \in R(x)$, and $\|x_1 - x_2\|_\alpha < \delta$, $t \in R$. By piecewise almost periodic of f and x, there exists a relatively dense set Ω of R such that the following conditions hold:

$$\begin{gathered} \|f(t + \tau, x_0) - f(t, x_0)\|_\alpha < \frac{\epsilon}{2}, \\ \|x(t + \tau) - x(t)\|_\alpha < \frac{\epsilon}{2}, \end{gathered} \tag{6}$$

for $x_0 \in R(x)$ and $t \in R$, $|t - t_i| > \epsilon$, $i \in Z$, $\tau \in \Omega$. Note that

$$\begin{aligned} & f(t + \tau, x(t + \tau)) - f(t, x(t)) \\ & \quad = f(t + \tau, x(t + \tau)) - f(t, x(t + \tau)) \\ & \qquad + f(t, x(t + \tau)) - f(t, x(t)). \end{aligned} \tag{7}$$

We have

$$\begin{aligned} & \|f(t + \tau, x(t + \tau)) - f(t, x(t))\|_\alpha \\ & \quad \leq \|f(t + \tau, x(t + \tau)) - f(t, x(t + \tau))\|_\alpha \\ & \qquad + \|f(t, x(t + \tau)) - f(t, x(t))\|_\alpha. \end{aligned} \tag{8}$$

We deduce from (5) and (6) that the following formula holds:

$$\begin{gathered} \|f(t + \tau, x(t + \tau)) - f(t, x(t))\|_\alpha \leq \epsilon, \\ t \in R, \quad |t - t_i| > \epsilon, \quad i \in Z, \quad \tau \in \Omega. \end{gathered} \tag{9}$$

That is, $f(\cdot, x(\cdot))$ is piecewise almost periodic. The proof is complete. □

Since the uniform continuity is weaker than the Lipschitz continuity, we obtain the following lemma as an immediate consequence of the previous lemma.

Lemma 7. *Let $f(t, x) \in AP_T(R \times X_\alpha, X_\alpha)$ and f is Lipschitz; that is, there is a positive number L such that*

$$\|f(t, x) - f(t, y)\|_\alpha \leq L\|x - y\|_\alpha, \tag{10}$$

for all $t \in R$ and $x, y \in X_\alpha$; if for any $x \in AP_T(R, X_\alpha)$, then $f(\cdot, x(\cdot)) \in AP_T(R, X_\alpha)$.

In this paper, we assume that the phase space $\mathfrak{B}$ is a linear space formed by functions mapping $(-\infty, 0]$ into X_α endowed with a norm $\|\cdot\|_{\mathfrak{B}}$ and satisfying the following conditions.

(1) If $x \in PC(R, X_\alpha)$, then $x_t \in \mathfrak{B}$ for all $t \in R$ and $\|x_t\|_{\mathfrak{B}} \leq L \sup_{s\leq t}\|x(s)\|_\alpha$, where $L > 0$ is a constant independent of $x(\cdot)$ and $t \in R$.

(2) The space $\mathfrak{B}$ is complete.

(3) If $\{\phi^n\}_{n\in N} \subset \mathfrak{B}$ is a uniformly bounded sequence in $PC((-\infty, 0], X_\alpha)$ formed by functions with compact support and $\phi^n \to \phi$ in the compact open topology, then $\phi \in \mathfrak{B}$ and $\|\phi^n - \phi\|_{\mathfrak{B}} \to 0$, as $n \to \infty$.

Lemma 8 (see [1, 15]). *Assume that $f \in AP_T(R, X_\alpha)$, the sequence $\{x_i : i \in Z\}$ is almost periodic in X_α, and $\{t_i^j = t_{i+j} - t_i\}$, $i \in Z$, $j = 0, \pm1, \pm2, \ldots$, are equipotentially almost periodic. Then for each $\epsilon > 0$, there are relatively dense sets Ω_{ϵ,f,x_i} of R and Q_{ϵ,f,x_i} of Z such that the following conditions hold.*

(i) *$\|f(t+\tau) - f(t)\|_\alpha < \epsilon$ for all $t \in R$, $|t - t_i| > \epsilon$, $\tau \in \Omega_{\epsilon,f,x_i}$, and $i \in Z$.*

(ii) *$\|x_{i+q} - x_i\|_\alpha < \epsilon$ for all $q \in Q_{\epsilon,f,x_i}$ and $i \in Z$.*

(iii) *For every $\tau \in \Omega_{\epsilon,f,x_i}$, there exists at least one number $q \in Q_{\epsilon,f,x_i}$ such that*

$$\left|t_i^q - \tau\right| < \epsilon, \quad i \in Z. \tag{11}$$

Definition 9. A bounded function $u \in PC(R, X_\alpha)$ is a mild solution of (1) if the following holds: $u_\sigma = \phi \in \mathfrak{B}$, the function $AT(t-s)g(s, u_s)$ is integrable, and for any $t \in R$, $t_i < t \leq t_{i+1}$,

$$\begin{aligned} u(t) = {} & T(t-\sigma)[\phi(0) + g(\sigma,\phi)] - g(t, u_t) \\ & - \int_\sigma^t AT(t-s)\, g(s, u_s)\, ds \\ & + \int_\sigma^t T(t-s)\, f(s, u_s)\, ds \\ & + \sum_{\sigma<t_i<t} T(t-t_i)\, I_i(u_{t_i}). \end{aligned} \tag{12}$$

Since $\|T(t-\sigma)\| \leq M_1 e^{-\delta(t-\sigma)}$ for all $t \geq \sigma$, let $\sigma \to -\infty$; then we have $\|T(t-\sigma)\| \to 0$, and the above formula can be replaced by

$$\begin{aligned} u(t) = {} & -g(t, u_t) - \int_{-\infty}^t AT(t-s)\, g(s, u_s)\, ds \\ & + \int_{-\infty}^t T(t-s)\, f(s, u_s)\, ds \\ & + \sum_{t_i<t} T(t-t_i)\, I_i(u_{t_i}). \end{aligned} \tag{13}$$

