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Research Article

New Results on Impulsive Functional Differential Equations with Infinite Delays

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We investigate the stability for a class of impulsive functional differential equations with infinite delays by using Lyapunov functions and Razumikhin-technique. Some new Razumikhin-type theorems on stability are obtained, which shows that impulses do contribute to the system's stability behavior. An example is also given to illustrate the importance of our results.

1. Introduction

Impulsive differential equations have attracted the interest of many researchers in recent years. It arises naturally from a wide variety of applications such as orbital transfer of satellite, ecosystems management, and threshold theory in biology. There has been a significant development in the theory of impulsive differential equations in the past several years ago, and various interesting results have been reported; see [1–4]. Recently, systems with impulses and time delay have received significant attention [5–16]. In fact, the system stability and convergence properties are strongly affected by time delays, which are often encountered in many industrial and natural processes due to measurement and computational delays, transmission, and transport lags. In [5, 6, 8], the authors considered the stability of impulsive differential equations with finite delay and got some results. In [7], by using Lyapunov functions and Razumikhin technique, some Razumikhin-type theorems on stability are obtained for a class of impulsive functional differential equations with infinite-delay. However, not much has been developed in the direction of the stability theory of impulsive functional differential systems, especially for infinite delays impulsive functional differential systems. As we know, there are a number of difficulties that one must face in developing the corresponding theory of impulsive functional differential systems with infinite-delay; for example, the interval $(-\infty, \sigma]$ is not compact, and the images of a solution map of closed and bounded sets in $C((-\infty, 0], R_n)$ space may not be compact. Therefore, it is an interesting and complicated problem to study the stability theory for impulsive functional differential systems with infinite delays.

In the present paper, we will consider the stability of impulsive infinite-delay differential equations by using Lyapunov functions and the Razumikhin technique, we get some new results. The effect of delay and impulses which do contribute to the equations's stability properties will be shown in this paper.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we get some criteria for uniform stability and uniform asymptotic stability for impulsive infinite-delay differential equations, and an example is given to illustrate our results. Finally, concluding remarks are given in Section 4.

2. Preliminaries

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers, and R^n the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. For any $t \ge t_0 \ge 0 > \alpha \ge -\infty$, let $f(t, x(s))$ where $s \in [t+\alpha, t]$ or $f(t, x(\cdot))$ be a Volterra-type functional. In the case when $\alpha = -\infty$, the interval $[t+\alpha, t]$ is understood to be replaced by $(-\infty, t]$.

We consider the impulsive functional differential equations

$$\begin{aligned} x'(t) &= f(t, x(\cdot)), \quad t \geq t_0,\ t \neq t_k, \\ \Delta x|_{t=t_k} &= x(t_k) - x(t_k^-) \\ &= I_k(x(t_k^-)), \quad k = 1, 2, \ldots, \end{aligned} \tag{1}$$

where the impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k \to +\infty} t_k = +\infty$ and x' denotes the right-hand derivative of x. $f \in C([t_{k-1}, t_k) \times C,\ R^n)$, $f(t, 0) = 0$. C is an open set in $\mathrm{PC}([\alpha, 0], R^n)$, where $\mathrm{PC}([\alpha, 0], R^n) = \{\psi : [\alpha, 0] \to R^n$ is continuous everywhere except at finite number of points t_k, at which $\psi(t_k^+)$ and $\psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)\}$. For each $k = 1, 2, \ldots$, $I(t, x) \in C([t_0, \infty) \times R^n, R^n)$, $I(t_k, 0) = 0$.

For any $\rho > 0$, there exists a $\rho_1 > 0$ $(0 < \rho_1 < \rho)$ such that $x \in S(\rho_1)$ implies that $x + I(t_k, x) \in S(\rho)$, where $S(\rho) = \{x : \|x\| < \rho,\ x \in R^n\}$.

Define $\mathrm{PCB}(t) = \{x \in C : x$ is bounded$\}$. For $\psi \in \mathrm{PCB}(t)$, the norm of ψ is defined by $\|\psi\| = \sup_{\alpha \leq \theta \leq 0} |\psi(\theta)|$. For any $\sigma \geq 0$, let $\mathrm{PCB}_\delta(\sigma) = \{\psi \in \mathrm{PCB}(\sigma) : \|\psi\| < \delta\}$.

For any given $\sigma \geq t_0$, the initial condition for system (1) is given by

$$x_\sigma = \phi, \tag{2}$$

where $\phi \in \mathrm{PC}([\alpha, 0], R^n)$.

We assume that the solution for the initial problems, (1)-(2) does exist and is unique which will be written in the form $x(t, \sigma, \phi)$; see [4, 10]. Since $f(t, 0) = 0$, $I(t_k, 0) = 0$, $k = 1, 2, \ldots$, then $x(t) = 0$ is a solution of (1)-(2), which is called the trivial solution. In this paper, we always assume that the solution $x(t, \sigma, \phi)$ of (1)-(2) can be continued to ∞ from the right of σ.

For convenience, we also have the following classes in later sections:

$K_1 = \{a \in C(R_+, R_+) \mid a(0) = 0$ and $a(s) > 0$ for $s > 0\}$;

$K_2 = \{a \in C(R_+, R_+) \mid a(0) = 0$ and $a(s) > 0$ for $s > 0$ and a is nondecreasing in $s\}$;

$\Delta V(t_k, \psi(0)) = V(t_k, \psi(0) + I_k(t_k, \psi)) - V(t_k^-, \psi(0))$, $k = 1, 2, \ldots$;

$\Delta t_k = t_k - t_{k-1}$, $k = 1, 2, \ldots$.

