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## Research Article

# Universality and Constant Scalar Curvature Invariants

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A classical solution is called universal if the quantum correction is a multiple of the metric. Therefore, universal solutions play an important role in the quantum theory. We show that in a spacetime which is universal all scalar curvature invariants are constant (i.e., the spacetime is CSI).

## 1. Universality

In [1, 2], metrics of holonomy  $\text{Sim}(n - 2)$  were investigated, and it was found that all 4-dimensional  $\text{Sim}(2)$  metrics (which belong to the subclass of Kundt-CSI spacetimes) are universal and consequently can be interpreted as metrics with vanishing quantum corrections and are automatically solutions to the quantum theory. A classical solution is called *universal* if the quantum correction is a multiple of the metric and therefore plays an important role in the quantum theory regardless of what the exact form of this theory might be.

That is, if the spacetime is universal, then every symmetric conserved rank-2 tensor,  $T_{ab}$ , which is constructed from the metric, Riemann tensor and its covariant derivatives, is of the form

$$T_{ab} = \mu g_{ab}, \quad (1.1)$$

where  $\mu$  is a constant. Now, for every scalar  $S$  that appears in the action (gravitational Lagrangian) we obtain by variation (since these geometrical tensors are automatically conserved due to the invariance of the actions under spacetime diffeomorphisms) a symmetric conserved rank-2 tensor  $S_{ab}$ . For each such tensor, we have from the condition of universality that  $S_{ab} = \hat{\mu} g_{ab}$ . By using an appropriate set of such scalars, we will show that the resulting spacetime is CSI. Since the resulting spacetime is automatically an Einstein space,

in effect we must show that all scalar contractions of the Weyl tensor and its derivatives are constants [3–5]. We utilize the results of FKWC [6–8] to obtain all conserved rank-2 tensors obtained from variations (by Noether’s theorem) of an elemental scalar Riemann polynomial.

There are a number of related results we would like to investigate in this paper. We will state these in terms of a conjecture and will corroborate this conjecture by proving a number of subresults.

**Conjecture 1.1.** *A Universal  $n$ -dimensional Lorentzian spacetime,  $(M, g)$ , has the following properties.*

- (1) *It is CSI.*
- (2) *It is a degenerate Kundt spacetime.*
- (3) *There exists a spacetime,  $(\widetilde{M}, \widetilde{g})$ , of Riemann type D having identical scalar polynomial invariants; consequently  $(\widetilde{M}, \widetilde{g})$  is spacetime homogeneous.*
- (4) *There exists a homogeneous isotropy-irreducible Riemannian spacetime  $(\widehat{M}, \widehat{g})$  having identical scalar polynomial invariants as  $(M, g)$ ; that is,  $(\widehat{M}, \widehat{g})$  is universal as a Riemannian space.*

In low dimensions, this conjecture can be proven; in particular, dimension 2 is trivial as there is only one independent component, namely, the Ricci scalar  $R$ . In dimension 3, there are only Ricci invariants and the conjecture can be proven by brute force using symmetric conserved tensors. Most of our investigation will focus on dimension 4, and unless stated otherwise, we will assume that there is a 4 dimensional manifold.

We will use different methods to substantiate the above conjecture. This will consist of partial proofs and other arguments.

## 2. The CSI Case

Let us first provide with results substantiating the claim that universal spacetimes are CSI. This is clearly the case in the Riemannian case where Bleecker [9] showed that the critical manifolds are homogeneous, hence, CSI.

### 2.1. The Direct Method

Field theoretic calculations on curved spacetimes are nontrivial due to the systematic occurrence, in the expressions involved, of Riemann polynomials. These polynomials are formed from the Riemann tensor by covariant differentiation, multiplication, and contraction. The results of these calculations are complicated because of the nonuniqueness of their final forms, since the symmetries of the Riemann tensor as well as Bianchi identities can not be used in a uniform manner and monomials formed from the Riemann tensor may be linearly dependent in nontrivial ways. In [6], Fulling, King, Wybourne, and Cummings (FKWC) systematically expanded the Riemann polynomials encountered in calculations on standard bases constructed from group theoretical considerations. They displayed such bases for scalar Riemann polynomials of order eight or less in the derivatives of the metric tensor and for tensorial Riemann polynomials of order six or less. We adopt the FKWC-notations  $\mathcal{R}_{s,q}^r$  and  $\mathcal{R}_{\{\lambda_1, \dots\}}^r$  to denote, respectively, the space of Riemann polynomials of rank  $r$  (number of free indices), order  $s$  (number of differentiations of the metric tensor), degree  $q$  (number of factors  $\nabla^p R_{\dots}$ ), and the space of Riemann polynomials of rank  $r$  spanned by contractions of products of the type  $\nabla^{\lambda_1} R_{\dots}$  [6]. The geometrical identities utilized to eliminate