In fact, for $t > \sigma$,

$$\begin{aligned} & -\int_\sigma^t AT(t-s)\, g(s,u_s)\, ds + \int_\sigma^t T(t-s)\, f(s,u_s)\, ds \\ & \quad + \sum_{\sigma<t_i<t} T(t-t_i)\, I_i(u_{t_i}) \\ & = -\int_{-\infty}^t AT(t-s)\, g(s,u_s)\, ds \\ & \quad + \int_{-\infty}^\sigma AT(t-s)\, g(s,u_s)\, ds \\ & \quad + \int_{-\infty}^t T(t-s)\, f(s,u_s)\, ds \\ & \quad - \int_{-\infty}^\sigma T(t-s)\, f(s,u_s)\, ds \\ & \quad + \sum_{-\infty<t_i<t} T(t-t_i)\, I_i(u_{t_i}) \\ & \quad - \sum_{-\infty<t_i<\sigma} T(t-t_i)\, I_i(u_{t_i}) \\ & = -g(t,u_t) - \int_{-\infty}^t AT(t-s)\, g(s,u_s)\, ds \\ & \quad + \int_{-\infty}^t T(t-s)\, f(s,u(s))\, ds \\ & \quad + \sum_{-\infty<t_i<t} T(t-t_i)\, I_i(u(t_i)) \\ & \quad + g(t,u_t) - T(t-\sigma) \\ & \quad \times \Bigg[-g(\sigma,u_\sigma) - \int_{-\infty}^\sigma AT(\sigma-s)\, g(s,u_s)\, ds \\ & \qquad + \int_{-\infty}^\sigma T(\sigma-s)\, f(s,u(s))\, ds \\ & \qquad + \sum_{-\infty<t_i<\sigma} T(\sigma-t_i)\, I_i(u(t_i)) + g(\sigma,u_\sigma) \Bigg] \\ & = u(t) + g(t,u_t) - T(t-\sigma)[u(\sigma) + g(\sigma,u_\sigma)] \\ & = u(t) + g(t,u_t) - T(t-\sigma)[u_\sigma(0) + g(\sigma,u_\sigma)] \\ & = u(t) + g(t,u_t) - T(t-\sigma)[\phi(0) + g(\sigma,\phi)], \end{aligned} \tag{14}$$

so that

$$\begin{aligned} u(t) = {} & T(t-\sigma)[\phi(0) + g(\sigma,\phi)] - g(t,u_t) \\ & - \int_\sigma^t AT(t-s)\, g(s,u_s)\, ds \\ & + \int_\sigma^t T(t-s)\, f(s,u_s)\, ds + \sum_{\sigma<t_i<t} T(t-t_i)\, I_i(u_{t_i}). \end{aligned} \tag{15}$$

In order to obtain our results, we need to introduce some additional notations. Let $h : R \to R$ be a continuous function such that $h(t) \geq 1$ for all $t \in R$ and $h(t) \to \infty$ as $|t| \to \infty$. We consider the space

$$(PC)_h^0(R, X_\alpha) = \left\{u \in PC(R, X_\alpha) : \lim_{|t|\to\infty} \frac{\|u(t)\|_\alpha}{h(t)} = 0\right\}. \tag{16}$$

Endowed with the norm $\|u\|_h = \sup_{t\in R}(\|u(t)\|_\alpha / h(t))$, it is a Banach space.

We recall here the following compactness criterion in these spaces, which we can refer to [7, 19–23].

Lemma 10. *A set $B \subseteq (PC)_h^0(R, X_\alpha)$ is a relatively compact set if and only if*

(1) $\lim_{|t|\to\infty}(\|x(t)\|_\alpha / h(t)) = 0$ *uniformly for $x \in B$;*

(2) $B(t) = \{x(t) : x \in B\}$ *is relatively compact in X_α for every $t \in R$;*

(3) *the set B is equicontinuous on each interval (t_i, t_{i+1}) $(i \in Z)$.*

Theorem 11 (Krasnoselskii's fixed-point theorem [24]). *Let M be a closed convex nonempty subset of a Banach space X; suppose that A and B map M into X such that*

(i) $Ax + By \in M$ $(\forall x, y \in M)$,

(ii) *A is completely continuous,*

(iii) *B is a contraction with constant $L < 1$.*

Then there is a $y \in M$ with $Ay + By = y$.

3. Main Results

In this section, we discuss the existence of piecewise almost periodic solutions for impulsive neutral functional differential equation (1). To begin, let us list the following hypotheses.

(A1) The operator A is the infinitesimal generator of an exponentially stable strongly continuous semigroup of linear operators $\{T(t)\}_{t\geq 0}$; that is, there exist constants $M_1 > 0$, $\delta > 0$ such that $\|T(t)\| \leq M_1 e^{-\delta t}$ for $t \geq 0$. Moreover, $T(t)$ is compact for $t > 0$.

(A2) $f(t, x) \in AP_T(R \times \mathfrak{B}, X_\alpha)$ is uniformly continuous in $x \in \mathfrak{B}$ uniformly in $t \in R$; $I_i(x)$ is almost periodic in $i \in Z$ uniformly in $x \in \mathfrak{B}$ and is a uniformly continuous function in x for all $i \in Z$. For every $l > 0$, $C_{1l} = \sup_{t\in R, \|x\|_{\mathfrak{B}} \leq lL} \|f(t, x)\|_\alpha < \infty$, $C_{2l} = \sup_{i\in Z, \|x\|_{\mathfrak{B}} \leq lL} \|I_i(x)\|_\alpha < \infty$. Moreover, there exist a number $L_0 > 0$, such that $M_1(C_1/\delta + C_2/(1 - e^{-\delta\gamma})) \leq L_0/2$, where $C_1 = C_{1L_0}$, $C_2 = C_{2L_0}$.

(A3) $g \in AP_T(R \times \mathfrak{B}, X_\beta)$, $g(t, 0) = 0$, and there exist a number $L_1 > 0$ such that

$$\|g(t, \phi_1) - g(t, \phi_2)\|_\beta \leq L_1 \|\phi_1 - \phi_2\|_{\mathfrak{B}} \tag{17}$$

for all $t \in R$, $\phi_1, \phi_2 \in \mathfrak{B}$.

(A4) Let $\{x_n\} \subseteq AP_T(R, X_\alpha)$ be uniformly bounded in R and uniformly convergent in each compact set of R; then $\{f(\cdot, x_n(\cdot))\}$ is relatively compact in $PC(R, X_\alpha)$.

Theorem 12. *Suppose that conditions (A1)–(A4) hold; then* (1) *has a piecewise almost periodic solution provided that $KLL_1 + (M_2\pi/\delta^{\beta-\alpha} \sin(\pi(1 + \alpha - \beta))\Gamma(1 + \alpha - \beta))L_1 L \leq 1/2$.*

Proof. Let

$$B = \{u \in AP_T(R, X_\alpha) : \|u\|_\alpha \leq L_0\}. \tag{18}$$

Note that B is a closed convex set of $AP_T(R, X_\alpha)$. By (A3) and Lemma 1, we have

$$\begin{aligned}
&\|AT(t-s)g(s, u_s)\|_\alpha \\
&\quad \leq \left\|A^{1+\alpha-\beta}T(t-s)\right\| \|g(s, u_s)\|_\beta \\
&\quad \leq M_2 e^{-\delta(t-s)}(t-s)^{-(1+\alpha-\beta)} \|g(s, u_s) - g(s, 0)\|_\beta \\
&\quad \leq M_2 e^{-\delta(t-s)}(t-s)^{-(1+\alpha-\beta)} L_1 \|u_s\|_{\mathfrak{B}} \\
&\quad \leq M_2 e^{-\delta(t-s)}(t-s)^{-(1+\alpha-\beta)} L_1 L \sup_{r\leq s} \|u(r)\|_\alpha \\
&\quad \leq M_2 e^{-\delta(t-s)}(t-s)^{-(1+\alpha-\beta)} L_1 L L_0,
\end{aligned} \tag{19}$$

and we infer that $s \to AT(t - s)g(s, u_s)$ is integrable on $(-\infty, t]$.