We introduce some definitions as follows.

Definition 1 (see [4]). The function $V : [\alpha, \infty) \times C \to R_+$ belongs to class ν_0 if

(A_1) V is continuous on each of the sets $[t_{k-1}, t_k) \times C$ and $\lim_{(t,\varphi) \to (t_k^-, \psi)} V(t, \varphi) = V(t_k^-, \psi)$ exists;

(A_2) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 2 (see [4]). Let $V \in \nu_0$, for any $(t, \psi) \in [t_{k-1}, t_k) \times C$, the upper right-hand Dini derivative of $V(t, x)$ along the solution of (1)-(2) is defined by

$$D^+V(t, \psi(0)) = \frac{\limsup_{h \to 0^+} \{V(t + h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))\}}{h}. \tag{3}$$

Similarly, we can define $D^-V(t, \psi(0))$, $D_-V(t, \psi(0)$, $D_+V(t, \psi(0))$. If $V \in C'$, then $DV(t, \psi(0)) = \dot{V}(t, \psi(0))$, where D is any of the four Dini derivatives.

For $V \in \nu_0$, $(t, \psi) \in [t_{k-1}, t_k) \times C$, the upper right-hand Dini derivative of $\dot{V}(t, x)$ along the solution of (1)-(2) is defined by

$$D^+\dot{V}(t, \psi(0)) = \frac{\limsup_{h \to 0^+} \{\dot{V}(t + h, \psi(0) + hf(t, \psi)) - \dot{V}(t, \psi(0))\}}{h}. \tag{4}$$

Similarly, we can define $D^-\dot{V}(t, \psi(0))$. If $V \in C''$, then these are simply the second derivative of V.

Definition 3 (see [4]). Assume $x(t) = x(t, \sigma, \phi)$ to be the solution of (1)-(2) through (σ, ϕ). Then, the zero solution of (1)-(2) is said to be

(1) uniformly stable, if for any $\varepsilon > 0$ and $\sigma \geq t_0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that $\phi \in \mathrm{PCB}_\delta(\sigma)$ implies $\|x(t)\| < \varepsilon$, $t \geq \sigma$.

(2) uniformly asymptotically stable, if it is uniformly stable, and there exists a $\delta > 0$ such that for any $\varepsilon > 0$, $\sigma \geq t_0$, there is a $T = T(\varepsilon) > 0$ such that $\phi \in \mathrm{PCB}_\delta(\sigma)$ implies $\|x(t)\| < \varepsilon$, $t \geq \sigma + T$.

3. Main Results

Theorem 4. *Assume that there exist functions $w_i \in K_1, g \in K_2$, $c_i, p, q \in C(R_+, R_+)$, $V(t, x) \in \nu_0$, $i = 1, 2$, and constants $m > 1$, such that the following conditions hold:*

(i) $w_1(\|x\|) \leq V(t, x) \leq w_2(\|x\|)$, $(t, x) \in [\alpha, \infty) \times S(\rho)$;

(ii) *for any $\sigma \geq t_0$ and $\psi \in PC([\alpha, 0], S(\rho))$, if $V(t, \psi(0)) \geq m^{-2} g(V(t + \theta, \psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \leq \theta \leq 0$, $t \neq t_k$, then*

$$D^+V(t, \psi(0)) \leq p(t)\, c_1(V(t, \psi(0))), \tag{5}$$

where $s/m \leq g(s) < s$ for any $s > 0$;

(iii) *for all $(t, \psi(0)) \in (t_{k-1}, t_k) \times PC([\alpha, 0], S(\rho_1))$,*

$$D^-\dot{V}(t, \psi(0)) \geq 0. \tag{6}$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$\Delta t_k \dot{V}(t_k^-, \psi(0)) + \Delta V(t_k, \psi(0)) \leq -\mu_k c_2(V(t_k, \psi(0))), \tag{7}$$

where $c_2(s) \leq s c_2'(s)$, $s > 0$, μ_k satisfies $\liminf_{k \to \infty} \mu_k \geq 2 \cdot \sup_{s > 0}(s / c_2(m^{-1} \cdot s))$;

(iv) *there exist constants* $M_1, M_2 > 0$ *such that the following inequalities hold:*

$$\sup_{t\geq 0}\int_t^{t+\tau} p(s)\,ds = M_1 < \infty,$$
$$\inf_{s>0}\int_{g(s)}^{s}\frac{dt}{c_1(t)} = M_2 > M_1, \tag{8}$$

where $\tau = \max_{k\geq 1}\{t_k - t_{k-1}\} < \infty$.

Then, the zero solution of (1)-(2) *is uniformly asymptotically stable.*

Proof. Condition (i) implies that $w_1(s) \leq w_2(s)$ for $s \in [0,\rho]$. So let W_1 and W_2 be continuous, strictly increasing functions satisfying $W_1(s) \leq w_1(s) \leq w_2(s) \leq W_2(s)$ for all $s \in [0,\rho]$. Then

$$W_1(\|x\|) \leq V(t,x) \leq W_2(\|x\|), \quad (t,x) \in [\alpha,\infty)\times S(\rho). \tag{9}$$

We first show uniform stability.

