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Research Article

A Shifted Jacobi-Gauss Collocation Scheme for Solving Fractional Neutral Functional-Differential Equations

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The shifted Jacobi-Gauss collocation (SJGC) scheme is proposed and implemented to solve the fractional neutral functional-differential equations with proportional delays. The technique we have proposed is based upon shifted Jacobi polynomials with the Gauss quadrature integration technique. The main advantage of the shifted Jacobi-Gauss scheme is to reduce solving the generalized fractional neutral functional-differential equations to a system of algebraic equations in the unknown expansion. Reasonable numerical results are achieved by choosing few shifted Jacobi-Gauss collocation nodes. Numerical results demonstrate the accuracy, and versatility of the proposed algorithm.

1. Introduction

Fractional differential equations (FDEs) have drawn the interest of many researchers in recent years [1–6], due to their useful applications in many fields of science. In fact, we may observe several applications in electrochemistry, viscoelasticity, electromagnetic, control, plasma physics, porous media, fluctuating environments, dynamical processes, and so on. In consequence, fractional differential equations are gaining much attention from the researchers. For some recent developments on this subject, see [7–15].

In the last decade or so, comprehensive research has been accomplished on the development of numerical algorithms which are numerically stable for both linear and nonlinear FDEs. Tripathi et al. [16] presented a new operational matrix of hat functions to solve linear FDEs. The spectral tau method was proposed in [17] to achieve an accurate solution of linear and nonlinear FDEs subject to multipoint conditions. In [18], the author proposed Bernstein polynomial to design a numerical algorithm for fractional Riccati equations. The authors of [19] investigated the spline collocation method for approximating the solution of nonlinear FDEs. Furthermore, the author of [20] transformed the time-dependent space FDE with variable coefficients into a system of ordinary differential equations, which is then solved by a standard numerical method. Baleanu et al. [21] developed the generalized Laguerre spectral tau and collocation approximations to solve FDEs on the half line. In [22], Ma and Huang developed spectral collocation method for solving linear fractional integrodifferential equations. Yang and Huang [23] analyzed and developed the Jacobi collocation scheme for pantograph integrodifferential equations with fractional orders in finite interval. In [24] Yin et al. proposed a new fractional-order Legendre function with spectral method to solve partial FDEs; based on the operational matrix of these functions, the same authors developed their approach in combination with variational iteration formula to solve a class of FDEs; see [25]. More recently, the Jacobi Galerkin method was extended in [26] to solve stochastic FDEs.

Polynomial approximations can be quite useful for expressing the solution of a differential equation. One such approach would be the spectral methods. An advantage of a spectral collocation method is that it gives high accurate solutions with relatively fewer spatial grid nodes when compared with other numerical techniques. In [27], the Jacobi rational collocation scheme was proposed and developed to solve generalized pantograph equations. In [28], the authors extended the application of Jacobi-Gauss-Lobatto collocation approximation to solve (1 + 1) nonlinear Schrödinger equations. Also, the generalized Laguerre-Legendre collocation method has been successfully applied to initial-boundary value problems [29]. In [30], approximate solutions of nonlinear Klein-Gordon and Sine-Gordon equations were provided using the Chebyshev tau meshless scheme. For some recent developments on spectral methods, see [31–34].

Neutral functional-differential equations play an important role in the mathematical modeling of several phenomena. It is well known that most of delay differential equations cannot be solved exactly. Therefore, numerical methods would be presented and developed to get approximate solutions of these equations. In this direction, Ishiwata and Muroya [35] applied the rational approximation scheme for solving a class of delay differential equations. In [36], Chen and Wang implemented the variational iteration scheme to obtain an analytical solution of the neutral functional-differential equation. Very recently, Heydari et al. [37] proposed a new numerical algorithm based on the operational matrix formulation of Chebyshev cardinal functions for solving delay differential equations arising in electrodynamics. In this paper we propose a numerical solution for a new class of delay differential equations, namely, fractional neutral functional-differential equations (FNFDEs) with proportional delay.

The main aim of this paper is to design a suitable way to approximate a new class of functional-differential equations with fractional orders on the interval $(0, L)$ using spectral collocation method. The spectral shifted Jacobi-Gauss collocation (SJGC) approximation is proposed to obtain the numerical solution $u_N(t)$. The SJGC approximation, which is more reliable, is employed to obtain approximate solution of FNFDEs with leading fractional order θ $(m-1<\theta<m)$ and m initial conditions. We choose the $(N-m+1)$ nodes of the shifted Jacobi-Gauss interpolation on $(0, L)$ as suitable collocation nodes. The Legendre and Chebyshev collocation approximations can be obtained as special cases from our general approach. Finally, the validity and effectiveness of the method are demonstrated by solving two numerical examples. Numerical examples are presented in the form of tables and graphs to make comparisons with the results obtained by other methods and with the exact solutions more easier.