Define the operator Υ on $(PC)_h^0(R, X_\alpha)$ by

$$\begin{aligned}
\Upsilon u(t) = &-g(t, u_t) - \int_{-\infty}^{t} AT(t-s)g(s, u_s)\,ds \\
&+ \int_{-\infty}^{t} T(t-s)f(s, u_s)\,ds \\
&+ \sum_{t_i<t} T(t-t_i) I_i(u_{t_i}).
\end{aligned} \tag{20}$$

In order to prove that Υ has a fixed point in $AP_T(R, X_\alpha)$, we introduce the decomposition $\Upsilon = \Upsilon_1 + \Upsilon_2$, where

$$\begin{aligned}
\Upsilon_1 u(t) &= -g(t, u_t) - \int_{-\infty}^{t} AT(t-s)g(s, u_s)\,ds, \\
\Upsilon_2 u(t) &= \int_{-\infty}^{t} T(t-s)f(s, u_s)\,ds + \sum_{t_i<t} T(t-t_i) I_i(u_{t_i}).
\end{aligned} \tag{21}$$

Our proof will be split into the following three steps.

Step 1. We claim that $\Upsilon_1 x + \Upsilon_2 y \in B$ (for all $x, y \in B$).

For any $x, y \in B$, by (A3) and Lemma 1, we have

$$\begin{aligned}
\|\Upsilon_1 x(t)\|_\alpha &= \left\| -g(t, x_t) - \int_{-\infty}^{t} AT(t-s)\, g(s, x_s)\, ds \right\|_\alpha \\
&\le \|g(t, x_t)\|_\alpha + \int_{-\infty}^{t} \|AT(t-s)\, g(s, x_s)\|_\alpha ds \\
&\le K\|g(t, x_t)\|_\beta + \int_{-\infty}^{t} \left\|A^{1+\alpha-\beta} T(t-s)\right\| \|g(s, x_s)\|_\beta ds \\
&= K\|g(t, x_t) - g(t, 0)\|_\beta + \int_{-\infty}^{t} \left\|A^{1+\alpha-\beta} T(t-s)\right\| \times \|g(s, x_s) - g(s, 0)\|_\beta ds \\
&\le KL_1 \|x_t\|_{\mathfrak{B}} + \int_{-\infty}^{t} \left\|A^{1+\alpha-\beta} T(t-s)\right\| L_1 \|x_s\|_{\mathfrak{B}} ds \\
&\le KL_1 L \sup_{r\le t} \|x(r)\|_\alpha + \int_{-\infty}^{t} M_2 e^{-\delta(t-s)} (t-s)^{-(1+\alpha-\beta)} \times L_1 L \sup_{r\le s} \|x(r)\|_\alpha ds \\
&\le KL_1 L \|x\|_\alpha + \int_{-\infty}^{t} M_2 e^{-\delta(t-s)} (t-s)^{-(1+\alpha-\beta)} \times L_1 L \|x\|_\alpha ds \\
&\le KL_1 L \|x\|_\alpha + \frac{M_2 \pi}{\delta^{\beta-\alpha} \sin(\pi(1+\alpha-\beta))\, \Gamma(1+\alpha-\beta)} \times L_1 L \|x\|_\alpha,
\end{aligned}$$

$$\begin{aligned}
\|\Upsilon_2 y(t)\|_\alpha &= \left\| \int_{-\infty}^{t} T(t-s)\, f(s, y_s)\, ds + \sum_{t_i<t} T(t-t_i)\, I_i(y_{t_i}) \right\|_\alpha \\
&\le \int_{-\infty}^{t} \|T(t-s)\| \, \|f(s, y_s)\|_\alpha ds + \sum_{t_i<t} \|T(t-t_i)\| \, \|I_i(y_{t_i})\|_\alpha \\
&\le \int_{-\infty}^{t} M_1 e^{-\delta(t-s)} C_1 ds + \sum_{t_i<t} M_1 e^{-\delta(t-t_i)} C_2 \\
&\le \frac{M_1}{\delta} C_1 + \sum_{t_i<t} M_1 e^{-\delta(t-t_i)} C_2.
\end{aligned} \tag{22}$$

In order to estimate the last part of the second term on the right hand side of the above formula, we assume that, for every $t \in R$, there exists $j \in Z$, such that $t_j \le t < t_{j+1}$,

$$\begin{aligned}
t - t_i &= (t - t_j) + (t_j - t_i) \ge (j - i)\gamma, \\
\sum_{t_i<t} M_1 e^{-\delta(t-t_i)} C_2 &\le \sum_{-\infty<i\le j} M_1 e^{-\delta(j-i)\gamma} C_2 \\
&= \sum_{0\le k=j-i<\infty} M_1 e^{-\delta k\gamma} C_2 = \frac{M_1}{1-e^{-\delta\gamma}} C_2,
\end{aligned} \tag{23}$$

so

$$\|\Upsilon_2 y(t)\|_\alpha \le \frac{M_1}{\delta} C_1 + \frac{M_1}{1-e^{-\delta\gamma}} C_2. \tag{24}$$

Then,

$$\begin{aligned}
\|\Upsilon_1 x(t) + \Upsilon_2 y(t)\|_\alpha &\le \|\Upsilon_1 x(t)\|_\alpha + \|\Upsilon_2 y(t)\|_\alpha \\
&\le KL_1 L \|x\|_\alpha + \frac{M_2 \pi}{\delta^{\beta-\alpha} \sin(\pi(1+\alpha-\beta))\, \Gamma(1+\alpha-\beta)} L_1 L \|x\|_\alpha \\
&\quad + \frac{M_1}{\delta} C_1 + \frac{M_1}{1-e^{-\delta\gamma}} C_2,
\end{aligned} \tag{25}$$

from $KL_1L + (M_2\pi/\delta^{\beta-\alpha} \sin(\pi(1+\alpha-\beta))\Gamma(1+\alpha-\beta))L_1L \le 1/2$, and we obtain that

$$\|\Upsilon_1 x(t) + \Upsilon_2 y(t)\|_\alpha \le L_0, \quad x, y \in B. \tag{26}$$