For any $\varepsilon > 0 (< \rho_1)$, one may choose a $\delta = \delta(\varepsilon) > 0$ such that $W_2(\delta) \leq g(W_1(\varepsilon))$. Let $x(t) = x(t,\sigma,\phi)$ be a solution of (1)-(2) through (σ,ϕ), $\sigma \geq t_0$. For any $\phi \in \mathrm{PCB}_\delta(\sigma)$, we will prove that $\|x(t)\| < \varepsilon, t \geq \sigma$.

For convenience, let $V(t) = V(t,x(t))$. Suppose that $\sigma \in [t_{l-1}, t_l)$, $l \in Z_+$. First, for $\sigma + \alpha \leq t \leq \sigma$, we have

$$W_1(\|x\|) \,\&\leq V(t) < W_2(\delta) \leq g(W_1(\varepsilon)) < W_1(\varepsilon). \tag{10}$$

So, $\|x(t)\| < \varepsilon < \rho_1$, $t \in [\sigma+\alpha,\sigma]$.

Next, we claim that

$$V(t) < W_1(\varepsilon), \quad t \in [\sigma, t_l). \tag{11}$$

Suppose on the contrary that there exists some $t \in [\sigma, t_l)$ such that $V(t) \geq W_1(\varepsilon)$. Since $V(\sigma) < W_1(\varepsilon)$, we can define $\hat{t} = \inf\{t \in [\sigma,t_l) \mid V(t) \geq W_1(\varepsilon)\}$. Thus, $\hat{t} \in (\sigma, t_l)$, $V(\hat{t}) = W_1(\varepsilon)$, and $V(t) < W_1(\varepsilon)$, $t \in [\sigma,\hat{t})$. Also, from (10) we obtain

$$V(t) < W_1(\varepsilon), \quad t \in [\sigma+\alpha, \hat{t}). \tag{12}$$

On the other hand, note that $V(\hat{t}) = W_1(\varepsilon) > g(W_1(\varepsilon))$ and $V(\sigma) < g(W_1(\varepsilon))$ in view of (10), we can define $t^* = \sup\{t \in [\sigma,\hat{t}] V(t) \leq g(W_1(\varepsilon))\}$; it is obvious that $t^* \in [\sigma,\hat{t}), V(t^*) = g(W_1(\varepsilon))$ and $V(t) > g(W_1(\varepsilon))$ for $t \in (t^*,\hat{t}]$. Therefore, combining (12), we have for $t \in (t^*,\hat{t})$

$$V(t) > g(W_1(\varepsilon)) > g(V(t+\theta)), \quad \alpha \leq \theta \leq 0; \tag{13}$$

that is,

$$V(t,\psi(0)) > m^{-2} g(V(t+\theta,\psi(\theta))),$$
$$\max\{\alpha, -q(V(t))\} \leq \theta \leq 0. \tag{14}$$

By assumption (ii), (iv), we have

$$\int_{V(t^*)}^{V(\hat{t})}\frac{ds}{c_1(s)} = \int_{g(W_1(\varepsilon))}^{W_1(\varepsilon)}\frac{ds}{c_1(s)} \geq M_2 > M_1. \tag{15}$$

However, we also have

$$\int_{V(t^*)}^{V(\hat{t})}\frac{ds}{c_1(s)} \leq \int_{t^*}^{\hat{t}} p(s)\,ds < \int_{t^*}^{t^*+\tau} p(s)\,ds \leq M_1, \tag{16}$$

which is a contradiction. So, (11) holds.

Hence, $W_1(\|x\|) \leq V(t) < W_1(\varepsilon)$, $t \in [\sigma, t_l)$ implies that $\|x(t_l^-)\| < \varepsilon < \rho_1$. Thus, $x(t_l) \in S(\rho)$.

On the other hand, from condition (iii), we note for $k = 1, 2, \ldots$,

$$\begin{aligned} V(t_k) - V(t_{k-1}) &= V(t_k) - V(t_k^-) + V(t_k^-) - V(t_{k-1}) \\ &= \Delta V(t_k) + \int_{t_{k-1}}^{t_k} \dot{V}(t)\,dt \\ &\leq \Delta V(t_k) + \Delta t_k \dot{V}(t_k^-) \\ &\leq -\mu_k c_2(V(t_k)) \leq 0. \end{aligned} \tag{17}$$

Hence, we obtain $V(t_k) \leq V(t_{k-1})$, $k = 1, 2, \ldots$. particularly, $V(t_l) \leq V(t_{l-1})$. In view of (10), we get

$$V(t_l) \leq V(t_{l-1}) < g(W_1(\varepsilon)) < W_1(\varepsilon). \tag{18}$$

Next, we claim that

$$V(t) < W_1(\varepsilon), \quad t \in [t_l, t_{l+1}). \tag{19}$$

Suppose on the contrary that there exists some $t \in [t_l, t_{l+1})$ such that $V(t) \geq W_1(\varepsilon)$. Then applying exactly the same argument as in the proof of (11) yields our desired contradiction.

By induction hypothesis, we may prove, in general, that for $t \in [t_{l+k}, t_{l+k+1})$, $k > 0$,

$$V(t) < W_1(\varepsilon). \tag{20}$$

In view of condition (i), we obtain that $\|x(t)\| < \varepsilon, t \geq \sigma$. Therefore, we have proved that the solutions of (1)-(2) are uniformly stable.

Next, we claim that they are uniformly asymptotically stable. Since the zero solution of (1)-(2) is uniformly stable, for any given constant $H > 0 (< \rho_1)$, then there exists $\delta > 0$ such that $\phi \in \mathrm{PCB}_\delta(\sigma)$ implies that $V(t) < W_1(H)$, $\|x(t)\| < \rho_1$, $t \geq \sigma$.