In the next section, we present an overview of shifted Jacobi polynomials and fractional calculus needed hereafter. Section 3 is devoted to present and implement the collocation scheme for solving FNFDEs with proportional delay using Jacobi polynomials. In Section 4, we introduce two numerical examples demonstrating the high accuracy and efficiency of the present numerical algorithm.

2. Preliminaries

Here, we state some preliminaries of fractional calculus [38] and some relevant properties of Jacobi polynomials. The most commonly used definition of fractional integral is the Riemann-Liouville operator.

Definition 1. The Riemann-Liouville fractional integral operator of order θ $(\theta>0)$ is defined as

$$J^{\theta} f(t)=\frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1} f(s)\,ds,\quad \theta>0,\ t>0,$$
$$J^{0} f(t)=f(t). \tag{1}$$

Definition 2. The Caputo fractional derivatives of order θ are defined as

$$\begin{aligned} D^{\theta} f(t) &= J^{m-\theta} D^{m} f(t) \\ &= \frac{1}{\Gamma(m-\theta)}\int_0^t (t-s)^{m-\theta-1}\frac{d^m}{ds^m} f(s)\,ds, \\ &\qquad m-1<\theta<m,\ t>0, \end{aligned} \tag{2}$$

where D^m is the classical differential operator of order m.

Also

$$D^{\theta} C = 0,\quad (C \text{ is aconstant}), \tag{3}$$

$$D^{\theta} t^{\mu} = \begin{cases} 0, & \text{for } \mu\in N_0,\ \mu<\lceil\theta\rceil, \\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu+1-\theta)} t^{\mu-\theta}, & \text{for } \mu\in N_0,\ \mu\geq\lceil\theta\rceil \text{ or } \mu\notin N,\ \mu>\lfloor\theta\rfloor, \end{cases} \tag{4}$$

where $N=\{1,2,\ldots\}$ and $N_0=\{0,1,2,\ldots\}$, while $\lceil\theta\rceil$ and $\lceil\theta\rceil$ are the floor and ceiling functions, respectively,

The Caputo's fractional differentiation is a linear operation, similar to the integer-order differentiation

$$D^{\theta}(\lambda f(t)+\eta g(t))=\lambda D^{\theta} f(t)+\eta D^{\theta} g(t), \tag{5}$$

where λ and η are constants.

Let $\nu>-1$, $\mu>-1$ and let $P_k^{(\nu,\mu)}(t)$ be the Jacobi polynomial of degree k; then we get

$$P_k^{(\nu,\mu)}(-t)=(-1)^k P_k^{(\nu,\mu)}(t),$$
$$P_k^{(\nu,\mu)}(-1)=\frac{(-1)^k\Gamma(k+\mu+1)}{k!\Gamma(\mu+1)}, \tag{6}$$
$$P_k^{(\nu,\mu)}(1)=\frac{\Gamma(k+\nu+1)}{k!\Gamma(\nu+1)}.$$

Besides,

$$D^m P_k^{(\nu,\mu)}(t)=2^{-m}\frac{\Gamma(m+k+\nu+\mu+1)}{\Gamma(k+\nu+\mu+1)} P_{k-m}^{(\nu+m,\mu+m)}(t). \tag{7}$$

Let $w^{(\nu,\mu)}(t)=(1-t)^{\nu}(1+t)^{\mu}$; then we define the weighted space $L^2_{w^{(\nu,\mu)}}[-1,1]$ as usual, equipped with the following inner product and norm:

$$(u,v)_{w^{(\nu,\mu)}}=\int_{-1}^{1} u(t)\,v(t)\,w^{(\nu,\mu)}(t)\,dt,$$
$$\|v\|_{w^{(\nu,\mu)}}=(v,v)^{1/2}_{w^{(\nu,\mu)}}. \tag{8}$$

The set of Jacobi polynomials forms a complete $L^2_{w^{\nu,\mu}}[-1,1]$-orthogonal system, and

$$\left\|P_k^{(\nu,\mu)}\right\|^2_{w^{(\nu,\mu)}} = h_k^{(\nu,\mu)} = \frac{2^{\nu+\mu+1}\Gamma(k+\nu+1)\Gamma(k+\mu+1)}{(2k+\nu+\mu+1)\Gamma(k+1)\Gamma(k+\nu+\mu+1)}. \tag{9}$$

Let us define the shifted Jacobi polynomial of degree k by $P_{L,k}^{(\nu,\mu)}(t) = P_k^{(\nu,\mu)}((2t/L)-1)$, $L>0$, and thanks to (6) and (7), yield

$$D^q P_{L,k}^{(\nu,\mu)}(0) = \frac{(-1)^{k-q}\Gamma(k+\mu+1)(k+\nu+\mu+1)_q}{L^q\Gamma(k-q+1)\Gamma(q+\mu+1)}, \tag{10}$$

$$D^m P_{L,k}^{(\nu,\mu)}(t) = \frac{\Gamma(m+k+\nu+\mu+1)}{L^m\Gamma(k+\nu+\mu+1)} P_{L,k-m}^{(\nu+m,\mu+m)}(t). \tag{11}$$