By (A2) and Lemma 6, $f(t, y_t) \in AP_T(R, X_\alpha)$, and, by (A3) and Lemma 7, $g(t, x_t) \in AP_T(R, X_\beta)$. By (A2) and [1, Lemma 37], $\{I_i(y_{t_i})\}$ is almost periodic. From [1, Theorem 73], for the two almost periodic functions $f(t, y_t)$, $g(t, x_t)$, there exists a relatively dense set of their common ϵ-translation numbers. Then by Lemma 8, for every $\epsilon > 0$, there exist relatively dense sets $\Omega_{\epsilon,f,g,I_i}$ of R and Q_{ϵ,f,g,I_i} of Z such that for $\tau \in \Omega_{\epsilon,f,g,I_i}$, $\exists\, q \in Q_{\epsilon,f,g,I_i}$,

$$\begin{gathered}
\|g(t+\tau, x_{t+\tau}) - g(t, x_t)\|_\beta \le \epsilon, \\
\|f(t+\tau, y_{t+\tau}) - f(t, y_t)\|_\alpha \le \epsilon, \\
\left\|I_{i+q}(y_{t_{i+q}}) - I_i(y_{t_i})\right\|_\alpha \le \epsilon, \qquad \left|t_i^q - \tau\right| \le \epsilon,
\end{gathered} \tag{27}$$

where $t \in R$, $|t - t_i| > \epsilon$, $i \in Z$. So for $\tau \in \Omega_{\epsilon,f,g,I_i}$, we know that

$$\begin{aligned} &\Upsilon_1 x(t+\tau) - \Upsilon_1 x(t) \\ &= -g(t+\tau, x_{t+\tau}) - \int_{-\infty}^{t+\tau} AT(t+\tau-s)\, g(s, x_s)\, ds \\ &\quad + g(t, x_t) + \int_{-\infty}^{t} AT(t-s)\, g(s, x_s)\, ds \\ &= -g(t+\tau, x_{t+\tau}) + g(t, x_t) \\ &\quad - \int_{-\infty}^{t} AT(t-s)\left[g(s+\tau, x_{s+\tau}) - g(s, x_s)\right] ds, \end{aligned} \tag{28}$$

so

$$\begin{aligned} &\|\Upsilon_1 x(t+\tau) - \Upsilon_1 x(t)\|_\alpha \\ &\le \|g(t+\tau, x_{t+\tau}) - g(t, x_t)\|_\alpha \\ &\quad + \int_{-\infty}^{t} \|AT(t-s)\left[g(s+\tau, x_{s+\tau}) - g(s, x_s)\right]\|_\alpha ds \\ &\le K\|g(t+\tau, x_{t+\tau}) - g(t, x_t)\|_\beta \\ &\quad + \int_{-\infty}^{t} \|A^{1+\alpha-\beta}T(t-s)\| \\ &\qquad \times \|g(s+\tau, x_{s+\tau}) - g(s, x_s)\|_\beta ds \\ &\le K\epsilon + \int_{-\infty}^{t} M_2 e^{-\delta(t-s)}(t-s)^{-(1+\alpha-\beta)}\epsilon ds \\ &\le \epsilon\left(K + \frac{M_2\pi}{\delta^{\beta-\alpha}\sin(\pi(1+\alpha-\beta))\Gamma(1+\alpha-\beta)}\right). \end{aligned} \tag{29}$$

That is,

$$\begin{aligned} &\|\Upsilon_1 x(t+\tau) - \Upsilon_1 x(t)\|_\alpha \\ &\quad \le \epsilon\Big(K + (M_2\pi) \\ &\qquad \times \left(\delta^{\beta-\alpha}\sin(\pi(1+\alpha-\beta))\right. \\ &\qquad\quad \left.\times\, \Gamma(1+\alpha-\beta)\right)^{-1}\Big), \end{aligned} \tag{30}$$

$$\begin{aligned} &\Upsilon_2 y(t+\tau) - \Upsilon_2 y(t) \\ &= \int_{-\infty}^{t+\tau} T(t+\tau-s)\, f(s, y_s)\, ds \\ &\quad + \sum_{t_i<t+\tau} T(t+\tau-t_i)\, I_i(y_{t_i}) \\ &\quad - \int_{-\infty}^{t} T(t-s)\, f(s, y_s)\, ds \\ &\quad - \sum_{t_i<t} T(t-t_i)\, I_i(y_{t_i}) \\ &= \int_{-\infty}^{t} T(t-s)\, f(s+\tau, y_{s+\tau})\, ds \\ &\quad + \sum_{t_i<t} T(t-t_i)\, I_{i+q}(y_{t_{i+q}}) \\ &\quad - \int_{-\infty}^{t} T(t-s)\, f(s, y_s)\, ds \\ &\quad - \sum_{t_i<t} T(t-t_i)\, I_i(y_{t_i}) \\ &= \int_{-\infty}^{t} T(t-s)\left[f(s+\tau, y_{s+\tau})\right. \\ &\qquad\qquad \left.-f(s, y_s)\right] ds \\ &\quad + \sum_{t_i<t} T(t-t_i)\left[I_{i+q}(y_{t_{i+q}}) - I_i(y_{t_i})\right], \end{aligned} \tag{31}$$

so we have

$$\begin{aligned} &\left\|\int_{-\infty}^{t} T(t-s)\left[f(s+\tau, y_{s+\tau}) - f(s, y_s)\right] ds\right\|_\alpha \\ &\quad \le \int_{-\infty}^{t} \|T(t-s)\|\, \|f(s+\tau, y_{s+\tau}) - f(s, y_s)\|_\alpha ds \\ &\quad \le \int_{-\infty}^{t} M_1 e^{-\delta(t-s)}\epsilon\, ds \le \frac{M_1}{\delta}\epsilon, \\ &\left\|\sum_{t_i<t} T(t-t_i)[I_{i+q}(y_{t_{i+q}}) - I_i(y_{t_i})]\right\|_\alpha \\ &\quad \le \sum_{t_i<t} \|T(t-t_i)\|\, \left\|I_{i+q}(y_{t_{i+q}}) - I_i(y_{t_i})\right\|_\alpha \\ &\quad \le \sum_{t_i<t} M_1 e^{-\delta(t-t_i)}\epsilon \le \frac{M_1}{1-e^{-\delta\gamma}}\epsilon, \end{aligned} \tag{32}$$

so

$$\|\Upsilon_2 y(t+\tau) - \Upsilon_2 y(t)\|_\alpha \le \left(\frac{M_1}{\delta} + \frac{M_1}{1-e^{-\delta\gamma}}\right)\epsilon. \tag{33}$$

Combining (26), (30), and (33), it follows that $\Upsilon_1 x + \Upsilon_2 y \in B$ (for all $x, y \in B$).

Step 2. Υ_1 is a contraction.