“spurious” Riemann monomials include (i) the commutation of covariant derivatives, (ii) the “symmetry” properties of the Ricci and the Riemann tensors (pair symmetry, antisymmetry, cyclic symmetry), and (iii) the Bianchi identity and its consequences obtained by contraction of index pairs.

In this paper, we actually use a slightly modified version of the FKWC-bases [7, 8], which are independent of the dimension of spacetime and provide irreducible expressions for all of our results. In addition, the results of [8] provide irreducible expressions for the metric variations (i.e., for the functional derivatives with respect to the metric tensor) of the action terms associated with the 17 basis elements for the scalar Riemann polynomials of order six in derivatives of the metric tensor (the so-called curvature invariants of order six).

### 2.1.1. Riemann Polynomials of Rank 0 (Scalars)

The most general expression for a scalar of order six or less in derivatives of the metric tensor is obtained by expanding it on the FKWC-basis for Riemann polynomials of rank 0 and order 6 or less [6].

The subbasis for Riemann polynomials of rank 0 and order 2 consists of a single element:  $R$  [ $\mathcal{R}_{2,1}^0$ ].

Choosing  $S$  to be the Ricci scalar,  $R$ , we find that the Einstein tensor is conserved and  $R_{ab} = \lambda g_{ab}$ , where  $\lambda$  is a constant, and the spacetime is necessarily an Einstein space:

$$R_{pq} = \lambda g_{pq}; \quad R_{pq;r} = 0. \quad (2.1)$$

Every scalar contraction of the Ricci tensor (or its covariant derivatives, which are in fact zero) will thus necessarily be constant. Every scalar contraction of the Riemann tensor and its derivatives with the Ricci tensor or its covariant derivatives will be constant. For example, for  $S = R_{ab}R^{ab}$  for an Einstein space we have that  $S_{ab} = 2(R_{acbd} - (1/4)g_{ab}R_{cd})R^{cd} = \tilde{\mu}g_{ab}$  (where  $\tilde{\mu} \approx \lambda^2$ ). Every mixed invariant (containing both the Ricci tensor and the Weyl tensor and their derivatives, will be constant or can be written entirely as a contraction of scalars involving just the Weyl tensor and its derivatives (up to an additive constant term).

Thus to prove that the resulting spacetimes are CSI, we must show that all scalar contractions of the Weyl tensor and its derivatives are constants.

The subbasis for Riemann polynomials of rank 0 and order 4 has 4 elements:  $\square R$  [ $\mathcal{R}_{4,1}^0$ ]:  $R^2$ ,  $R_{pq}R^{pq}$ ,  $R_{pqrs}R^{pqrs}$  [ $\mathcal{R}_{4,2}^0$ ].

From (2.1), there is only one rank 0/order 4 independent scalar,  $C^2 \equiv C_{pqrs}C^{pqrs}$ . By varying  $S = C^2$ , we obtain a symmetric conserved rank-2 tensor which depends on polynomial contractions of the Weyl tensor quadratically, which by universality is proportional to the metric:

$$C_{almn}C_b^{lmn} + C_{blmn}C_a^{lmn} = 2\hat{\lambda}g_{ab}. \quad (2.2)$$

Hence we have that

$$C^2 = \hat{\lambda}. \quad (2.3)$$

Indeed, by choosing  $S$  to be a polynomial contraction of the Weyl tensor alone (higher than quadratic), we find that by varying  $S$  we obtain symmetric conserved rank-2 tensors which depend on polynomial contractions of the Weyl tensor which by universality are

proportional to the metric, and hence all zeroth-order invariants constructed from the Weyl tensor are constant (and the spacetime is said to be CSI<sub>0</sub>). We note that in higher dimensions, all Lovelock tensors are divergence free and consequently (by universality) proportional to the metric. However, we will not proceed in this way here.