For any $\varepsilon \in (0, H)$, let

$$\begin{aligned} d &< \min\{\hat{d}, W_1(\varepsilon)\}, \\ \hat{d} &= \inf\{s - g(s) \mid m^{-1}W_1(\varepsilon) \leq s \leq W_1(H)\}, \\ h &= \sup\{q(s) \mid m^{-1}W_1(\varepsilon) \leq s \leq W_1(H)\}, \\ n_0 &= \frac{W_1(H)}{2\cdot\sup_{s>0}(s/c_2(m^{-1}s))\,c_2(m^{-1}W_1(\varepsilon))} + 1. \end{aligned} \tag{21}$$

From condition (iii), we get that there exists a $n_1 > 0$ such that for $k > n_1$,

$$\mu_k \geq 2\cdot\sup_{s>0}\frac{s}{c_2(m^{-1}\cdot s)}. \tag{22}$$

Choose a positive integer N satisfying

$$W_1(\varepsilon) + (N-1)d < W_1(H) \le W_1(\varepsilon) + Nd, \quad (23)$$

and define $T = N(h + n_0\tau) + n_1$, we will prove that $\phi \in \mathrm{PCB}_\delta(\sigma)$ implies $\|x(t)\| < \varepsilon$, $t \ge \sigma + T$.

First, we prove that there exists $\hat{t} \in [\sigma+h+n_1, \sigma+h+n_1+n_0\tau]$ such that

$$V(\hat{t}) < m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]. \quad (24)$$

Suppose on the contrary that for all $t \in [\sigma+h+n_1, \sigma+h+n_1+n_0\tau]$,

$$V(t) \ge m^{-1}\left[W_1(\varepsilon) + (N-1)d\right] \ge m^{-1}W_1(\varepsilon). \quad (25)$$

Let $t_{k_1} = \min\{t_k : t_k \ge \sigma + h + n_1\}$, from (17), we get

$$\begin{aligned} V(t_{k_1}) - V(t_{k_1-1}) &\le -\mu_{k_1} c_2(V(t_{k_1})) \\ &\le -\mu_{k_1} c_2(m^{-1}W_1(\varepsilon)), \\ V(t_{k_1+1}) - V(t_{k_1}) &\le -\mu_{k_1+1} c_2(m^{-1}W_1(\varepsilon)), \\ &\vdots \\ V(t_{k_1+n_0}) - V(t_{k_1+n_0-1}) &\le -\mu_{k_1+n_0} c_2(m^{-1}W_1(\varepsilon)), \end{aligned} \quad (26)$$

In general, combining (22), we deduce that

$$\begin{aligned} V(t_{k_1+n_0}) &\le V(t_{k_1-1}) - \sum_{s=0}^{n_0} \mu_{k_1+s} c_2(m^{-1}W_1(\varepsilon)) \\ &\le W_1(H) - 2(n_0+1) \\ &\quad \cdot \sup_{s>0} \frac{s}{c_2(m^{-1}s)} c_2(m^{-1}W_1(\varepsilon)) \\ &= -4 \cdot \sup_{s>0} \frac{s}{c_2(m^{-1}s)} c_2(m^{-1}W_1(\varepsilon)) < 0, \end{aligned} \quad (27)$$

which is a contradiction. So, (24) holds.

Suppose $\hat{t} \in [t_{l-1}, t_l)$, $l > 1$. Furthermore, we can prove that for $t \in [\hat{t}, t_l)$

$$V(t) < W_1(\varepsilon) + (N-1)d. \quad (28)$$

Suppose this assertion is false, then there exists some $t \in [\hat{t}, t_l)$ such that $V(t) \ge W_1(\varepsilon) + (N-1)d$. Since $V(\hat{t}) < m^{-1}[W_1(\varepsilon) + (N-1)d] < W_1(\varepsilon) + (N-1)d$, so define

$$t^* = \inf\{t \in [\hat{t}, t_l) \mid V(t) \ge W_1(\varepsilon) + (N-1)d\}; \quad (29)$$

then $t^* \in (\hat{t}, t_l)$, $V(t^*) = W_1(\varepsilon) + (N-1)d$ and $V(t) < W_1(\varepsilon) + (N-1)d$, $t \in (\hat{t}, t^*)$. Note that

$$\begin{aligned} V(t^*) &= W_1(\varepsilon) + (N-1)d > g(W_1(\varepsilon) + (N-1)d), \\ V(\hat{t}) &< m^{-1}\left[W_1(\varepsilon) + (N-1)d\right] < g(W_1(\varepsilon) + (N-1)d); \end{aligned} \quad (30)$$

thus, we can define

$$\bar{t} = \sup\{t \in [\hat{t}, t^*] \mid V(t) \le g(W_1(\varepsilon) + (N-1)d)\}, \quad (31)$$

then $\bar{t} \in [\hat{t}, t^*)$, $V(\bar{t}) = g(W_1(\varepsilon) + (N-1)d)$ and $V(t) > g(W_1(\varepsilon) + (N-1)d)$ for $t \in (\bar{t}, t^*]$.

