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Research Article

MHD Equations with Regularity in One Direction

Zujin Zhang

School of Mathematics and Computer Science, Gannan Normal University, Ganzhou 341000, China

Correspondence should be addressed to Zujin Zhang; zhangzujin361@163.com

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We consider the 3D MHD equations and prove that if one directional derivative of the fluid velocity, say, $\partial_3\mathbf{u} \in L^p(0,T;L^q(\mathbf{R}^3))$, with $2/p + 3/q = \gamma \in [1, 3/2)$, $3/\gamma \le q \le 1/(\gamma-1)$, then the solution is in fact smooth. This improves previous results greatly.

1. Introduction

We consider the following three-dimensional (3D) magnetohydrodynamic (MHD) equations:

$$\begin{gathered}
\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\mathbf{b}\cdot\nabla)\mathbf{b} - \Delta\mathbf{u} + \nabla p = \mathbf{0},\\
\partial_t\mathbf{b} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \Delta\mathbf{b} = \mathbf{0},\\
\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{b} = 0,\\
\mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{b}(0) = \mathbf{b}_0.
\end{gathered} \tag{1}$$

Here, $\mathbf{u}$ and $\mathbf{b}$ are the fluid velocity and magnetic fields, respectively; $\mathbf{u}_0$ and $\mathbf{b}_0$ are the corresponding initial data satisfying the compatibility conditions

$$\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0; \tag{2}$$

p is a scalar pressure. The MHD system (1) is a mathematical model for electronically conducting fluids such as plasma and salted water, which governs the dynamics of the fluid velocity and the magnetic fields.

There have been extensive studies on (1). In particular, Duvaut and Lions [1] constructed a class of global weak solutions with finite energy, which is similar to the Leray-Hopf weak solutions (see [2, 3]) for the Navier-Stokes equations ($\mathbf{b} = \mathbf{0}$ in (1)). However, the issue of uniqueness and regularity for a given weak solution remains a challenging open problem. Initiated by He and Xin [4] and Zhou [5], a lot of literatures have been devoted to the study of conditions which would ensure the smoothness of the solutions to (1) and which involve only the fluid velocity field. Such conditions are called regularity criteria. The readers, who are interested in the regularity criteria for the Navier-Stokes equations, are referred to [4–18] and references cited therein.

For the Navier-Stokes equations, the authors have established that the regularity of the velocity in one direction (say, $\partial_3\mathbf{u}$), one component of the velocity (say, u_3), or some other partial components of the velocity, velocity gradient, velocity Hessian, vorticity, pressure, and so forth, would guarantee the regularity of the weak solutions; see [19–29] and references therein. Many of these regularity criteria have been proved to be enjoyed by the MHD equations (1); see [30–33]. However, due to the strong coupling of the fluid velocity and the magnetic fields, the scaling dimensions for the MHD equations are not as good (large) as that for the Navier-Stokes equations.

In this paper, we would like to improve the regularity criterion

$$\partial_3\mathbf{u} \in L^p\left(0,T;L^q\left(\mathbf{R}^3\right)\right), \quad \text{with } \frac{2}{p} + \frac{3}{q} = 1,\ 3 < q \le \infty, \tag{3}$$

shown in [30]. That is, we enlarge the scaling dimension from 1 to (almost) 3/2. Precisely, we show that the condition

$$\begin{gathered}
\partial_3\mathbf{u} \in L^p\left(0,T;L^q\left(\mathbf{R}^3\right)\right), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \gamma \in \left[1, \frac{3}{2}\right),\\
\frac{3}{\gamma} \le q \le \frac{1}{\gamma-1},
\end{gathered} \tag{4}$$

is enough to ensure the smoothness of the solution. The key idea is a multiplicative Sobolev inequality, which is in spirit similar to that in [20]; see Lemma 2.

The rest of this paper is organized as follows. In Section 2, we recall the weak formulation of (1) and establish the fundamental Sobolev inequality. Section 3 is devoted to stating and proving the main result.