The most general expression for a gravitational Lagrangian of order six in derivatives of the metric tensor is obtained by expanding it on the FKWC-basis for *Riemann polynomials of order 6 and rank 0*. This subbasis consists of the 17 following elements [6]:  $\square\square R [\mathcal{R}_{6,1}^0]$ :  $R\square R$ ,  $R_{;pq}R^{pq}$ ,  $R_{pq}\square R^{pq}$ ,  $R_{pq;rs}R^{pqrs} [\mathcal{R}_{\{2,0\}}^0]$ :  $R_{;p}R^{;p}$ ,  $R_{pq;r}R^{pq;r}$ ,  $R_{pq;r}R^{pr;q}$ ,  $R_{pqrs;t}R^{pqrs;t} [\mathcal{R}_{\{1,1\}}^0]$ :  $R^3$ ,  $RR_{pq}R^{pq}$ ,  $R_{pq}R^p{}_{;r}R^{qr}$ ,  $R_{pq}R_{rs}R^{pqrs}$ ,  $RR_{pqrs}R^{pqrs}$ ,  $R_{pq}R^p{}_{rst}R^{qrst}$ ,  $R_{pqrs}R^{pquv}$ ,  $R^{rs}{}_{uv}$ , and  $R_{pqrs}R^p{}_{;u}{}^q{}_{;v}R^{rsuv} [\mathcal{R}_{6,3}^0]$ .

In general, only 10 of these give rise to independent variations. The other 7 depend on these via total divergences (and Stokes theorem); the functional derivatives (i.e., conserved tensors) with respect to the metric tensor of the 7 remaining action terms can then be obtained in a straightforward manner.

In the case of an Einstein space satisfying the conditions (2.1), (2.2), and (2.3), there are only three independent rank 0/order 6 scalars:

$$(\nabla C)^2 \equiv C_{pqrs;t}C^{pqrs;t}, \quad C_1^3 \equiv C_{pqrs}C^{pquv}C^{rs}{}_{uv}, \quad C_2^3 \equiv C_{pqrs}C^p{}_{;u}{}^q{}_{;v}C^{rsuv}, \quad (2.4)$$

(where, for example,  $(\nabla C)^2 \equiv R_{pqrs;t}R^{pqrs;t} = C_{pqrs;t}C^{pqrs;t}$ ).

Variations of the last four scalars in the list above give rise to 4 independent conserved rank-2 tensors (although  $RR_{pqrs}R^{pqrs}$  and  $R_{pq}R^p{}_{rst}R^{qrst}$  are equivalent to  $\lambda\hat{\lambda}$ , their variations are nontrivial). Note that  $R_{pqrs;t}R^{pqrs;t}$  depends on the other 4 scalars via a total divergence (and Stokes theorem).

### 2.1.2. Conserved Rank 2 Tensors of Order Six

The functional derivatives of the ten independent action terms on the FKWC basis were expanded in [7]: for an Einstein space satisfying the conditions (2.1), (2.2), and (2.3), we obtain the following 4 independent explicit irreducible expressions for the metric variations of the action terms constructed from the 17 scalar Riemann monomials of order six:

$$\begin{aligned} H_{ab}^{(6,3)(7)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pqrs} R^{pquv} R^{rs}{}_{uv} \\ &= 24 R^p{}_{(a}{}^{;qr} R_{|pqr|b)} - 12 R^p{}_{a;q} R_{pb}{}^{;q} + 12 R^p{}_{a;q} R^q{}_{b;p} \\ &\quad + 3 R^{pqrs}{}_{;a} R_{pqrs;b} - 6 R^{pqr}{}_{a;s} R_{pqrb}{}^{;s} - 6 R^{pq} R^{rs}{}_{pa} R_{rsqb} \\ &\quad + 12 R^{pqrs} R^t{}_{pqa} R_{trsb} + \frac{1}{2} g_{ab} [R_{pqrs} R^{pquv} R^{rs}{}_{uv}], \end{aligned} \quad (2.5)$$

which implies that (using (2.1), (2.2)-(2.3))

$$3 C^{pqrs}{}_{;a} C_{pqrs;b} - 6 C^{pqr}{}_{a;s} C_{pqrb}{}^{;s} + 12 C^{pqrs} C^t{}_{pqa} C_{trsb} + \frac{1}{2} g_{ab} [C_{pqrs} C^{pquv} C^{rs}{}_{uv}] = \lambda_1 g_{ab}. \quad (2.6)$$