Hence, we get for $t \in (\bar{t}, t^*]$

$$\begin{aligned} V(t) &> g(W_1(\varepsilon) + (N-1)d) \\ &\ge m^{-1}\left[W_1(\varepsilon) + (N-1)d\right] \\ &\ge m^{-1}W_1(\varepsilon), \end{aligned} \quad (32)$$

which implies that for $t \in (\bar{t}, t^*]$

$$\begin{aligned} V(t) &\ge g(V(t)) + d \ge m^{-1}V(t) + d \\ &> \frac{mV(t)}{m^2} + \frac{d}{m^2} \ge \frac{W_1(\varepsilon) + Nd}{m^2} \\ &\ge \frac{W_1(H)}{m^2} \ge \frac{V(s)}{m^2} > \frac{g(V(s))}{m^2}, \quad t + \alpha < s \le t. \end{aligned} \quad (33)$$

Thus, $V(t) \ge (1/m^2)g(V(t+\theta, \psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \le \theta \le 0$.

By assumption, (ii), (iv), we have for $t \in (\bar{t}, t^*)$,

$$\int_{V(\bar{t})}^{V(t^*)} \frac{ds}{c_1(s)} = \int_{g(W_1(\varepsilon)+(N-1)d)}^{W_1(\varepsilon)+(N-1)d} \frac{ds}{c_1(s)} \ge M_2 > M_1. \quad (34)$$

However, we also have

$$\int_{V(\bar{t})}^{V(t^*)} \frac{ds}{c_1(s)} < \int_{\bar{t}}^{t^*} p(s)\,ds < \int_{\bar{t}}^{\bar{t}+\tau} p(s)\,ds < M_1, \quad (35)$$

which is a contradiction. So, (28) holds.

On the other hand, it is easy to prove that the functions $s/c_2(m^{-1}s)$ are nonincreasing for $s \in (0, +\infty)$ in view of condition $c_2(s) \le s c_2'(s)$ for any $s > 0$.

Hence, the following inequalities hold: for $k > n_1$,

$$\begin{aligned} \frac{W_1(\varepsilon) + (N-i)d}{c_2(m^{-1}(W_1(\varepsilon) + (N-i-1)d))} &\le \frac{W_1(\varepsilon) + d}{c_2(m^{-1}W_1(\varepsilon))} \\ &< \frac{2W_1(\varepsilon)}{c_2(m^{-1}W_1(\varepsilon))} \\ &\le \mu_k, \quad i = 1, 2, \ldots, N-1. \end{aligned} \quad (36)$$

Next, we claim that

$$V(t_l) < m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]. \quad (37)$$

Or else, then $V(t_l) \ge m^{-1}[W_1(\varepsilon) + (N-1)d]$; from (17), we get

$$\begin{aligned} V(t_l) - V(t_{l-1}) &\le -\mu_l c_2(V(t_l)) \\ &\le -\mu_l c_2(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]). \end{aligned} \quad (38)$$

Considering (36), it holds that

$$\begin{aligned} V(t_l) &\le V(t_{l-1}) - \mu_l c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\le W_1(H) - \mu_l c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\le W_1(\varepsilon) + Nd - \mu_l c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\le c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right) \\ &\quad \times \left\{\frac{W_1(\varepsilon) + Nd}{c_2\left(m^{-1}\left[W_1(\varepsilon) + (N-1)d\right]\right)} - \mu_l\right\} \\ &< 0, \end{aligned} \tag{39}$$

which is a contradiction and (37) holds.

Next, we can prove that for $t \in [t_l, t_{l+1})$

$$V(t) < W_1(\varepsilon) + (N-1)d. \tag{40}$$

Suppose that this assertion is false, then there exists some $t \in [\hat{t}, t_l)$ such that $V(t) \ge W_1(\varepsilon) + (N-1)d$. Then applying exactly the same argument as in the proof of (24) and (28) yields our desired contradiction. Here, we omit it.

By induction hypothesis, we may prove, for $t \in [t_{l+k}, t_{l+k+1})$, $k = 1, 2, \ldots,$

$$V(t) < W_1(\varepsilon) + (N-1)d; \tag{41}$$

that is,

$$V(t) < W_1(\varepsilon) + (N-1)d, \quad t \ge \hat{t}. \tag{42}$$

Hence, we obtain

$$V(t) < W_1(\varepsilon) + (N-1)d, \quad t \ge \sigma + h + n_1 + n_0\tau. \tag{43}$$

Next, we prove that there exists $\hat{t}_2 \in [\sigma + 2h + n_1 + n_0\tau, \sigma + 2h + n_1 + 2n_0\tau]$ such that

$$V(\hat{t}_2) < m^{-1}\left[W_1(\varepsilon) + (N-2)d\right]. \tag{44}$$

Suppose that for all $t \in [\sigma + 2h + n_1 + n_0\tau, \sigma + 2h + n_1 + 2n_0\tau]$,

$$V(t) \ge m^{-1}\left[W_1(\varepsilon) + (N-2)d\right] \ge m^{-1}W_1(\varepsilon). \tag{45}$$

Using the same argument as in the proof of (24), we get

$$\begin{aligned} V\left(t_{k_2+n_0}\right) &\le V\left(t_{k_2-1}\right) - \sum_{s=0}^{n_0}\mu_{k_2+s}c_2\left(m^{-1}W_1(\varepsilon)\right) \\ &\le W_1(H) - 2\left(n_0+1\right) \\ &\quad \cdot \sup_{s>0}\frac{s}{c_2\left(m^{-1}s\right)}c_2\left(m^{-1}W_1(\varepsilon)\right) \\ &= -4 \cdot \sup_{s>0}\frac{s}{c_2\left(m^{-1}s\right)}c_2\left(m^{-1}W_1(\varepsilon)\right) < 0, \end{aligned} \tag{46}$$

where $t_{k_2} = \min\{t_k : k \ge \sigma + 2h + n_1 + n_0\tau\}$.