2. Preliminaries

In this section, we first recall the weak formulation of (1).

Definition 1. Let $(\mathbf{u}_0, \mathbf{b}_0) \in L^2(\mathbf{R}^3)$ satisfying $\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0$, $T > 0$ be given. A measurable pair $(\mathbf{u}_0, \mathbf{b}_0)$ on $(0, T)$ is said to be a weak solution to (1) provided that

(1) $(\mathbf{u}, \mathbf{b}) \in L^\infty(0, T; L^2(\mathbf{R}^3)) \cap L^2(0, T; H^1(\mathbf{R}^3))$;

(2) $(1)_{1,2}$ are satisfied in the sense of distributions;

(3) the energy inequality is given as

$$\begin{aligned} &\|\mathbf{u}(t)\|_{L^2(\mathbf{R}^3)}^2 + \|\mathbf{b}(t)\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + 2\int_0^t \left(\|\nabla\mathbf{u}(s)\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla\mathbf{b}(s)\|_{L^2(\mathbf{R}^3)}^2\right) ds \\ &\leq \|\mathbf{u}_0\|_{L^2(\mathbf{R}^3)}^2 + \|\mathbf{b}_0\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{5}$$

for all $t \in [0, T]$.

Then a fundamental Sobolev inequality is given.

Lemma 2. *Suppose that $f, g \in C_c^\infty(\mathbf{R}^3)$; then, one has*

$$\begin{aligned} &\int_{\mathbf{R}^3} |f|^2 |g|^2 dx \\ &\leq C\|f\|_{L^\alpha(\mathbf{R}^3)}^{(2r-1)/r} \|\partial_3 f\|_{L^q(\mathbf{R}^3)}^{1/r} \|g\|_{L^2(\mathbf{R}^3)}^{2(r-1)/r} \|(\partial_1, \partial_2) g\|_{L^2(\mathbf{R}^3)}^{2/r}, \end{aligned} \tag{6}$$

where C is a generic constant independent of f and g; $1 \leq \alpha$, $q \leq \infty$, and $1 < r \leq \infty$ satisfy

$$\frac{2r-1}{\alpha} + \frac{1}{q} = 1. \tag{7}$$

Proof. Consider the following

$$\begin{aligned} &\int_{\mathbf{R}^3} |f|^2 |g|^2 dx \\ &\leq \int_{\mathbf{R}^2} \left[\max_{x_3} |f|^2 \cdot \int_{\mathbf{R}} |g|^2 dx_3\right] dx_1\, dx_2 \\ &\leq \left[\int_{\mathbf{R}^2} \max_{x_3} |f|^{2r} dx_1\, dx_2\right]^{1/r} \\ &\quad \cdot \left[\int_{\mathbf{R}^2} \left(\int_{\mathbf{R}} |g|^2 dx_3\right)^{r/(r-1)} dx_1\, dx_2\right]^{(r-1)/r} \\ &\quad (\text{Hölder inequality}) \\ &\leq C\left[\int_{\mathbf{R}^3} |f|^{2r-1} |\partial_3 f|\, dx\right]^{1/r} \\ &\quad \cdot \int_{\mathbf{R}} \left(\int_{\mathbf{R}^2} |g|^{2r/(r-1)} dx_1\, dx_2\right)^{(r-1)/r} dx_3 \\ &\quad (\text{Minkowski inequality}) \\ &\leq C\|f\|_{L^\alpha(\mathbf{R}^3)}^{(2r-1)/r} \|\partial_3 f\|_{L^q(\mathbf{R}^3)}^{1/r} \|g\|_{L^2(\mathbf{R}^3)}^{2(r-1)/r} \|(\partial_1, \partial_2) g\|_{L^2(\mathbf{R}^3)}^{2/r}, \end{aligned} \tag{8}$$

where in the last inequality we have used Hölder inequality with

$$\frac{2r-1}{\alpha} + \frac{1}{q} = 1 \tag{9}$$

and the Gagliardo-Nirenberg inequality. □

3. The Main Result and Its Proof

In this section, we state and prove our main regularity criterion.