In addition,

$$\begin{aligned}
H_{ab}^{(6,3)(5)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R R_{pqrs} R^{pqrs}, \\
H_{ab}^{(6,3)(6)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pq} R^p{}_{rst} R^{qrst}, \\
H_{ab}^{(6,3)(8)} &\equiv \frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{ab}} \int_{\mathcal{M}} d^D x \sqrt{-g} R_{pqrs} R^p{}_{u^q}{}_{v} R^{rusv}
\end{aligned} \tag{2.7}$$

yield (respectively)

$$\begin{aligned}
&2 C^{pqrs} C_{pqrs;(ab)} + 2 C^{pqrs}{}_{;a} C_{pqrs;b} \\
&\quad + g_{ab} [-2 C_{pqrs;t} C^{pqrs;t} + 2 C_{pqrs} C^{pquv} C^{rs}{}_{uv} + 8 C_{prqs} C^p{}_{u^q}{}_{v} C^{rusv}] = \lambda_2 g_{ab}, \\
&\frac{1}{2} C^{pqrs} C_{pqrs;(ab)} + \frac{1}{2} C^{pqrs}{}_{;a} C_{pqrs;b} - C^{pqr}{}_{a;s} C_{pqrb}{}^{;s} \\
&\quad + C^{pqrs} C_{pqta} C_{rs}{}^t{}_b + 4 C^{prqs} C^t{}_{pqa} C_{trsb} - C^{pqr}{}_{s} C_{pqrt} C^s{}_{a^t}{}_b \\
&\quad + \frac{1}{4} g_{ab} [-C_{pqrs;t} C^{pqrs;t} + C_{pqrs} C^{pquv} C^{rs}{}_{uv} + 4 C_{prqs} C^p{}_{u^q}{}_{v} C^{rusv}] = \lambda_3 g_{ab}, \\
&-\frac{3}{4} C^{pqrs} C_{pqrs;(ab)} + \frac{3}{4} C^{pqrs}{}_{;a} C_{pqrs;b} - \frac{3}{2} C^{pqrs} C_{pqta} C_{rs}{}^t{}_b - 9 C^{prqs} C^t{}_{pqa} C_{trsb} \\
&\quad + \frac{3}{2} C^{pqr}{}_{s} C_{pqrt} C^s{}_{a^t}{}_b + \frac{1}{2} g_{ab} [C_{prqs} C^p{}_{u^q}{}_{v} C^{rusv}] = \lambda_4 g_{ab}.
\end{aligned} \tag{2.8}$$

Contracting (2.6), (2.8), and using  $C^{pqrs} \square C_{pqrs} = -C_1^3 - 4C_2^3 + 2\lambda\hat{\lambda}$  (etc.) [7], we then obtain

$$\begin{aligned}
-3(\nabla C)^2 + 2C_1^3 + 12C_2^3 &= 4\lambda_1, \\
-3(\nabla C)^2 + 3C_1^3 + 4C_2^3 &= 2\lambda_2 - 2\lambda\hat{\lambda}, \\
-3(\nabla C)^2 + 3C_1^3 + 12C_2^3 &= 8\lambda_3 - 2\lambda\hat{\lambda}, \\
-3(\nabla C)^2 + 3C_1^3 + 16C_2^3 &= -16\lambda_4 - 6\lambda\hat{\lambda},
\end{aligned} \tag{2.9}$$

and hence the 3 independent scalars of order 6 are all constant:

$$(\nabla C)^2 = \mu_1, \quad C_1^3 = \mu_2, \quad C_2^3 = \mu_3. \tag{2.10}$$

Since all of the basis scalars of order six are constant, then all scalars of order six are constant.