Let $x, y \in B$; by (A3) and Lemma 1, we have

$$\begin{aligned} &\|\Upsilon_1 x(t) - \Upsilon_1 y(t)\|_\alpha \\ &\quad = \Big\| -g(t, x_t) - \int_{-\infty}^{t} AT(t-s)\, g(s, x_s)\, ds \\ &\qquad + g(t, y_t) + \int_{-\infty}^{t} AT(t-s)\, g(s, y_s)\, ds \Big\|_\alpha \\ &\quad \le \|g(t, x_t) - g(t, y_t)\|_\alpha \end{aligned}$$

$$
\begin{aligned}
&+\int_{-\infty}^{t}\left\|AT(t-s)\left[g(s,x_s)-g(s,y_s)\right]\right\|_{\alpha}ds \\
&\le K\|g(t,x_t)-g(t,y_t)\|_{\beta} \\
&+\int_{-\infty}^{t}\left\|A^{1+\alpha-\beta}T(t-s)\right\|\|g(s,x_s)-g(s,y_s)\|_{\beta}ds \\
&\le KL_1\|x_t-y_t\|_{\mathfrak{B}} \\
&+\int_{-\infty}^{t}\left\|A^{1+\alpha-\beta}T(t-s)\right\|L_1\|x_s-y_s\|_{\mathfrak{B}}ds \\
&\le KL_1L\sup_{r\le t}\|x(r)-y(r)\|_{\alpha} \\
&+\int_{-\infty}^{t}\left\|A^{1+\alpha-\beta}T(t-s)\right\|L_1L\sup_{r\le s}\|x(r)-y(r)\|_{\alpha}ds \\
&\le KL_1L\|x-y\|_{\alpha} \\
&+\int_{-\infty}^{t}M_2e^{-\delta(t-s)}(t-s)^{-(1+\alpha-\beta)}L_1L\|x-y\|_{\alpha}ds \\
&\le KL_1L\|x-y\|_{\alpha} \\
&+\frac{M_2\pi}{\delta^{\beta-\alpha}\sin(\pi(1+\alpha-\beta))\Gamma(1+\alpha-\beta)} \\
&\times L_1L\|x-y\|_{\alpha}.
\end{aligned}
\tag{34}
$$

Therefore,

$$
\begin{aligned}
&\|\Upsilon_1x-\Upsilon_1y\|_{\alpha} \\
&\quad\le\left[KL_1L+\frac{M_2\pi}{\delta^{\beta-\alpha}\sin(\pi(1+\alpha-\beta))\Gamma(1+\alpha-\beta)}L_1L\right] \\
&\qquad\times\|x-y\|_{\alpha}.
\end{aligned}
\tag{35}
$$

Since $KL_1L+(M_2\pi/\delta^{\beta-\alpha}\sin(\pi(1+\alpha-\beta))\Gamma(1+\alpha-\beta))L_1L\le 1/2<1$, it follows that Υ_1 is a contraction.

Step 3. Υ_2 is completely continuous.

Claim 1. Υ_2 is continuous.

Let $\{x_n\}\subseteq AP_T(R,X_\alpha)$, $x_n\to x$ in $AP_T(R,X_\alpha)$ as $n\to\infty$; by Lemma 5, we may find a compact subset $B_0\subseteq X_\alpha$ such that $x_n(t),x(t)\in B_0$ for all $t\in R$, $n\in N$; here we assume $B\subseteq B_0$. By (A2), for the given ϵ, there exist $\delta>0$ such that $x,y\in\mathfrak{B}$, $\|x-y\|_{\mathfrak{B}}\le\delta$, implies that

$$
\begin{aligned}
\|f(t,x_t)-f(t,y_t)\|_{\alpha}&\le\epsilon,\\
\left\|I_i\left(x_{t_i}\right)-I_i\left(y_{t_i}\right)\right\|_{\alpha}&\le\epsilon.
\end{aligned}
\tag{36}
$$

For the above δ, there exists $n_0\in N$ such that

$$
\|x_n(t)-x(t)\|_{\alpha}\le\frac{\delta}{L}
\tag{37}
$$

for $n>n_0$ and $t\in R$; then

$$
\begin{aligned}
\|(x_n)_t-x_t\|_{\mathfrak{B}}&\le\delta,\\
\|f(t,(x_n)_t)-f(t,x_t)\|_{\alpha}&\le\epsilon,\\
\left\|I_i((x_n)_{t_i})-I_i\left(x_{t_i}\right)\right\|_{\alpha}&\le\epsilon,
\end{aligned}
\tag{38}
$$

for $n>n_0$ and $t\in R$. Hence,

$$
\begin{aligned}
&\|\Upsilon_2(x_n)(t)-\Upsilon_2x(t)\|_{\alpha} \\
&\quad=\left\|\int_{-\infty}^{t}T(t-s)f(s,(x_n)_s)ds+\sum_{t_i<t}T(t-t_i)I_i\left((x_n)_{t_i}\right)-\int_{-\infty}^{t}T(t-s)f(s,x_s)ds-\sum_{t_i<t}T(t-t_i)I_i\left(x_{t_i}\right)\right\|_{\alpha} \\
&\quad\le\int_{-\infty}^{t}\|T(t-s)\|\|f(s,(x_n)_s)-f(s,x_s)\|_{\alpha}ds+\sum_{t_i<t}\|T(t-t_i)\|\left\|I_i\left((x_n)_{t_i}\right)-I_i\left(x_{t_i}\right)\right\|_{\alpha} \\
&\quad\le\int_{-\infty}^{t}M_1e^{-\delta(t-s)}\epsilon\,ds+\sum_{t_i<t}M_1e^{-\delta(t-t_i)}\epsilon \\
&\quad=\frac{M_1}{\delta}\epsilon+\frac{M_1}{1-e^{\delta\gamma}}\epsilon,
\end{aligned}
\tag{39}
$$

for $n>n_0$ and $t\in R$, from which it follows that Υ_2 is continuous.

Claim 2. $\{\Upsilon_2u(t):u\in B\}$ is a relatively compact subset of X_α for each $t\in R$.