Theorem 3. *Let $(\mathbf{u}_0, \mathbf{b}_0) \in L^2(\mathbf{R}^3)$ satisfying $\nabla\cdot\mathbf{u}_0 = \nabla\cdot\mathbf{b}_0 = 0$, $T > 0$ be given. Assume that the measurable pair $(\mathbf{u}, \mathbf{b})$ is a weak solution as in Definition 1 on $(0, T)$. If*

$$\begin{aligned} &\partial_3 \mathbf{u} \in L^p\left(0, T; L^q\left(\mathbf{R}^3\right)\right), \quad \text{with } \frac{2}{p} + \frac{3}{q} = \gamma \in \left[1, \frac{3}{2}\right), \\ &\qquad \frac{3}{\gamma} \leq q \leq \frac{1}{\gamma - 1}, \end{aligned} \tag{10}$$

then $(\mathbf{u}, \mathbf{b}) \in C^\infty((0, T) \times \mathbf{R}^3)$.

Proof. For any $\varepsilon \in (0, T)$, we can find a $\delta \in (0, \varepsilon)$ such that

$$(\nabla\mathbf{u}(\delta), \nabla\mathbf{b}(\delta)) \in L^2\left(\mathbf{R}^3\right) \tag{11}$$

since $(\mathbf{u}, \mathbf{b}) \in L^2(0, T; H^1(\mathbf{R}^3))$ as in Definition 1. Our strategy is to show that under condition (10) the weak solution is in fact strong; that is,

$$(\mathbf{u}, \mathbf{b}) \in L^\infty\left(\delta, T; H^1\left(\mathbf{R}^3\right)\right) \cap L^2\left(\delta, T; H^2\left(\mathbf{R}^3\right)\right), \tag{12}$$

which would imply the smoothness of the solution via standard energy estimates and Sobolev embeddings. Due to the arbitrariness of ε, we complete the proof.

To prove (12), we multiply $(1)_1$ by $-\Delta\mathbf{u}$ and $(1)_2$ by $-\Delta\mathbf{b}$ to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] + \|\Delta\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 + \|\Delta\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad = \int_{\mathbf{R}^3} [(\mathbf{u}\cdot\nabla)\mathbf{u}] \cdot \Delta\mathbf{u} dx - \int_{\mathbf{R}^3} [(\mathbf{b}\cdot\nabla)\mathbf{b}] \cdot \Delta\mathbf{u} dx \\ &\quad\quad + \int_{\mathbf{R}^3} [(\mathbf{u}\cdot\nabla)\mathbf{b}] \cdot \Delta\mathbf{b} dx - \int_{\mathbf{R}^3} [(\mathbf{b}\cdot\nabla)\mathbf{u}] \cdot \Delta\mathbf{b} dx. \end{aligned} \tag{13}$$

Integration by parts formula together with the divergence-free conditions $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0$ yields

$$\begin{aligned} &-\int_{\mathbf{R}^3} [(\mathbf{b}\cdot\nabla)\mathbf{b}]\cdot\Delta\mathbf{u}dx - \int_{\mathbf{R}^3} [(\mathbf{b}\cdot\nabla)\mathbf{u}]\cdot\Delta\mathbf{b}dx \\ &= \int_{\mathbf{R}^3} [(\partial_i\mathbf{b}\cdot\nabla)\mathbf{b}]\cdot\partial_i\mathbf{u}dx + \int_{\mathbf{R}^3} [(\partial_i\mathbf{b}\cdot\nabla)\mathbf{u}]\cdot\partial_i\mathbf{b}dx \\ &= -\int_{\mathbf{R}^3} [(\Delta\mathbf{b}\cdot\nabla)\mathbf{b} + (\partial_i\mathbf{b}\cdot\nabla)\partial_i\mathbf{b}]\cdot\mathbf{u}dx \\ &\quad - \int_{\mathbf{R}^3} [(\partial_i\mathbf{b}\cdot\nabla)\partial_i\mathbf{b}]\cdot\mathbf{u}dx, \end{aligned} \tag{14}$$

where we use the summation convention; that is, the repeated index (say, i here) is automatically summed over $\{1, 2, 3\}$.