We now proceed with the higher-order scalars: orders (8,10,12) were considered in [6]. In particular, there is a subbasis of scalar (rank-0) order 8 polynomials consisting

of 92 elements given in [6, Appendix B] from which, by variation, we can obtain a set of independent conserved rank-2 tensors of order 8. For an Einstein space satisfying (2.1), (2.2)-(2.3) and (2.10), there are only 11 independent scalars:  $C^{pqrs;tu}C_{pqrs;tu}$  and  $C^{pqrs}C_p{}^{tuv}C_{qtru;sv}$ , 3 scalars (involving squares of the first covariant derivative) of the form  $C^{pqrs}C^{tuv}{}_{p;q}C_{tuwr;s}$ , and 6 algebraic fourth-order polynomials of the form  $C^{pqrs}C_{pqr}{}^t C^{uvw}{}_s C_{uvw}{}_t$ . By obtaining the set of (more than 12) independent conserved rank-2 tensors of order 8, it follows that all of these 11 independent scalars are constant. In particular, the 3 scalars involving the first covariant derivative of the Weyl tensor are constant, and we are well on our way to show that the spacetime is  $CSI_1$ . Indeed, in four dimensions, this is sufficient to show that the resulting spacetime is  $CSI$  [3–5]. Continuing in this way, we obtain the result that in a universal spacetime all scalar curvature invariants are constant.

## 2.2. The Slice Theorem

Let  $I_i$  be all possible curvature invariants. Then we can generate a corresponding set of conserved symmetric tensors,  $T_{i,\mu\nu}$ , by considering the variation of  $S[I_i] = \int I_i \sqrt{-g} d^N x$ .

Assume that the spacetime under consideration is universal. If the spacetime is strongly universal, then all of these symmetric tensors are zero:  $T_{i,\mu\nu} = 0$ . If there is a  $T_{i,\mu\nu}$  which is nonzero, then the spacetime is weakly universal, and, assuming that  $T_{1,\mu\nu} = \lambda_1 g_{\mu\nu} \neq 0$ , then we can define the equivalent set of invariants:

$$\tilde{I}_1 = I_1 + 2\lambda_1, \quad \tilde{I}_i = I_i - \frac{\lambda_i}{\lambda_1} I_1. \quad (2.11)$$

We notice that for this new set of invariants, the corresponding conserved tensors are all zero:  $\tilde{T}_{i,\mu\nu} = 0$ .

This means that we have a full set of invariants all of which has a zero variation:  $\delta S / \delta g_{\mu\nu} = 0$ . This is a signal that the universal metric has degeneracy in their curvature structure. In particular, consider a metric variation  $\delta g_{\mu\nu} = \epsilon h_{\mu\nu}$  where  $h_{\mu\nu} g^{\mu\nu} = 0$  (traceless). Then this implies that the variation with respect to the metric is zero implying that the variation of all the invariants in the direction of  $h_{\mu\nu}$  is zero. The metric is therefore a fixed point of all possible actions.

For the degenerate Kundt metrics, there exists a one-parameter family of metrics  $g_\tau$  such that  $I_i[g] = I_i[g_\tau]$ . Clearly, this implies that  $\lim_{\tau \rightarrow 0} ((S[g] - S[g_\tau]) / \tau) = 0$ ; that is,  $\delta S / \delta g_{\mu\nu}$  vanishes along  $h_{\mu\nu} \equiv \lim_{\tau \rightarrow 0} ((g_{\mu\nu} - g_{\tau,\mu\nu}) / \tau)$ . We note that for the Kundt spacetimes this metric deformation can always be chosen to be traceless (indeed, nilpotent). The universality condition puts additional conditions since *all* variations of the metric are required to be zero. However, degenerate Kundt metrics are particularly promising candidates for universal metrics [10].