This is a contradiction. So, (44) holds.

Suppose $\hat{t}_2 \in [t_{k-1}, t_k), k > l$. Furthermore, we claim that for $t \in [\hat{t}_2, t_k)$

$$V(t) < W_1(\varepsilon) + (N-2)d. \tag{47}$$

Suppose on the contrary, that there exists some $t \in [\hat{t}_2, t_k)$ such that $V(t) \ge W_1(\varepsilon) + (N-2)d$. We define

$$t^* = \inf\left\{t \in [\hat{t}_2, t_k) \mid V(t) \ge W_1(\varepsilon) + (N-2)d\right\}, \tag{48}$$

since $V(\hat{t}_2) < m^{-1}[W_1(\varepsilon) + (N-2)d] < W_1(\varepsilon) + (N-2)d$ in view of (44). Thus, $t^* \in (\hat{t}_2, t_k)$, $V(t^*) = W_1(\varepsilon) + (N-2)d$ and $V(t) < W_1(\varepsilon) + (N-2)d$, $t \in (\hat{t}_2, t^*)$. Note that

$$\begin{aligned} V(t^*) &= W_1(\varepsilon) + (N-2)d > g\left(W_1(\varepsilon) + (N-2)d\right), \\ V(\hat{t}_2) &< m^{-1}\left[W_1(\varepsilon) + (N-2)d\right] < g\left(W_1(\varepsilon) + (N-2)d\right); \end{aligned} \tag{49}$$

furthermore, we can define

$$\tilde{t} = \sup\left\{t \in [\hat{t}_2, t^*] \mid V(t) \le g\left(W_1(\varepsilon) + (N-2)d\right)\right\}, \tag{50}$$

then $\tilde{t} \in [\hat{t}_2, t^*)$, $V(\tilde{t}) = g(W_1(\varepsilon) + (N-2)d)$ and $V(t) > g(W_1(\varepsilon) + (N-2)d)$ for $t \in (\tilde{t}, t^*]$.

Hence, we get for $t \in (\tilde{t}, t^*]$

$$\begin{aligned} V(t) &> g\left(W_1(\varepsilon) + (N-2)d\right) \\ &\ge m^{-1}\left[W_1(\varepsilon) + (N-2)d\right] \\ &\ge m^{-1}W_1(\varepsilon); \end{aligned} \tag{51}$$

considering the definition of d and (43), we get for $t \in (\tilde{t}, t^*]$

$$\begin{aligned} V(t) &\ge g(V(t)) + d \ge m^{-1}V(t) + d \\ &> \frac{mV(t)}{m^2} + \frac{d}{m^2} \ge \frac{W_1(\varepsilon) + (N-1)d}{m^2} \\ &\ge \frac{V(s)}{m^2} > \frac{g(V(s))}{m^2}, \quad t - h < s \le t. \end{aligned} \tag{52}$$

Thus, $V(t) \ge (1/m^2)g(V(t+\theta, \psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \le \theta \le 0$.

Using assumptions (ii), (iv), we have

$$\int_{V(\tilde{t})}^{V(t^*)}\frac{ds}{c_1(s)} = \int_{g(W_1(\varepsilon)+(N-2)d)}^{W_1(\varepsilon)+(N-2)d}\frac{ds}{c_1(s)} \ge M_2 > M_1. \tag{53}$$

However,

$$\int_{V(\tilde{t})}^{V(t^*)}\frac{ds}{c_1(s)} < \int_{\tilde{t}}^{t^*}p(s)\,ds < \int_{\tilde{t}}^{\tilde{t}+\tau}p(s)\,ds < M_{1,} \tag{54}$$

giving us a contradiction. So, (47) holds.

Next, we claim that

$$\begin{aligned} V(t_l) &< m^{-1}\left[W_1(\varepsilon) + (N-1)d\right], \\ V(t) &< W_1(\varepsilon) + (N-1)d, \quad t \in [t_l, t_{l+1}), \end{aligned} \tag{55}$$

whose arguments are the same as was employed in the proof of (36), (37). there we omit it.

Repeating this process, it is easy to check that

$$V(t) < W_1(\varepsilon) + (N-2)d, \quad t \geq \sigma + 2h + n_1 + 2n_0\tau. \quad (56)$$

By induction hypothesis, we have

$$V(t) \leq W_1(\varepsilon) + (N-i)d, \quad t \geq \sigma + ih + n_1 + in_0\tau. \quad (57)$$

Let $i = N$, then for $t \geq \sigma + N(h + n_0\tau) + n_1$,

$$V(t) < W_1(\varepsilon). \quad (58)$$

Therefore, we arrive at $\|x(t)\| < \varepsilon$, $t \geq T$. The proof of Theorem 4 is complete. □

Corollary 5. *Assume that there exist functions $w_i \in K_1$, $g \in K_2$, $c, p \in C(R_+, R_+)$, $V(t,x) \in v_0$, $i = 1, 2$, and constants $m > 1$, such that the following conditions hold:*

(i) $w_1(\|x\|) \leq V(t,x) \leq w_2(\|x\|)$, $(t,x) \in [\alpha, \infty) \times S(\rho)$;