Substituting (14) into (13) and using a simple Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] + \|\Delta\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 + \|\Delta\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2 \\ &\le \int_{\mathbf{R}^3} |\mathbf{u}|\cdot|\nabla\mathbf{u}|\cdot|\Delta\mathbf{u}|\,dx + 4\int_{\mathbf{R}^3} |\mathbf{u}|\cdot|\nabla\mathbf{b}|\cdot\left|\nabla^2\mathbf{b}\right|dx \\ &\le \frac{1}{2}\int_{\mathbf{R}^3} |\mathbf{u}|^2\cdot|\nabla\mathbf{u}|^2dx + \|\Delta\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + C_\varepsilon\int_{\mathbf{R}^3} |\mathbf{u}|^2\cdot|\nabla\mathbf{b}|^2dx + \varepsilon\left\|\nabla^2\mathbf{b}\right\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{15}$$

where $0 < \varepsilon \ll 1$ is to be determined later on.

Due to the Calderón-Zygmund inequality,

$$\left\|\nabla^2\mathbf{b}\right\|_{L^2(\mathbf{R}^3)}^2 = \sum_{i,j=1}^{3}\left\|R_iR_j\Delta\mathbf{b}\right\|_{L^2(\mathbf{R}^3)}^2 \le C_1\|\Delta\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2, \tag{16}$$

(R_j being the Riesz transform) we may take that the ε in (15) equals $1/2C_1$ to get

$$\begin{aligned} &\frac{d}{dt}\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 + \|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] + \|\Delta\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 + \|\Delta\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2 \\ &\le C\int_{\mathbf{R}^3} |\mathbf{u}|^2\cdot|\nabla\mathbf{u}|^2dx + C\int_{\mathbf{R}^3} |\mathbf{u}|^2\cdot|\nabla\mathbf{b}|^2dx \equiv I_1 + I_2. \end{aligned} \tag{17}$$

To further bound I_1, I_2, we introduce some notations. Denote

$$r = \frac{5}{2} - \gamma, \tag{18}$$

$$\alpha = \frac{2q(2-\gamma)}{q-1}. \tag{19}$$

Then, by (10), we have

$$r \in (1, \infty), \qquad \alpha \in [2, 6]. \tag{20}$$

Invoking Lemma 2 with q as in (10), r as in (18), and α as in (19), we may estimate I_1 as

$$\begin{aligned} I_1 &= C\int_{\mathbf{R}^3} |\mathbf{u}|^2\cdot|\nabla\mathbf{u}|^2dx \\ &\le C\|\mathbf{u}\|_{L^\alpha(\mathbf{R}^3)}^{(2r-1)/r}\|\partial_3\mathbf{u}\|_{L^q(\mathbf{R}^3)}^{1/r}\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^{2(r-1)/r}\left\|\nabla^2\mathbf{u}\right\|_{L^2(\mathbf{R}^3)}^{2/r} \\ &\le C\|\mathbf{u}\|_{L^\alpha(\mathbf{R}^3)}^{(2r-1)/(r-1)}\|\partial_3\mathbf{u}\|_{L^q(\mathbf{R}^3)}^{1/(r-1)}\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{u}\right\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{21}$$

where C_1 is as in (16). Applying Hölder inequality with

$$\begin{aligned} &\frac{2q(r-1)-(q\gamma-3)}{2q(r-1)} + \frac{q\gamma-3}{2q(r-1)} = 1, \\ &\frac{q\gamma-3}{2q(r-1)} = \frac{q\gamma-3}{q(3-2\gamma)} \in [0,1] \quad (\text{by (18) and (10)}), \end{aligned} \tag{22}$$