The shifted Jacobi polynomials of degree i on the interval $[0,L]$ are given by

$$\begin{aligned} P_{L,i}^{(\nu,\mu)}(t) &= \sum_{k=0}^{i}(-1)^{i-k}\Big(\big(\Gamma(i+\mu+1)\Gamma(i+k+\nu+\mu+1)t^k\big) \\ &\quad\times\big(\Gamma(k+\mu+1)\Gamma(i+\nu+\mu+1) \\ &\quad\times(i-k)!k!L^k\big)^{-1}\Big) \quad i=0,1,\ldots. \end{aligned} \tag{12}$$

Since the analytic form of $P_{L,i}^{(\nu,\mu)}(t)$ is given by (12), with the use of (4), (5), and (12), we obtain

$$\begin{aligned} D^\theta P_{L,i}^{(\nu,\mu)}(t) &= \sum_{k=0}^{i}(-1)^{i-k}\Big(\big(\Gamma(i+\mu+1)\Gamma(i+k+\nu+\mu+1)\big) \\ &\quad\times\big(\Gamma(k+\mu+1)\Gamma(i+\nu+\mu+1) \\ &\quad\times(i-k)!k!L^k\big)^{-1}\Big)D^\theta t^k \\ &= \sum_{k=\lceil\theta\rceil}^{i}(-1)^{i-k}\Big(\big(\Gamma(i+\mu+1)\Gamma(i+k+\nu+\mu+1)\big) \\ &\quad\times\big(\Gamma(k+\mu+1)\Gamma(i+\nu+\mu+1) \\ &\quad\times(i-k)!\Gamma(k-\theta+1)L^k\big)^{-1}\Big)t^{k-\theta}, \\ &\qquad i=\lceil\theta\rceil,\ldots,N. \end{aligned} \tag{13}$$

Now, approximate $t^{k-\theta}$ by $N+1$ terms of shifted Jacobi series; we have

$$t^{k-\theta} = \sum_{j=0}^{N} b_j P_{L,j}^{(\nu,\mu)}(t), \tag{14}$$

where b_j is given from (18) with $u(x)=t^{k-\theta}$, and

$$\begin{aligned} b_j &= \frac{L^{\nu+\mu-\theta+1}\ \Gamma(j+\mu+1)}{h_j^{(\nu,\mu)}\Gamma(j+\nu+\mu+1)} \\ &\quad\times\sum_{\ell=0}^{j}\Big(\big((-1)^{j-\ell}\Gamma(j+\ell+\nu+\mu+1)\Gamma(\nu+1) \\ &\quad\times\Gamma(\ell+k-\theta+\nu+\ell+1)\big) \\ &\quad\times\big(\Gamma(\ell+\mu+1)(j-\ell)!(\ell)! \\ &\quad\times\Gamma(\ell+k-\theta+\nu+\ell+2\Gamma)\big)^{-1}\Big). \end{aligned} \tag{15}$$

Employing (13)–(15) we get

$$D^\theta P_{L,i}^{(\nu,\mu)}(t) = \sum_{j=0}^{N} S_\theta(i,j) P_{L,j}^{(\nu,\mu)}(t), \quad i=\lceil\theta\rceil,\ldots,N, \tag{16}$$

where $S_\theta(i,j)=\sum_{k=\lceil\theta\rceil}^{i}\vartheta_{ijk}$, and

$$\begin{aligned} \vartheta_{ijk} &= \big((-1)^{i-k}L^{\nu+\mu-\theta+1}\Gamma(j+\mu+1) \\ &\quad\times\Gamma(i+\mu+1)\Gamma(i+k+\nu+\mu+1)\big) \\ &\quad\times\big(h_j^{(\nu,\mu)}\Gamma(j+\nu+\mu+1)\Gamma(k+\mu+1) \\ &\quad\times\Gamma(i+\nu+\mu+1)\Gamma(k-\theta+1)(i-k)!\big)^{-1} \\ &\quad\times\sum_{\ell=0}^{j}\Big(\big((-1)^{j-\ell}\Gamma(j+\ell+\nu+\mu+1)\Gamma(\nu+1) \\ &\quad\times\Gamma(\ell+k-\theta+\nu+\ell+1)\big) \\ &\quad\times\big(\Gamma(\ell+\mu+1)(j-\ell)!(\ell)! \\ &\quad\times\Gamma(\ell+k-\theta+\nu+\ell+2)\big)^{-1}\Big). \end{aligned} \tag{17}$$