In the Riemannian case, the slice theorem was used by Bleecker [9] to show many results regarding critical metrics. The slice theorem considers the manifold of metrics modulo the diffeomorphism group. Ebin [11] proved the slice theorem for compact Riemannian case. The Lorentzian case is a bit more problematic, and its validity is questionable for the general case but Isenberg and Marsden [12] showed a slice theorem for solutions to the Einstein equations given some assumptions (essentially, global hyperbolicity and compact spatial sections). In its infinitesimal version, it states that a symmetric tensor can be split as follows:

$$S_{\mu\nu} = \mathcal{L}_X g_{\mu\nu} + T_{\mu\nu}, \quad (2.12)$$

for a vector field  $X$ , and  $T$  is conserved:  $\nabla^\mu T_{\nu\mu} = 0$ . The vector field  $X$  can be interpreted as the generator of the diffeomorphism group and thus is a ‘‘gauge freedom’’.

Consider the case when the slice theorem is valid. One can now show that universality implies CSI (following parts of Bleecker’s argument). Assume therefore that the spacetime is *not* CSI. Then there must exist a nonconstant invariant  $I$ . In particular, there must exist a nontrivial interval  $[a, b]$  onto which the invariant  $I$  is onto. Choose therefore a sufficiently small and smooth function  $f(I)$ . The space of such functions is clearly infinite dimensional. Construct then the tensor deformation  $\tilde{g}_{\mu\nu} = (1 + f(I))g_{\mu\nu}$ . By the slice theorem, there exists a diffeomorphism  $\phi$  such that  $\phi^*\tilde{g}_{\mu\nu}$  is conserved. Clearly,  $\phi^*\tilde{g}_{\mu\nu}$  is an invariant tensor and thus, by universality,  $\phi^*\tilde{g}_{\mu\nu} = \lambda g_{\mu\nu}$ . This implies that the metric deformation is a conformal transformation. However, the space of conformal transformations is finite, thus, it must be possible to choose an  $f(I)$  such that  $\phi^*\tilde{g}_{\mu\nu} \neq \lambda g_{\mu\nu}$ . Consequently, the space is not universal. To summarise, not CSI implies not universal, thus *universality implies CSI*.

Note that in the compact Riemannian case the slice theorem holds thus universality implies CSI. In the Riemannian case, CSI implies locally homogeneous, and thus this provides a slightly alternate proof to that of Bleecker [9].

This result depends crucially on the validity of the slice theorem and it is unclear to the authors for which Lorentzian spaces it holds. However, the result is important as one can see that there is a clear link between universality and CSI spaces and thus supports our conjecture.

### 3. Kundt CSI Metrics

In [5], we proved that if a 4D spacetime is CSI, then either the spacetime is locally homogeneous or the spacetime is a Kundt spacetime. The Kundt-CSI spacetimes are of particular interest since they are solutions of supergravity or superstring theory when supported by appropriate bosonic fields [13]. It is plausible that a wide class of CSI solutions are exact solutions to string theory nonperturbatively [14]. In the context of string theory, it is of considerable interest to study higher-dimensional Lorentzian CSI spacetimes. In particular, a number of higher-dimensional CSI spacetimes are also known to be solutions of supergravity theory [13]. The supersymmetric properties of CSI spacetimes have also been studied, particularly those that admit a null covariantly constant vector (CCNV).

A Kundt CSI can be written in the form [15]

$$ds^2 = 2du \left[ dv + H(v, u, x^k) du + W_i(v, u, x^k) dx^i \right] + g_{ij}^\perp(x^k) dx^i dx^j, \quad (3.1)$$

where the metric functions  $H$  and  $W_i$ , requiring  $CSI_0$ , are given by

$$\begin{aligned} W_i(v, u, x^k) &= v W_i^{(1)}(u, x^k) + W_i^{(0)}(u, x^k), \\ H(v, u, x^k) &= v^2 \tilde{\sigma} + v H^{(1)}(u, x^k) + H^{(0)}(u, x^k), \\ \tilde{\sigma} &= \frac{1}{8} \left( 4\sigma + W^{(1)i} W_i^{(1)} \right), \end{aligned} \quad (3.2)$$

where  $\sigma$  is a constant. The remaining equations for  $CSI_0$  that need to be solved are (hatted indices refer to an orthonormal frame in the transverse space):