(21) becomes

$$I_1 \le C\|\mathbf{u}\|_{L^6(\mathbf{R}^3)}^2\|\partial_3\mathbf{u}\|_{L^q(\mathbf{R}^3)}^{1/(r-1)}\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 + \frac{1}{2C_1}\left\|\nabla^2\mathbf{u}\right\|_{L^2(\mathbf{R}^3)}^2, \tag{23}$$

when $(q\gamma-3)/2q(r-1) = 0$, or

$$\begin{aligned} I_1 &\le C\Big(\|\mathbf{u}\|_{L^\alpha(\mathbf{R}^3)}^{((2r-1)/(r-1))\cdot(2q(r-1)/(2q(r-1)-(q\gamma-3)))} \\ &\qquad + \|\partial_3\mathbf{u}\|_{L^q(\mathbf{R}^3)}^{(1/(r-1))\cdot(2q(r-1)/(q\gamma-3))}\Big)\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{u}\right\|_{L^2(\mathbf{R}^3)}^2 \\ &= C\left(\|\mathbf{u}\|_{L^\alpha(\mathbf{R}^3)}^{2q(2r-1)/(2q(r-1)-(q\gamma-3))} + \|\partial_3\mathbf{u}\|_{L^q(\mathbf{R}^3)}^{2q/(q\gamma-3)}\right)\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{u}\right\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{24}$$

when $(q\gamma-3)/2q(r-1) \in (0,1)$, or

$$\begin{aligned} I_1 &\le C\|\mathbf{u}\|_{L^2(\mathbf{R}^3)}^{(2r-1)/(r-1)}\|\partial_3\mathbf{u}\|_{L^q(\mathbf{R}^3)}^{2q/(q\gamma-3)}\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{u}\right\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{25}$$

when $(q\gamma-3)/2q(r-1) = 1$.

Similarly, I_2 can be dominated as

$$\begin{aligned} I_2 &\le C\|\mathbf{u}\|_{L^6(\mathbf{R}^3)}^2\|\partial_3\mathbf{u}\|_{L^q(\mathbf{R}^3)}^{1/(r-1)}\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{b}\right\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{26}$$

when $(q\gamma-3)/2q(r-1)=0$, or

$$\begin{aligned} I_2 &\le C\left(\|\mathbf{u}\|_{L^\alpha(\mathbf{R}^3)}^{((2r-1)/(r-1))\cdot(2q(r-1)/(2q(r-1)-(q\gamma-3)))} + \left\|\partial_3\mathbf{u}\right\|_{L^q(\mathbf{R}^3)}^{(1/(r-1))\cdot(2q(r-1)/(q\gamma-3))}\right)\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{b}\right\|_{L^2(\mathbf{R}^3)}^2 \\ &= C\left(\|\mathbf{u}\|_{L^\alpha(\mathbf{R}^3)}^{2q(2r-1)/(2q(r-1)-(q\gamma-3))} + \left\|\partial_3\mathbf{u}\right\|_{L^q(\mathbf{R}^3)}^{2q/(q\gamma-3)}\right)\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{b}\right\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{27}$$

when $(q\gamma-3)/2q(r-1)\in(0,1)$, or

$$\begin{aligned} I_2 &\le C\|\mathbf{u}\|_{L^2(\mathbf{R}^3)}^{(2r-1)/(r-1)}\left\|\partial_3\mathbf{u}\right\|_{L^q(\mathbf{R}^3)}^{2q/(q\gamma-3)}\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2 \\ &\quad + \frac{1}{2C_1}\left\|\nabla^2\mathbf{b}\right\|_{L^2(\mathbf{R}^3)}^2, \end{aligned} \tag{28}$$

when $(q\gamma-3)/2q(r-1)=1$.