For any $\epsilon>0$, let

$$
\begin{aligned}
\Upsilon_2^{\epsilon}u(t)&=\int_{-\infty}^{t-\epsilon}T(t-s)f(s,u_s)ds+\sum_{t_i<t-\epsilon}T(t-t_i)I_i\left(u_{t_i}\right) \\
&=T(\epsilon)\left[\int_{-\infty}^{t-\epsilon}T(t-\epsilon-s)f(s,u_s)ds+\sum_{t_i<t-\epsilon}T(t-\epsilon-t_i)I_i\left(u_{t_i}\right)\right] \\
&=T(\epsilon)\Upsilon_2u(t-\epsilon),
\end{aligned}
\tag{40}
$$

where $\{\Upsilon_2 u(t-\epsilon) : u \in B\}$ is uniformly bounded in X_α and $T(\epsilon)$ is compact, so $\{\Upsilon_2^\epsilon u(t) : u \in B\}$ is relatively compact in X_α. Moreover,

$$\begin{aligned}
&\left\|\Upsilon_2 u(t) - \Upsilon_2^\epsilon u(t)\right\|_\alpha \\
&\quad = \left\| \int_{t-\epsilon}^{t} T(t-s) f(s, x_s)\, ds + \sum_{t-\epsilon<t_i<t} T(t-t_i) I_i(u_{t_i}) \right\|_\alpha \\
&\quad \le \int_{t-\epsilon}^{t} \|T(t-s)\| \, \|f(s,x_s)\|_\alpha ds + \sum_{t-\epsilon<t_i<t} \|T(t-t_i)\| \left\|I_i(u_{t_i})\right\|_\alpha \\
&\quad \le \int_{t-\epsilon}^{t} M_1 e^{-\delta(t-s)} \|f(s,x_s)\|_\alpha ds + \sum_{t-\epsilon<t_i<t} M_1 e^{-\delta(t-t_i)} \left\|I_i(u_{t_i})\right\|_\alpha \\
&\quad \le \epsilon M_1 C_1 + \frac{\epsilon}{\gamma} M_1 C_2,
\end{aligned} \tag{41}$$

so $\{\Upsilon_2 u(t) : u \in B\}$ is a relatively compact subset of X_α for each $t \in R$.

Claim 3. $\{\Upsilon_2 u : u \in B\}$ is equicontinuous on each interval (t_i, t_{i+1}) $(i \in Z)$.

Let $t'' < t'$, $t'', t' \in (t_i, t_{i+1})$, $i \in Z$, $u \in B$,

$$\begin{aligned}
&\Upsilon_2 u(t') - \Upsilon_2 u(t'') \\
&\quad = \int_{-\infty}^{t'} T(t'-s) f(s,u_s)\, ds + \sum_{t_i<t'} T(t'-t_i) I_i(u_{t_i}) \\
&\qquad - \int_{-\infty}^{t''} T(t''-s) f(s,u_s)\, ds - \sum_{t_i<t''} T(t''-t_i) I_i(u_{t_i}) \\
&\quad = \int_{-\infty}^{t''} \left[T(t'-s) - T(t''-s)\right] f(s,u_s)\, ds + \int_{t''}^{t'} T(t'-s) f(s,u_s)\, ds \\
&\qquad + \sum_{t_i<t''} \left[T(t'-t_i) - T(t''-t_i)\right] I_i(u_{t_i}) + \sum_{t''<t_i<t'} T(t'-t_i) I_i(u_{t_i}).
\end{aligned} \tag{42}$$

Moreover,

$$\begin{aligned}
&\int_{-\infty}^{t''} \left[T(t'-s) - T(t''-s)\right] f(s,u_s)\, ds \\
&\quad = \int_0^\infty \left[T(t'-t''+s) - T(s)\right] f(t''-s, u_{t''-s})\, ds \\
&\quad = \int_0^\infty \left[T(t'-t'') - I\right] T(s) f(t''-s, u_{t''-s})\, ds.
\end{aligned} \tag{43}$$

By (A1), for the given $\epsilon > 0$, there exists $\mu(\epsilon) < \epsilon/4M_1(C_1 + C_2/\gamma)$, such that if $0 < t' - t'' < \mu$, then

$$\left\|T(t'-t'') - I\right\| < \min\left\{\frac{\delta\epsilon}{4M_1C_1}, \frac{(1-e^{-\delta\gamma})\epsilon}{4M_1C_2}\right\}. \tag{44}$$

So

$$\begin{aligned}
&\left\|\int_{-\infty}^{t''} \left[T(t'-s) - T(t''-s)\right] f(s,u(s))\, ds\right\|_\alpha \\
&\quad \le \int_0^\infty \left\|T(t'-t'') - I\right\| \|T(s)\| \times \left\|f(t''-s, u(t''-s))\right\|_\alpha ds \\
&\quad \le \int_0^\infty \frac{\delta\epsilon}{4M_1C_1} M_1 e^{-\delta s} C_1 ds < \frac{\epsilon}{4}, \\
&\left\|\int_{t''}^{t'} T(t'-s) f(s,u(s)) ds\right\|_\alpha \\
&\quad \le \int_{t''}^{t'} \left\|T(t'-s)\right\| \|f(s,u(s))\|_\alpha ds \\
&\quad < \mu M_1 C_1 < \frac{\epsilon}{4}.
\end{aligned} \tag{45}$$

Similarly,

$$\begin{aligned}
&\left\|\sum_{t_i<t''} \left[T(t'-t_i) - T(t''-t_i)\right] I_i(u(t_i))\right\|_\alpha \\
&\quad = \left\|\sum_{t_i<t''} \left[T(t'-t'') T(t''-t_i) - T(t'',t_i)\right] I_i(u(t_i))\right\|_\alpha \\
&\quad = \left\|\sum_{t_i<t''} \left[T(t'-t'') - I\right] T(t''-t_i) I_i(u(t_i))\right\|_\alpha \\
&\quad \le \sum_{t_i<t''} \left\|T(t'-t'') - I\right\| \left\|T(t''-t_i)\right\| \|I_i(u(t_i))\|_\alpha \\
&\quad \le \sum_{t_i<t''} \frac{(1-e^{-\delta\gamma})\epsilon}{4M_1C_2} M_1 e^{-\delta(t''-t_i)} C_2
\end{aligned}$$

$$\leq \frac{\epsilon}{4},$$

$$\begin{aligned}
&\left\| \sum_{t''<t_i<t'} T\left(t'-t_i\right) I_i\left(u\left(t_i\right)\right) \right\|_\alpha \\
&\quad \leq \sum_{t''<t_i<t'} \left\| T\left(t', t_i\right) \right\| \left\| I_i\left(u\left(t_i\right)\right) \right\|_\alpha \\
&\quad < \frac{\mu}{\gamma} M_1 C_2 < \frac{\epsilon}{4}.
\end{aligned} \tag{46}$$

So that, for $u \in B, t', t'' \in (t_i, t_{i+1})$, $i \in Z$, for any $\epsilon > 0$, there exists a positive number $\mu(\epsilon)$, $\mu < \epsilon / 4M_1(C_1 + C_2/\gamma)$; if $0 < t' - t'' < \mu$,

$$\left\| \Upsilon_2 u(t') - \Upsilon_2 u(t'') \right\|_\alpha \leq \epsilon. \tag{47}$$

That is, $\{\Upsilon_2 u : u \in B\}$ is equicontinuous on each interval (t_i, t_{i+1}) $(i \in Z)$.

Since $\{\Upsilon_2 u : u \in B\} \subseteq (PC)_h^0(R, X_\alpha)$ and $\{\Upsilon_2 u : u \in B\}$ satisfies the conditions of Lemma 10, Υ_2 is completely continuous.