$$\begin{aligned} W_{[\hat{i}\hat{j}]}^{(1)} &= a_{\hat{i}\hat{j}}, \\ W_{(\hat{i}\hat{j})}^{(1)} - \frac{1}{2} \left( W_{\hat{i}}^{(1)} \right) \left( W_{\hat{j}}^{(1)} \right) &= s_{\hat{i}\hat{j}}, \end{aligned} \tag{3.3}$$

and the components  $R_{\hat{i}\hat{j}\hat{m}\hat{n}}^\perp$  are all constants (i.e.,  $dS_H^2 = g_{ij}^\perp(x^k)dx^i dx^j$  is curvature homogeneous). In four dimensions,  $g_{ij}^\perp(x^k)dx^i dx^j$  is 2 dimensional, which immediately implies  $g_{ij}^\perp(x^k)dx^i dx^j$  is a 2-dimensional locally homogeneous space and, in fact, maximally symmetric space. Up to scaling, there are (locally) only 3 such, namely, the sphere,  $S^2$ ; the flat plane,  $\mathbb{E}^2$ ; and the hyperbolic plane,  $\mathbb{H}^2$ .

Equation (3.3) now gives a set of differential equations for  $W_{\hat{i}}^{(1)}$ . These equations determine uniquely  $W_{\hat{i}}^{(1)}$  up to initial conditions (which may be free functions in  $u$ ). Also, requiring  $CSI_1$  gives an additional set of constraints:

$$\begin{aligned} \alpha_{\hat{i}} &= \sigma W_{\hat{i}}^{(1)} - \frac{1}{2} \left( s_{\hat{j}\hat{i}} + a_{\hat{j}\hat{i}} \right) W^{(1)\hat{j}}, \\ \beta_{\hat{i}\hat{j}\hat{k}} &= W^{(1)\hat{n}} R_{\hat{m}\hat{i}\hat{j}\hat{k}}^\perp - W_{\hat{i}}^{(1)} a_{\hat{j}\hat{k}} + \left( s_{\hat{i}\hat{j}} + a_{\hat{i}\hat{j}} \right) W_{\hat{k}}^{(1)}, \end{aligned} \tag{3.4}$$

where  $\alpha_i$  and  $\beta_{\hat{i}\hat{j}\hat{k}}$  are constants determined from the curvature invariants. We note that for a four-dimensional Kundt spacetime,  $CSI_1$  implies  $CSI$  [5].

The relation between  $CSI$  spacetimes and those that are universal is strong.

**Theorem 3.1.** *A universal spacetime of Petrov type D, II, or III, is Kundt CSI.*

*Proof.* Consider type D first. Here, assuming the spacetime is Einstein, we have that the spacetime is necessarily  $CSI_0$  (which follows from previous discussion). This implies that the b.w. 0 components are constants also in the canonical frame. Using the Bianchi identities, it immediately follows that it is Kundt also. Since the previous analysis also implies that it is  $CSI_1$ , then we have that the spacetime is Kundt- $CSI$ .

For type II, the analysis is almost identical to the type D analysis. For type III, it requires to calculate some conserved tensors. Using the Weyl type III canonical form, the Bianchi identities imply that  $\kappa = \sigma = 0$  (and  $\rho = \epsilon$ ,  $\beta = \tau$ ,  $\alpha = -2\pi$ ,  $\gamma = -2\mu$ ). Requiring also that  $H_{ab}^{(6,3)(8)} = \lambda g_{ab}$ , say, gives the additional equation:  $\rho^2 = 0$ . Clearly,  $\rho = 0$  and the spacetime is Kundt. Since  $CSI_1$  implies  $CSI$  for Kundt spacetimes, the theorem follows.  $\square$

Although the theorem does not include Weyl type N and I, it is believed that these are Kundt. For type I, the expressions are so messy for the conserved tensors to be manageable, and for type N it is necessary to compute a particular order 16 conserved tensor.

Thus proves the first two statements in the Conjecture 1.1 for Petrov types D, II and III in 4 dimensions. However, we can also see that the last two statements are true.