A function $u(t)\in L^2_{w^{(\nu,\mu)}}(0,L)$ may be expressed in terms of shifted Jacobi polynomials as

$$u(t) = \sum_{j=0}^{\infty} a_j P_{L,j}^{(\nu,\mu)}(t),$$
$$a_j = \frac{1}{h_k^{(\nu,\mu)}}\int_0^L u(t)P_{L,j}^{(\nu,\mu)}(t)w^{(\nu,\mu)}(t)\,dt, \quad j=0,1,2,\ldots. \tag{18}$$

In practice, only the first $(N+1)$ terms shifted Jacobi polynomials are considered. Then we have

$$u_N(t) = \sum_{j=0}^{N} a_j P_{L,j}^{(\nu,\mu)}(t). \tag{19}$$

Next, let $w_L^{(\nu,\mu)}(t) = (L-t)^{\nu} t^{\mu}$; then we define the weighted space $L^2_{w_L^{(\nu,\mu)}}[0,L]$ in the usual way, with the following inner product and norm:

$$(u,v)_{w_L^{(\nu,\mu)}} = \int_0^L u(t)\, v(t)\, w_L^{(\nu,\mu)}(t)\, dt,$$
$$\|v\|_{w_L^{(\nu,\mu)}} = (v,v)^{1/2}_{w_L^{(\nu,\mu)}}. \quad (20)$$

The set of shifted Jacobi polynomials forms a complete $L^2_{w_L^{(\nu,\mu)}}[0,L]$-orthogonal system. Moreover, and due to (9), we have

$$\left\| P_{L,k}^{(\nu,\mu)} \right\|^2_{w_L^{(\nu,\mu)}} = \left(\frac{L}{2}\right)^{\nu+\mu+1} h_k^{(\nu,\mu)} = h_{L,k}^{(\nu,\mu)}. \quad (21)$$

3. Shifted Jacobi Collocation Approximation for FNFDEs

In this section, we propose the shifted Jacobi collocation method with the Jacobi-Gauss quadrature nodes to solve numerically the following FNFDEs with proportional delay:

$$D^{\theta}\left(u(t) + a(t)\, u(p_m t)\right)$$
$$= \mu u(t) + \sum_{n=0}^{m-1} b_n(t)\, D^{\gamma_n} u(p_n t) + f(t), \quad t \ge 0, \quad (22)$$

with the initial conditions

$$\sum_{n=0}^{m-1} c_{in} u^{(n)}(0) = \lambda_i, \quad i = 0,1,\ldots,m-1. \quad (23)$$

Here, a and b_n $(n = 0,1,\ldots,m-1)$ are given analytical functions, $m-1 < \theta \le m$, $0 < \gamma_0 < \gamma_1 < \cdots < \gamma_{m-1} < \theta$ and $\mu, p_n, c_{in}, \lambda_i$ denote given constants with $0 < p_n < 1$ $(n = 0,1,\ldots,m)$. By using the shifted Jacobi-Gauss collocation method [39], we can approximate the fractional neutral functional-differential equations with proportional delays, without any artificial boundary and variable transformation. Let us first introduce some basic notation that will be used in the sequel.

Now we introduce the Jacobi-Gauss-Lobatto quadratures in two different intervals $(-1,1)$, and $(0,L)$. Denoting by $t_{N,j}^{(\nu,\mu)} (t_{L,N,j}^{(\nu,\mu)})$, $0 \leqslant j \leqslant N$, and $\varpi_{N,j}^{(\nu,\mu)} (\varpi_{L,N,j}^{(\nu,\mu)})$, $(0 \le i \le N)$, the nodes and Christoffel numbers of the standard (shifted) Jacobi-Gauss-Lobatto quadratures on $(-1,1)$, $(0,L)$, respectively. Therefore, we can deduce that

$$t_{L,N,j}^{(\nu,\mu)} = \frac{L}{2}\left(t_{N,j}^{(\nu,\mu)} + 1\right), \quad 0 \le j \le N,$$
$$\varpi_{L,N,j}^{(\nu,\mu)} = \left(\frac{L}{2}\right)^{\nu+\mu+1} \varpi_{N,j}^{(\nu,\mu)}, \quad 0 \le j \le N. \quad (24)$$

Let $S_N(0,L)$ be the set of all polynomials of degree $\le N$; then, for any $\phi \in S_{2N+1}(0,L)$, we have

$$\int_0^L w_L^{(\nu,\mu)}(t)\, \phi(t)\, dt$$
$$= \left(\frac{L}{2}\right)^{\nu+\mu+1} \int_{-1}^{1} (1-t)^{\nu} (1+t)^{\mu} \phi\left(\frac{L}{2}(t+1)\right) dt$$
$$= \left(\frac{L}{2}\right)^{\nu+\mu+1} \sum_{j=0}^{N} \varpi_{N,j}^{(\nu,\mu)} \phi\left(\frac{L}{2}\left(t_{N,j}^{(\nu,\mu)} + 1\right)\right)$$
$$= \sum_{j=0}^{N} \varpi_{L,N,j}^{(\nu,\mu)} \phi\left(t_{L,N,j}^{(\nu,\mu)}\right). \quad (25)$$