By Krasnoselskii's fixed-point theorem (Theorem 11), we know that Υ has a fixed point $u \in B$; that is, (1) has a piecewise almost periodic solution $u(t)$. The proof is complete. □

Note that the condition of uniformly continuous is weaker than that of Lipschitz continuous, so if assumption (A2) is replaced by the following assumption:

(A2′) $f(t,x) \in AP_T(R \times \mathfrak{B}, X_\alpha)$, $f(t,0) = 0$, $I_i(0) = 0$, $I_i(x)$ is almost periodic in $i \in Z$ uniformly in $x \in \mathfrak{B}$ and

$$\begin{aligned}
\left\| f(t,x) - f(t,y) \right\|_\alpha &\leq L_2 \left\| x - y \right\|_{\mathfrak{B}}, \\
\left\| I_i(x) - I_i(y) \right\|_\alpha &\leq L_3 \left\| x - y \right\|_{\mathfrak{B}},
\end{aligned} \tag{48}$$

for all $x, y \in \mathfrak{B}$.

We can get the almost periodic solution of (1) by means of contraction mapping principle.

Corollary 13. *Suppose that conditions (A1), (A2′), and (A3)-(A4) hold;* (1) *has a piecewise almost periodic solution provided that*

$$\begin{aligned}
&\frac{M_1 L_2 L}{\delta} + \frac{M_1 L_3 L}{1 - e^{-\delta r}} \\
&\quad + \left(K + \frac{M_2 \pi}{\delta^{\beta-\alpha} \sin\left(\pi\left(1+\alpha-\beta\right)\right) \Gamma\left(1+\alpha-\beta\right)} \right) \\
&\quad \times L_1 L < 1.
\end{aligned} \tag{49}$$

Proof. As the same discussion as Step 2 of Theorem 12, we can prove that Υ_1 is a contraction, and it remains to show that Υ_2 is a contraction. By Step 1 of Theorem 12, $\Upsilon_2 B \subseteq B$. For $x, y \in B$,

$$\begin{aligned}
&\left\| \Upsilon_2 x(t) - \Upsilon_2 y(t) \right\|_\alpha \\
&\quad = \left\| \int_{-\infty}^t T(t-s)\left[f\left(s, x_s\right) - f\left(s, y_s\right)\right] ds \right. \\
&\qquad \left. + \sum_{t_i<t} T\left(t-t_i\right)\left[I_i\left(x_{t_i}\right) - I_i\left(y_{t_i}\right)\right] \right\|_\alpha \\
&\quad \leq \int_{-\infty}^t \left\| T(t-s) \right\| \left\| f(s,x_s) - f(s,y_s) \right\|_\alpha ds \\
&\qquad + \sum_{t_i<t} \left\| T\left(t-t_i\right) \right\| \left\| I_i(x_{t_i}) - I_i(y_{t_i}) \right\|_\alpha \\
&\quad \leq \int_{-\infty}^t M_1 e^{-\delta(t-s)} L_2 \left\| x_s - y_s \right\|_{\mathfrak{B}} ds \\
&\qquad + \sum_{t_i<t} M_1 e^{-\delta(t-t_i)} L_3 \left\| x_{t_i} - y_{t_i} \right\|_{\mathfrak{B}} \\
&\quad \leq \int_{-\infty}^t M_1 e^{-\delta(t-s)} L_2 L \sup_{r\leq s} \left\| x(r) - y(r) \right\|_\alpha ds \\
&\qquad + \sum_{t_i<t} M_1 e^{-\delta(t-t_i)} L_3 L \sup_{r\leq t_i} \left\| x(r) - y(r) \right\|_\alpha \\
&\quad \leq M_1 L \left(\frac{L_2}{\delta} + \frac{L_3}{1 - e^{-\delta\gamma}} \right) \left\| x - y \right\|_\alpha.
\end{aligned} \tag{50}$$

The proof is complete. □

4. Application

Consider the following impulsive neutral differential system:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[u(t,\xi) + \int_{-\infty}^0 \int_0^\pi b_1(s,\eta,\xi) u(t+s,\eta) \, d\eta \, ds\right] \\
&\quad = \frac{\partial^2}{\partial \xi^2} u(t,\xi) + b_2(\xi) u(t,\xi) \\
&\qquad + \int_{-\infty}^0 b_3(s) u(t+s,\xi) \, ds + b_4(t,\xi), \\
&\qquad\qquad (t,\xi) \in R \times [0,\pi], \\
&u(t,0) = u(t,\pi) = 0, \quad t \in R, \\
&\Delta u(t_i)(\xi) = \int_0^\pi a_i(t_i - s) u(s,\xi) \, ds, \\
&\qquad\qquad i \in Z, \quad \xi \in [0,\pi],
\end{aligned} \tag{51}$$

where $\{t_i^j = t_{i+j} - t_i\}$, $i \in Z$, $j = 0, \pm 1, \pm 2, \ldots$, are equipotentially almost periodic such that $\gamma = \inf_{i\in Z}\{t_{i+1} - t_i\} > 0$. The system (51) arises, for example, in control systems described by abstract retarded functional differential equations with feedback control governed by proportional integrodifferential law; see [25, 26] for details.

Let $X = L^2([0,\pi])$ and A be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ which is compact for $t > 0$ and given by $Au = u''$ with domain $D(A) = \{u \in L^2([0,\pi]) : u'' \in L^2([0,\pi]), u(0) = u(\pi) = 0\}$. The semigroup $\{T(t)\}_{t\geq 0}$ is defined for $u \in L^2([0,\pi])$ by

$$T(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, \phi_n \rangle \phi_n, \tag{52}$$

where $\{\phi_n, n \in Z\}$ is an orthonormal basis of X; then $|T(t)u| \leq e^{-t}|u|$, $0 \neq u \in L^2([0,\pi])$. For $u \in L^2([0,\pi])$, $\alpha \in (0,1)$, $(-A)^{-\alpha}u = \sum_{n=1}^{\infty} n^{-2\alpha}\langle u, \phi_n\rangle\phi_n$, X_α is the Banach space endowed with the norm $\|\cdot\|_\alpha$ between X and $D(A)$.

In this paper, we assume $\alpha = 1/4$, $\beta = 3/4$.

To study the system (51), we make the following assumptions.

(i) The functions $(\partial^i/\partial\xi^i)b_1(\tau,\eta,\xi), i = 0,1$ are Lebesgue measurable, $b_1(\tau,\eta,0) = b_1(\tau,\eta,\pi) = 0$ and

$$\begin{aligned} L_1 &:= \max\left\{\int_0^{\pi}\int_{-\infty}^{0}\int_0^{\pi}\left(\frac{\partial^i}{\partial\xi^i}b_1(\tau,\eta,\xi)\right)^2 d\eta\, d\tau\, d\xi : i = 0,1\right\} \\ &< \infty. \end{aligned} \tag{53}$$

(ii) The functions b_i $(i = 2,3,4)$, a_i $(i \in Z)$ are continuous, and the sequence of functions $\{a_i, i \in Z\}$ is almost periodic.