**Proposition 3.2.** *Given a 4D Kundt CSI spacetime  $(M, g)$ . If  $(M, g)$  is universal then Conjecture 1.1 is true.*



*Proof.* The proof utilises the results from [5]. Assuming Einstein and Kundt CSI reduces to the cases where the corresponding homogeneous spacetime  $(\widetilde{M}, \widetilde{g})$  is locally one of the following: Minkowski, de Sitter, anti-de Sitter,  $dS_2 \times S^2$ , or  $AdS_2 \times H^2$ . These again have the corresponding Riemannian counterparts,  $(\widehat{M}, \widehat{g})$ , with identical invariants: flat space,  $S^4$ ,  $H^4$ ,  $S^2 \times S^2$ ,  $H^2 \times H^2$ .  $\square$

*Note.* The opposite is not true namely, that for every Riemannian universal spacetime there is a Lorentzian spacetime with the same invariants. For example, the symmetric spaces  $CP^2$  and  $H^2_{\mathbb{C}}$  (with the corresponding Fubini-Study and Bargmann metrics, respectively) do not have Lorentzian counterparts. Thus the Conjecture 1.1 is signature dependent.

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## References

- [1] A. A. Coley, G. W. Gibbons, S. Hervik, and C. N. Pope, "Metrics with vanishing quantum corrections," *Classical and Quantum Gravity*, vol. 25, no. 14, Article ID 145017, 17 pages, 2008.
- [2] G. W. Gibbons and C. N. Pope, "Time-dependent multi-centre solutions from new metrics with holonomy  $Sim(n-2)$ ," *Classical and Quantum Gravity*, vol. 25, no. 12, 2008.
- [3] A. Coley, S. Hervik, and N. Pelavas, "Lorentzian spacetimes with constant curvature invariants in three dimensions," *Classical and Quantum Gravity*, vol. 25, no. 2, Article ID 025008, 2008.
- [4] A. Coley, S. Hervik, and N. Pelavas, "On spacetimes with constant scalar invariants," *Classical and Quantum Gravity*, vol. 23, no. 9, pp. 3053–3074, 2006.
- [5] A. Coley, S. Hervik, and N. Pelavas, "Lorentzian spacetimes with constant curvature invariants in four dimensions," *Classical and Quantum Gravity*, vol. 26, no. 12, Article ID 125011, 17 pages, 2009.
- [6] S. A. Fulling, R. C. King, B. G. Wybourne, and C. J. Cummins, "Normal forms for tensor polynomials. I. The Riemann tensor," *Classical and Quantum Gravity*, vol. 9, no. 5, pp. 1151–1197, 1992.
- [7] Y. Décanini and A. Folacci, "FKWC-bases and geometrical identities for classical and quantum field theories in curved spacetime," <http://arxiv.org/abs/0805.1595>.
- [8] Y. Décanini and A. Folacci, "Irreducible forms for the metric variations of the action terms of sixth-order gravity and approximated stress-energy tensor," *Classical and Quantum Gravity*, vol. 24, no. 18, pp. 4777–4799, 2007.
- [9] D. D. Bleeker, "Critical Riemannian manifolds," *Journal of Differential Geometry*, vol. 14, no. 4, pp. 599–608, 1979.
- [10] A. Coley, S. Hervik, and N. Pelavas, "Spacetimes characterized by their scalar curvature invariants," *Classical and Quantum Gravity*, vol. 26, no. 2, Article ID 025013, 33 pages, 2009.
- [11] D. G. Ebin, "The manifold of Riemannian metrics," in *Proceedings of Symposia in Pure Mathematics*, vol. 15, pp. 11–40, Berkeley, Calif, USA, 1968.
- [12] J. Isenberg and J. E. Marsden, "A slice theorem for the space of solutions of Einstein's equations," *Physics Reports*, vol. 89, no. 2, pp. 179–222, 1982.
- [13] A. Coley, A. Fuster, and S. Hervik, "Supergravity solutions with constant scalar invariants," *International Journal of Modern Physics A*, vol. 24, no. 6, pp. 1119–1133, 2009.
- [14] A. A. Coley, "A class of exact classical solutions to string theory," *Physical Review Letters*, vol. 89, no. 28, Article ID 281601, 3 pages, 2002.
- [15] A. Coley, "Classification of the Weyl tensor in higher dimensions and applications," *Classical and Quantum Gravity*, vol. 25, no. 3, Article ID 033001, 29 pages, 2008.



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