We set

$$S_N(0,L) = \operatorname{span}\left\{P_{L,0}^{(\nu,\mu)}(t), P_{L,1}^{(\nu,\mu)}(t), \ldots, P_{L,N}^{(\nu,\mu)}(t)\right\}, \quad (26)$$

and the inner product and norm are defined as

$$(u,v)_{w_N^{(\nu,\mu)}} = \sum_{j=0}^{N} u\left(t_{L,N,j}^{(\nu,\mu)}\right) v\left(t_{L,N,j}^{(\nu,\mu)}\right) \varpi_{L,N,j}^{(\nu,\mu)},$$
$$\|u\|_{w_N^{(\nu,\mu)}} = \sqrt{(u,u)_{w_N^{(\nu,\mu)}}}. \quad (27)$$

Obviously,

$$(u,v)_{w_N^{(\nu,\mu)}} = (u,v)_{w^{(\nu,\mu)}}, \quad \forall u,v \in S_{2N+1}. \quad (28)$$

Thus, for any $u \in S_N(0,L)$, the norms $\|u\|_{w_N^{(\nu,\mu)}}$ and $\|u\|_{w_N^{(\nu,\mu)}}$ coincide.

Associating with this quadrature rule, we denote by $I_N^{P_L^{(\nu,\mu)}}$ the Jacobi-Gauss interpolation operator

$$I_N^{P_L^{(\nu,\mu)}} u\left(t_{L,N,j}^{(\nu,\mu)}\right) = u\left(t_{L,N,j}^{(\nu,\mu)}\right), \quad 0 \le k \le N. \quad (29)$$

The shifted Jacobi-Gauss collocation method for solving (22) and (23) is to seek $u_N(x) \in S_N(0,L)$, such that

$$D^{\theta}\left(u\left(t_{L,N-m,k}^{(\nu,\mu)}\right) + a\left(t_{L,N-m,k}^{(\nu,\mu)}\right) u\left(p_m t_{L,N-m,k}^{(\nu,\mu)}\right)\right)$$
$$= \mu u\left(t_{L,N-m,k}^{(\nu,\mu)}\right) + \sum_{n=0}^{m-1} b_n\left(t_{L,N-m,k}^{(\nu,\mu)}\right) D^{\gamma_n} u\left(p_n t_{L,N-m,k}^{(\nu,\mu)}\right)$$
$$+ f\left(t_{L,N-m,k}^{(\nu,\mu)}\right), \quad k = 0,1,\ldots,N-m,$$
$$\sum_{n=0}^{m-1} c_{in} u^{(n)}(0) = \lambda_i, \quad i = 0,1,\ldots,m-1. \quad (30)$$

We now derive the algorithm for solving (22) and (23). To do this, let

$$u_N(t) = \sum_{h=0}^{N} a_h P_{L,h}^{(\nu,\mu)}(t), \quad \mathbf{a} = (a_0, a_1, \ldots, a_N)^T. \quad (31)$$

We first approximate $D^{\theta}u(t)$ and $D^{\gamma_n}u(t)$, $n = 0, 1, \ldots, m-1$, using (31). By substituting this approximation in (22), we get

$$D^{\theta}\left(\sum_{h=0}^{N} a_h P_{L,h}^{(\nu,\mu)}(t) + a(t)\sum_{h=0}^{N} a_h P_{L,h}^{(\nu,\mu)}(p_m t)\right) = \mu\sum_{h=0}^{N} a_h P_{L,h}^{(\nu,\mu)}(t) + \sum_{n=0}^{m-1}\sum_{h=0}^{N} a_h b_n(t) D^{\gamma_n} P_{L,h}^{(\nu,\mu)}(p_n t) + f(t). \tag{32}$$

Making use of (16), we deduce that

$$\begin{aligned}&\sum_{h=0}^{N}\sum_{\sigma=0}^{N} a_h S_{\theta}(h,\sigma) P_{L,\sigma}^{(\nu,\mu)}(t) + a(t)\sum_{h=0}^{N}\sum_{\sigma=0}^{N} a_h S_{\theta}(h,\sigma) P_{L,\sigma}^{(\nu,\mu)}(p_m t)\\ &= \mu\sum_{h=0}^{N} a_h P_{L,h}^{(\nu,\mu)}(t) + \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{\sigma=0}^{N} a_h b_n(t) S_{\gamma_n}(h,\sigma) P_{L,\sigma}^{(\nu,\mu)}(p_n t) + f(t).\end{aligned} \tag{33}$$

Also, by substituting (31) in (23) we obtain

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M} a_{in} D^{(n)} P_{L,f}^{(\nu,\mu)}(0) = \lambda_i. \tag{34}$$