(ii) *for any $\sigma \geq t_0$ and $\psi \in PC([\alpha, 0], S(\rho))$, if $V(t, \psi(0)) \geq g(V(t+\theta, \psi(\theta)))$, $\alpha \leq \theta \leq 0$, $t \neq t_k$, then*

$$D^+V(t, \psi(0)) \leq p(t)\, c(V(t, \psi(0))), \quad (59)$$

where $(s/m) \leq g(s) < s$ for any $s > 0$;

(iii) *for all $(t, \psi(0)) \in (t_{k-1}, t_k) \times PC([\alpha, 0], S(\rho_1))$,*

$$D^-\dot{V}(t, \psi(0)) \geq 0. \quad (60)$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$\Delta t_k \dot{V}(t_k^-, \psi(0)) + \Delta V(t_k, \psi(0)) \leq 0; \quad (61)$$

(iv) *there exist constants $M_1, M_2 > 0$ such that the following inequalities hold:*

$$\sup_{t\geq 0} \int_t^{t+\tau} p(s)\, ds = M_1 < \infty,$$
$$\inf_{s>0} \int_{g(s)}^{s} \frac{dt}{c(t)} = M_2 > M_1, \quad (62)$$

where $\tau = \max_{k\geq 1}\{t_k - t_{k-1}\} < \infty$.

Then the zero solution of (1)-(2) is uniformly stable.

Theorem 4 has a dual result when $\dot{V}$ is nonincreasing on (t_{k-1}, t_k). Here, we only give the results whose proof is very similar to the proof of Theorem 4.

Theorem 6. *Assume that there exist functions $w_i \in K_1$, $g \in K_2$, c_i, p, $q \in C(R_+, R_+)$, $V(t,x) \in v_0$, $i = 1, 2$, and constants $m > 1$, such that the following conditions hold:*

(i) $w_1(\|x\|) \leq V(t,x) \leq w_2(\|x\|)$, $(t,x) \in [\alpha, \infty) \times S(\rho)$;

(ii) *for any $\sigma \geq t_0$ and $\psi \in PC([\alpha, 0], S(\rho))$, if $V(t, \psi(0)) \geq m^{-2} g(V(t+\theta, \psi(\theta)))$, $\max\{\alpha, -q(V(t))\} \leq \theta \leq 0$, $t \neq t_k$, then*

$$D^+V(t, \psi(0)) \leq p(t)\, c_1(V(t, \psi(0))), \quad (63)$$

where $(s/m) \leq g(s) < s$ for any $s > 0$;

(iii) *for all $(t, \psi(0)) \in (t_{k-1}, t_k) \times PC([\alpha, 0], S(\rho_1))$,*

$$D^-\dot{V}(t, \psi(0)) \leq 0. \quad (64)$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$\Delta t_k \dot{V}(t_{k-1}^-, \psi(0)) + \Delta V(t_k, \psi(0)) \leq -\mu_k c_2(V(t_k, \psi(0))), \quad (65)$$

where $c_2(s) \leq s c_2'(s)$, $s > 0$, μ_k satisfies $\liminf_{k\to\infty} \mu_k \geq 2 \cdot \sup_{s>0}(s/c_2(m^{-1}\cdot s))$;

(iv) *there exist constants $M_1, M_2 > 0$ such that the following inequalities hold:*

$$\sup_{t\geq 0} \int_t^{t+\tau} p(s)\, ds = M_1 < \infty, \quad (66)$$

$$\inf_{s>0} \int_{g(s)}^{s} \frac{dt}{c_1(t)} = M_2 > M_1, \quad (67)$$

where $\tau = \max_{k\geq 1}\{t_k - t_{k-1}\} < \infty$.

Then the zero solution of (1)-(2) is uniformly asymptotically stable.

Example 7. Consider the impulsive delay differential equations:

$$x'(t) = ax(t) - b\int_{-\infty}^{0} e^s x(t+s)\, ds, \quad t \geq 0, \ t \neq t_k,$$
$$\Delta x|_{t=t_k} = I_k(x), \quad k = 1, 2, \ldots, \quad (68)$$
$$x_0 = \phi > 0,$$

where $a \in (0, 3]$, $b \in (0, 2]$, $|x + I_k(x)| \leq \sqrt{\lambda} \cdot |x|$, $k = 1, 2, \ldots, \lambda \in (0, 1)$. For any given $\phi > 0$, we always suppose that (68) has and only has positive solutions, and assume without loss of generality that $x(t) = x(t, 0, \phi)$ is a solutions of (68) through $(0, \phi)$. Suppose that there exists $m > 1$ such that the following inequalities hold:

$$\tau < \min\left\{\frac{\ln m}{2(a - b\sqrt{m})}, \frac{1 - \lambda - 2\lambda m}{2a}\right\}, \quad a > 2b - 1, \quad (69)$$

where $\tau = \max_{k\geq 1}\{t_k - t_{k-1}\} < \infty$. Then, the zero solution of (68) is uniformly asymptotically stable.

In fact, let $V(t,x) = x^2/2$, $g(s) = m^{-1}s (m > 1)$, and $c_1(s) = s$ then $V(t, x(t)) > g(V(s, x(s)))$, $-\infty \leq s \leq t$ implies that $\sqrt{m} \cdot |x(t)| > |x(s)|$, $-\infty \leq s \leq t$. Thus, for $t \neq t_k$

$$\begin{aligned} D^+V(t, x(\cdot)) &= x(t)\, x'(t) \\ &= x(t)\left\{ax(t) - b\int_{-\infty}^{0} e^s x(t+s)\, ds\right\} \\ &\leq x^2(t)\left\{a - b\sqrt{m}\int_{-\infty}^{0} e^s ds\right\} \\ &= x^2(t)\{a - b\sqrt{m}\} \\ &= p(t)\, V(t, x(t)), \end{aligned} \quad (70)$$

where $p(t) = 2(a - b\sqrt{m})$.