Under these conditions, we define $f, g : R\times\mathfrak{B} \to X$, $I_i : \mathfrak{B} \to X$ $(i \in Z)$ by

$$\begin{aligned} g(t,\phi) &= \int_{-\infty}^{0}\int_0^{\pi} b_1(s,\eta,\xi)\,\phi(t+s,\eta)\,d\eta\,ds, \\ &\qquad t \in R, \quad \xi \in [0,\pi], \\ f(t,\phi) &= b_2(\xi)\,\phi(t,\xi) + \int_{-\infty}^{0} b_3(s)\,\phi(t+s,\xi)\,ds + b_4(t,\xi), \\ &\qquad t \in R, \quad \xi \in [0,\pi], \\ I_i(\phi) &= \int_0^{\pi} a_i(t_i - s)\,\phi(s,\xi)\,ds, \quad i \in Z,\ \xi \in [0,\pi]. \end{aligned} \tag{54}$$

We assume further that f, g satisfy the following condition.

(iii) $g \in AP_T(R\times\mathfrak{B}, X_\beta)$ and $f \in AP_T(R\times\mathfrak{B}, X_\alpha)$ are uniformly continuous in $x \in \mathfrak{B}$ uniformly in $t \in R$.

Under the above assumptions, we can rewrite (51) as the abstract form (1) and verify that the assumptions of Theorem 12 hold; then we can get the next result, which is a consequence of Theorem 12.

Proposition 14. *Assume that the previous conditions are verified, if*

$$L_1 L(K + \sqrt{\pi}) < 1. \tag{55}$$

The system (51) *has a piecewise almost periodic solution.*

Conflict of Interests

The authors declare that they have no conflict of interests.

References

[1] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[2] G. T. Stamov, J. O. Alzabut, P. Atanasov, and A. G. Stamov, "Almost periodic solutions for an impulsive delay model of price fluctuations in commodity markets," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3170–3176, 2011.

[3] G. T. Stamov and I. M. Stamova, "Almost periodic solutions for impulsive neural networks with delay," *Applied Mathematical Modelling*, vol. 31, no. 7, pp. 1263–1270, 2007.

[4] J. O. Alzabut, G. T. Stamov, and E. Sermutlu, "On almost periodic solutions for an impulsive delay logarithmic population model," *Mathematical and Computer Modelling*, vol. 51, no. 5-6, pp. 625–631, 2010.

[5] Q. Wang, H. Zhang, M. Ding, and Z. Wang, "Global attractivity of the almost periodic solution of a delay logistic population model with impulses," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 12, pp. 3688–3697, 2010.

[6] L. Pan and J. Cao, "Anti-periodic solution for delayed cellular neural networks with impulsive effects," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3014–3027, 2011.

[7] C. Cuevas, E. Hernandez, and M. Rabelo, "The existence of solutions for impulsive neutral functional differential equations," *Computers & Mathematics with Applications. An International Journal*, vol. 58, no. 4, pp. 744–757, 2009.

[8] A. Anguraj and K. Karthikeyan, "Existence of solutions for impulsive neutral functional differential equations with nonlocal conditions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2717–2721, 2009.

[9] E. Hernández, "Global solutions for abstract impulsive neutral differential equations," *Mathematical and Computer Modelling*, vol. 53, no. 1-2, pp. 196–204, 2011.

[10] O. Abdelghani, "Local and global existence and uniqueness results for impulsive functional differential equations with multiple delay," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 1, pp. 456–472, 2006.

[11] M. E. Hernandez, M. Rabello, and H. R. Henriquez, "Existence of solutions for impulsive partial neutral functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 2, pp. 1135–1158, 2007.

[12] Y. Chang, A. Anguraj, and M. M. Arjunan, "Existence results for impulsive neutral functional differential equations with infinite delay," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 1, pp. 209–218, 2008.

[13] R. Ye, "Existence of solutions for impulsive partial neutral functional differential equation with infinite delay," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 73, no. 1, pp. 155–162, 2010.

[14] M. E. Hernández, H. R. Henríquez, and M. A. McKibben, "Existence results for abstract impulsive second-order neutral functional differential equations," *Nonlinear Analysis*, vol. 70, no. 7, pp. 2736–2751, 2009.

[15] H. R. Henríquez, B. de Andrade, and M. Rabelo, "Existence of almost periodic solutions for a class of abstract impulsive differential equations," *ISRN Mathematical Analysis*, vol. 2011, Article ID 632687, 21 pages, 2011.

[16] J. W. Liu and C. Y. Zhang, "Existence and stability of almost periodic solutions for impulsive differential equations," *Advances in Difference Equations*, vol. 34, p. 14, 2012.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY. USA, 1983.

[18] H. R. Henríquez and C. Lizama, "Compact almost automorphic solutions to integral equations with infinite delay," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 71, no. 12, pp. 6029–6037, 2009.

[19] R. P. Agarwal, C. Cuevas, and J. P. dos Santos, "Analytic resolvent operator and existence results for fractional integro-differential equations," *Journal of Abstract Differential Equations and Applications*, vol. 2, no. 2, pp. 26–47, 2012.

[20] S. Sivasankaran, M. M. Arjunan, and V. Vijayakumar, "Existence of global solutions for second order impulsive abstract partial differential equations," *Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal. Series A: Theory and Methods*, vol. 74, no. 17, pp. 6747–6757, 2011.

[21] E. M. Hernandez, S. M. T. Aki, and H. Henríquez, "Global solutions for impulsive abstract partial differential equations," *Computers & Mathematics with Applications*, vol. 56, no. 5, pp. 1206–1215, 2008.

[22] R. P. Agarwal, B. de Andrade, and C. Cuevas, "Weighted pseudo-almost periodic solutions of a class of semilinear fractional differential equations," *Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal*, vol. 11, no. 5, pp. 3532–3554, 2010.

[23] Y. Chang, R. Zhang, and G. M. N'Guérékata, "Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations," *Computers and Mathematics with Applications*, vol. 64, no. 10, pp. 3160–3170, 2012.

[24] T. A. Burton, "A fixed-point theorem of Krasnoselskii," *Applied Mathematics Letters*, vol. 11, no. 1, pp. 85–88, 1998.

[25] E. Hernández and H. R. Henrıquez, "Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay," *Journal of Mathematical Analysis and Applications*, vol. 221, no. 2, pp. 499–522, 1998.

[26] T. Diagana, E. Hernandez, and M. Rabello, "Pseudo almost periodic solutions to some non-autonomous neutral functional differential equations with unbounded delay," *Mathematical and Computer Modelling*, vol. 45, no. 9-10, pp. 1241–1252, 2007.