Now, we collocate (33) at the $(N-m+1)$ shifted Jacobi-Gauss interpolation points, yielding

$$\begin{aligned}&\sum_{h=0}^{N}\sum_{\sigma=0}^{N} a_h S_{\theta}(h,\sigma) P_{L,\sigma}^{(\nu,\mu)}\left(t_{L,N-m,k}^{(\nu,\mu)}\right) + a\left(t_{L,N-m,k}^{(\nu,\mu)}\right) \times \sum_{h=0}^{N}\sum_{\sigma=0}^{N} a_h S_{\theta}(h,\sigma) P_{L,\sigma}^{(\nu,\mu)}\left(p_m t_{L,N-m,k}^{(\nu,\mu)}\right)\\ &= \mu\sum_{h=0}^{N} a_h P_{L,h}^{(\nu,\mu)}\left(t_{L,N-m,k}^{(\nu,\mu)}\right) + \sum_{n=0}^{m-1}\sum_{h=0}^{N}\sum_{\sigma=0}^{N} a_h b_n\left(t_{L,N-m,k}^{(\nu,\mu)}\right) S_{\gamma_n} \times (h,\sigma) P_{L,\sigma}^{(\nu,\mu)}\left(p_n t_{L,N-m,k}^{(\nu,\mu)}\right) + f\left(t_{L,N-m,k}^{(\nu,\mu)}\right).\end{aligned} \tag{35}$$

Next (34), after using (10), can be written as

$$\sum_{n=0}^{m-1}\sum_{f=0}^{M}(-1)^q a_{in}\frac{(-1)^{f-q}\Gamma(f+\mu+1)(f+\nu+\mu+1)_q}{L^q\Gamma(f-q+1)\Gamma(q+\mu+1)} = \lambda_i. \tag{36}$$

Table 1: Absolute errors using SJGC method at $N = 16$ for Example 3.

x	$\nu = \mu = -1/2$	$\nu = \mu = 0$	$\nu = \mu = 1/2$
0.5	$1.040 \cdot 10^{-3}$	$2.799 \cdot 10^{-3}$	$2.393 \cdot 10^{-4}$
1.0	$1.025 \cdot 10^{-3}$	$4.986 \cdot 10^{-3}$	$2.967 \cdot 10^{-4}$
1.5	$4.512 \cdot 10^{-3}$	$4.006 \cdot 10^{-3}$	$6.043 \cdot 10^{-4}$
2.0	$3.660 \cdot 10^{-3}$	$2.487 \cdot 10^{-3}$	$1.693 \cdot 10^{-4}$
2.5	$7.554 \cdot 10^{-3}$	$3.817 \cdot 10^{-3}$	$8.245 \cdot 10^{-4}$
3.0	$2.356 \cdot 10^{-3}$	$2.975 \cdot 10^{-3}$	$5.338 \cdot 10^{-4}$
3.5	$8.775 \cdot 10^{-3}$	$8.662 \cdot 10^{-3}$	$6.166 \cdot 10^{-4}$
4.0	$4.706 \cdot 10^{-3}$	$5.449 \cdot 10^{-3}$	$3.453 \cdot 10^{-4}$
4.5	$3.180 \cdot 10^{-3}$	$2.261 \cdot 10^{-3}$	$8.389 \cdot 10^{-5}$
5.0	$6.099 \cdot 10^{-3}$	$1.002 \cdot 10^{-2}$	$2.260 \cdot 10^{-3}$

Finally, relations (35) and (36) generate $(N+1)$ set of algebraic equations which can be solved for the unknown coefficients a_j, $j = 0, 1, 2, \ldots, N$, by using any standard solver technique.

4. Numerical Results

In this section, two fractional neutral functional-differential equations with proportional delays are solved by the SJGC method. We implement the method presented in this paper for these two examples to demonstrate the accuracy and capability of the proposed algorithm.

Example 3. Consider the following FNFDEs with proportional delay:

$$\begin{aligned}u^{1/2}(t) &= -u(t) + \frac{1}{4}u\left(\frac{t}{3}\right) + \frac{1}{3}u^{1/2}\left(\frac{t}{3}\right) + g(t),\\ u(0) &= 1, \quad t \in [0,5],\end{aligned} \tag{37}$$

where

$$\begin{aligned}g(t) = {}&\frac{1}{\Gamma(1/2)}\int_0^t (t-s)^{-1/2} e^s ds\\ &+ e^t - \frac{1}{4}e^{t/3} - \frac{1}{3\Gamma(1/2)}\int_0^t (t-s)^{-1/2} e^{s/3} ds,\end{aligned} \tag{38}$$

and the exact solution is given by $u(t) = e^t$.

Table 1 lists the results obtained by the shifted Jacobi collocation method in terms of absolute errors at $N = 16$ with $\nu = \mu = -1/2$ (first kind shifted Chebyshev collocation method), $\nu = \mu = 0$ (shifted Legendre collocation method), and $\nu = \mu = 1/2$ (second kind shifted Chebyshev collocation method). In the case of $\nu = 1/2$, $\mu = -1/2$, the approximate solution by the presented method is shown in Figure 1, to make it easier to compare with the analytic solution.