In view of condition (69), we note

$$\sup_{t\geq 0}\int_t^{t+\tau} p(s)\,ds = 2\tau(a-b\sqrt{m}) < \ln m = \inf_{s>0}\int_{g(s)}^{s}\frac{dt}{c_1(t)}. \quad (71)$$

So, condition (iv) in Corollary 5 holds.

On the other hand, we have for $t\neq t_k$

$$\begin{aligned} D^-\dot{V}(t,x(\cdot)) &= (x(t)x'(t))' \\ &= x(t)x''(t) + (x'(t))^2 \\ &= (x'(t))^2 \\ &\quad + x(t)\left(ax(t) - b\int_{-\infty}^{0} e^s x(t+s)\,ds\right)' \\ &= (x'(t))^2 + ax(t)x'(t) \\ &\quad - bx(t)\int_{-\infty}^{0} e^s x(t+s)\,ds - bx^2(t) \\ &= (x'(t))^2 + ax(t)x'(t) + ax^2(t) \\ &\quad - x(t)x'(t) - bx^2(t) \\ &= (x'(t))^2 + (a-1)x(t)x'(t) \\ &\quad + (a-b)x^2(t) \\ &\geq (x'(t))^2 - \frac{(x'(t))^2 + x^2(t)}{2}(a-1) \\ &\quad + (a-b)x^2(t) \\ &= \frac{3-a}{2}(x'(t))^2 + \left(\frac{a+1}{2} - b\right)x^2(t) \\ &\geq 0, \end{aligned} \quad (72)$$

in view of condition $a > 2b-1$. Also, considering $x(t)$ to be a positive solution of (68), we get

$$\begin{aligned} &\Delta t_k\dot{V}(t_k^-,\psi(0)) + \Delta V(t_k,\psi(0)) \\ &\quad\leq a\tau x^2(t_k^-) + (\lambda-1)\frac{x^2(t_k^-)}{2} \\ &\quad= (2a\tau+\lambda-1)x^2(t_k^-)\frac{x^2(t_k^-)}{2} \\ &\quad= (2a\tau+\lambda-1)V(t_k^-) \\ &\quad\leq -\frac{1-2a\tau-\lambda}{\lambda}V(t_k) \\ &\quad= -\mu_k c_2(V(t_k,\psi(0))), \end{aligned} \quad (73)$$

where $c_2 = s$, $\mu_k = (1-2a\tau-\lambda)/\lambda$.

Note that

$$\sup_{s>0}\frac{2s}{c_2(m^{-1}\cdot s)} = 2m < \frac{1-2a\tau-\lambda}{\lambda} = \mu_k, \quad (74)$$

in view of (69). So, the zero solution of (68) is uniformly stable by Corollary 5.

Furthermore, choose $q(s) = -\ln(1-1/m)$ (positive constants), which implies that $\int_{-q(s)}^{0} e^s ds = m^{-1}$. On the other hand, since $V(t,x(t)) > m^{-2}g(V(s,x(s)))$, $\max\{\alpha, t-q(V(t))\}\leq s\leq t$, implying that $m^{3/2}|x(t)| > |x(s)|$, $\max\{\alpha,t-q(V(t))\}\leq s\leq t$, then

$$\begin{aligned} &D^+V|_{(68)}(t,x(\cdot)) \\ &\quad\leq ax^2(t) - bx(t)\int_{-\infty}^{0} e^s|x(t+s)|\,ds \\ &\quad\leq ax^2(t) - bx(t)\int_{-\infty}^{t} e^{s-t}|x(s)|\,ds \\ &\quad\leq ax^2(t) - bx(t)\int_{t-q(V(t,x(\cdot)))}^{t} e^{s-t}|x(s)|\,ds \\ &\qquad - x(t)\int_{-\infty}^{t-q(V(t,x(\cdot)))} e^{s-t}|x(s)|\,ds \\ &\quad\leq ax^2(t) - bx(t)\int_{t-q(V(t,x(\cdot)))}^{t} e^{s-t}|x(s)|\,ds \\ &\quad\leq x^2(t)\left\{a - bm^{3/2}\int_{-q(V(t,x(\cdot)))}^{0} e^s ds\right\} \\ &\quad\leq x^2(t)\{a-b\sqrt{m}\} \\ &\quad= c(V(t,x(t)))\,p(t). \end{aligned} \quad (75)$$

By Theorem 4, we obtain that if (69) holds, then the zero solution of (68) is uniformly asymptotically stable.

Remark 8. In fact, $x(t) = \phi(0)e^t$ is a positive solution of (68) through $(0,\phi)$ in the absence of impulses. It is obvious that the solution is unstable. However, the solution is uniformly asymptotically stable under proper impulses effect, which shows that impulses do contribute to the system's stability behavior.

4. Conclusion

In this work, we have considered the stability of impulsive infinite-delay differential systems. By using Lyapunov functions and the Razumikhin technique, we have obtained some new results. We can see that impulses and delay do contribute to the system's stability behavior.

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