Thus, if $q=3/\gamma$, then, combing (23) and (26), we deduce from (17) that

$$\begin{aligned} &\frac{d}{dt}\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] \\ &\quad + \frac{1}{2}\left[\|\Delta\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\Delta\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] \\ &\le C\|\mathbf{u}\|_{L^6(\mathbf{R}^3)}^2\left\|\partial_3\mathbf{u}\right\|_{L^q(\mathbf{R}^3)}^{1/(r-1)} \\ &\quad \times\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right]. \end{aligned} \tag{29}$$

Applying Gronwall inequality and noting that $\mathbf{u}\in L^2(\delta,T;H^1(\mathbf{R}^3))$ as in Definition 1, we obtain (12) as desired.

If $3/\gamma<q<1/(\gamma-1)$, we gather (24) and (27) into (17) to get

$$\begin{aligned} &\frac{d}{dt}\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] \\ &\quad + \frac{1}{2}\left[\|\Delta\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\Delta\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] \\ &\le C\left[\|\mathbf{u}\|_{L^\alpha(\mathbf{R}^3)}^{2q(2r-1)/(2q(r-1)-(q\gamma-3))}+\left\|\partial_3\mathbf{u}\right\|_{L^q(\mathbf{R}^3)}^p\right] \\ &\quad \times\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right]. \end{aligned} \tag{30}$$

To deduce (12) by applying Gronwall inequality to (30), we need only to show that

$$\mathbf{u}\in L^{2q(2r-1)/(2q(r-1)-(q\gamma-3))}\left(\delta,T;L^\alpha\left(\mathbf{R}^3\right)\right). \tag{31}$$

This is indeed true. First, simple interpolation inequality together with the fact that

$$\mathbf{u}\in L^\infty\left(\delta,T;L^2\left(\mathbf{R}^3\right)\right)\cap L^2\left(\delta,T;H^1\left(\mathbf{R}^3\right)\right) \tag{32}$$

yields

$$\mathbf{u}\in L^a\left(\delta,T;L^b\left(\mathbf{R}^3\right)\right),\quad \text{with } \frac{2}{a}+\frac{3}{b}=\frac{3}{2},\ 2\le b\le 6. \tag{33}$$

Second, the integrability indices as in (31) satisfy

$$\begin{aligned} \frac{2q(r-1)-(q\gamma-3)}{q(2r-1)}+\frac{3}{\alpha} &= \frac{q(3-2\gamma)-(q\gamma-3)}{2q(2-\gamma)} \\ &\quad + \frac{3(q-1)}{2q(2-\gamma)}=\frac{3}{2}, \end{aligned} \tag{34}$$

and $\alpha\in(2,6)$, by (18) and (19).

If, however, $q=1/(\gamma-1)$, then $(q\gamma-3)/2q(r-1)=1$, and, combining (25) and (28), we deduce from (17) that

$$\begin{aligned} &\frac{d}{dt}\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] \\ &\quad + \frac{1}{2}\left[\|\Delta\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\Delta\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right] \\ &\le C\|\mathbf{u}\|_{L^2(\mathbf{R}^3)}^{(2r-1)/(r-1)}\left\|\partial_3\mathbf{u}\right\|_{L^q(\mathbf{R}^3)}^p\left[\|\nabla\mathbf{u}\|_{L^2(\mathbf{R}^3)}^2+\|\nabla\mathbf{b}\|_{L^2(\mathbf{R}^3)}^2\right], \end{aligned} \tag{35}$$

where we recall $p=2q/(q\gamma-3)$ from (10). Applying Gronwall inequality and noting that $\mathbf{u}\in L^\infty(\delta,T;L^2(\mathbf{R}^3))$ as in Definition 1, we obtain (12) as desired.

The proof of Theorem 3 is completed. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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