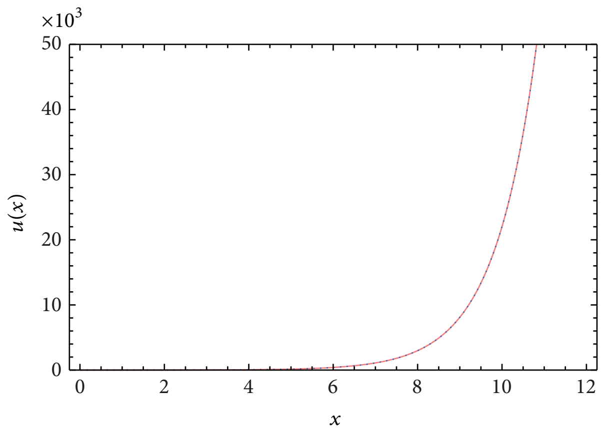

FIGURE 1: Graph of exact solution and approximate solution for $\nu = 1/2$, $\mu = -1/2$ at $N = 16$ for Example 3.

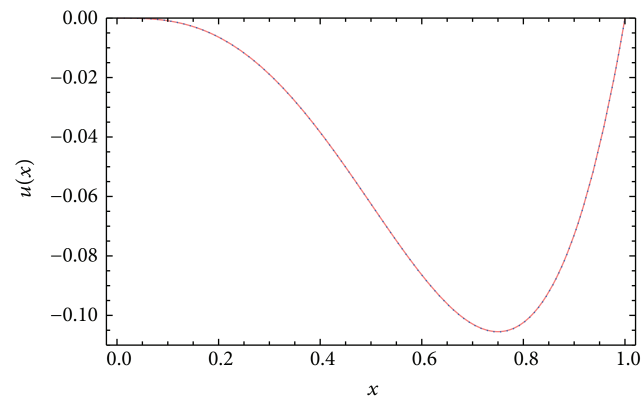

FIGURE 2: Graph of exact solution and approximate solution for $\nu = \mu = 1/2$ at $N = 16$ for Example 4.

TABLE 2: Absolute errors using SJGC method at $N = 16$ for Example 4.

x	$\nu = \mu = 3/2$	$\nu = \mu = 0$	$\nu = \mu = -1/2$
0.1	$1.462 \cdot 10^{-5}$	$3.794 \cdot 10^{-5}$	$2.092 \cdot 10^{-5}$
0.2	$7.095 \cdot 10^{-5}$	$1.641 \cdot 10^{-4}$	$9.636 \cdot 10^{-5}$
0.3	$1.717 \cdot 10^{-4}$	$3.733 \cdot 10^{-4}$	$1.947 \cdot 10^{-4}$
0.4	$3.355 \cdot 10^{-4}$	$6.819 \cdot 10^{-4}$	$3.394 \cdot 10^{-4}$
0.5	$5.649 \cdot 10^{-4}$	$1.134 \cdot 10^{-3}$	$6.780 \cdot 10^{-4}$
0.6	$8.468 \cdot 10^{-4}$	$1.705 \cdot 10^{-3}$	$1.203 \cdot 10^{-3}$
0.7	$1.183 \cdot 10^{-3}$	$2.411 \cdot 10^{-3}$	$1.650 \cdot 10^{-3}$
0.8	$1.592 \cdot 10^{-3}$	$3.201 \cdot 10^{-3}$	$1.934 \cdot 10^{-3}$
0.9	$2.083 \cdot 10^{-3}$	$4.128 \cdot 10^{-3}$	$2.295 \cdot 10^{-3}$
1.0	$2.681 \cdot 10^{-3}$	$1.790 \cdot 10^{-3}$	$2.903 \cdot 10^{-3}$

Example 4. Consider the following FNFDEs with proportional delay:

$$\begin{aligned} u^{5/2}(t) = {} & u(t) + u^{1/2}\left(\frac{t}{2}\right) + u^{3/2}\left(\frac{t}{3}\right) \\ & + \frac{1}{2}u^{5/2}\left(\frac{t}{4}\right) + \frac{\Gamma(5)}{\Gamma(5/2)}t^{3/2} - \frac{\Gamma(4)}{\Gamma(3/2)}t^{1/2} - x^4 \\ & + t^3 - \frac{\Gamma(5)}{\Gamma(9/2)}\left(\frac{t}{2}\right)^{7/2} + \frac{\Gamma(4)}{\Gamma(7/2)}\left(\frac{t}{2}\right)^{5/2} \\ & - \frac{\Gamma(5)}{\Gamma(7/2)}\left(\frac{t}{3}\right)^{5/2} + \frac{\Gamma(4)}{\Gamma(5/2)}\left(\frac{t}{3}\right)^{3/2} \\ & - \frac{\Gamma(5)}{2\Gamma(5/2)}\left(\frac{t}{4}\right)^{3/2} + \frac{\Gamma(4)}{2\Gamma(3/2)}\left(\frac{t}{4}\right)^{1/2}, \quad t \in [0,1], \end{aligned} \tag{39}$$

subject to

$$u(0) = 0, \qquad u'(0) = 0, \qquad u''(0) = 0, \tag{40}$$

with exact solution $u(t) = t^4 - t^3$.

In Table 2, we list the absolute errors obtained by the shifted Jacobi collocation method, with several values of ν, μ and at $N = 16$. It is clear that, for all Jacobi polynomials parameters, the results are stable. Meanwhile, Figure 2 presents the SJGC solution with $\nu = \mu = 1/2$ at $N = 16$ and exact solution, which are found to be in excellent agreement.

5. Conclusion

In this paper, we have proposed a numerical algorithm to solve a class of fractional delay differential equations. The Jacobi collocation approximation was developed to solve this problem. A number of collocation techniques can be obtained as special cases from the proposed technique. Numerical results were given to demonstrate the accuracy and applicability of the presented method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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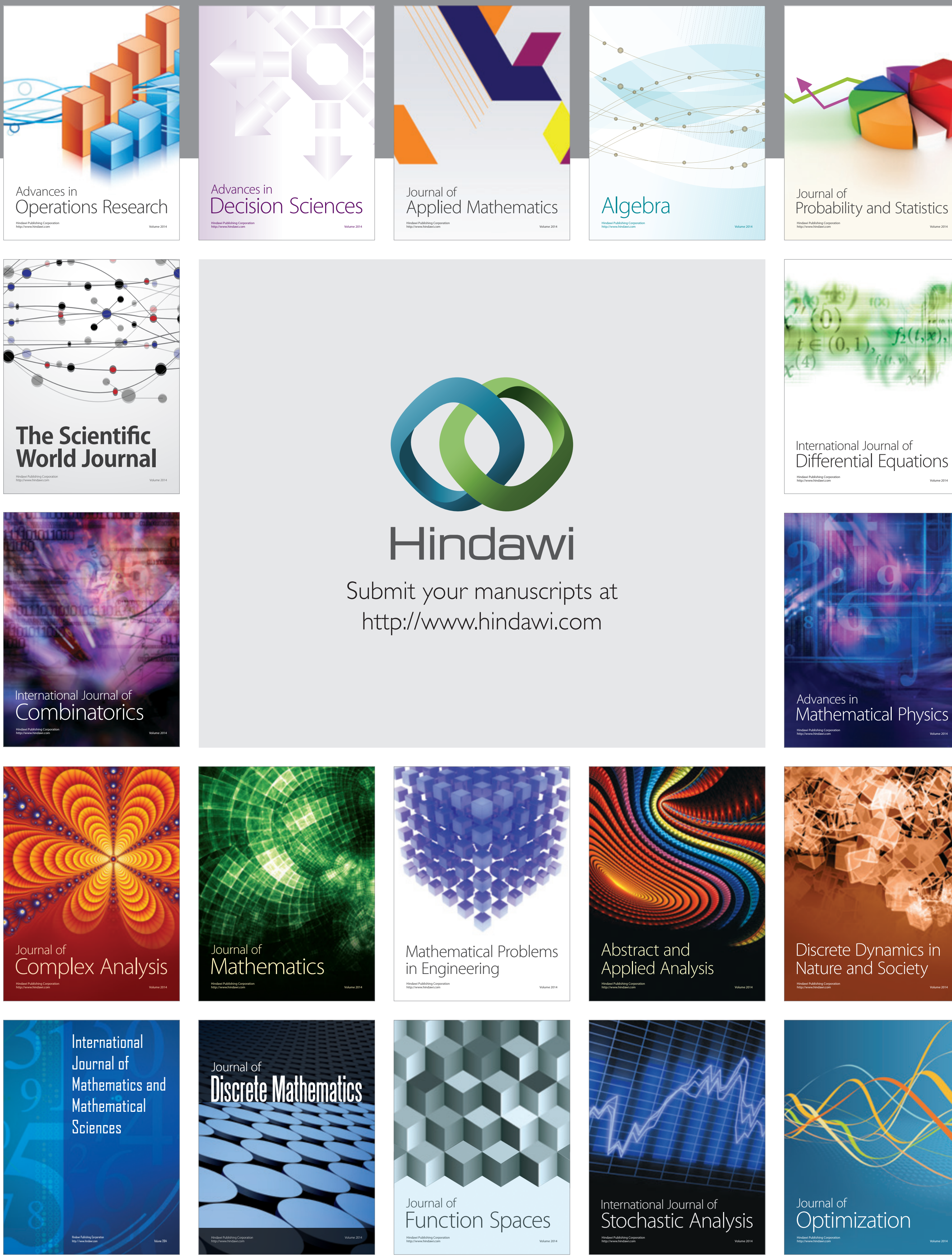

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