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Enriching some recent coincidence theorems for nonlinear contractions in ordered metric spaces

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Abstract
In this article, we generalize some frequently used metrical notions such as: completeness, continuity, *g*-continuity, and compatibility to order-theoretic setting especially in ordered metric spaces besides introducing some new notions such as: the *ICC* property, *DCC* property, *MCC* property *etc.* and utilize these relatively weaker notions to prove some coincidence theorems for *g*-increasing Boyd-Wong type contractions which enrich some recent results due to Alam *et al.* (Fixed Point Theory Appl. 2014:216, 2014).

MSC: 47H10; 54H25

Keywords: ordered metric space; O-completeness; O-continuity; *MCC* property

1 Introduction

In recent years, a multitude of order-theoretic metrical fixed point theorems have been proved for order-preserving contractions. This trend was essentially initiated by Turinici [1, 2]. After over two decades, Ran and Reurings [3] proved a slightly more natural version of the corresponding fixed point theorems of Turinici (*cf.* [1, 2]) for continuous monotone mappings with some applications to matrix equations. In the same lieu, Nieto and Rodríguez-López [4] proved some variants of the Ran and Reuring fixed point theorem for increasing mappings, which were generalized by many authors (*e.g.* [5–16]) in recent years. Most recently, Alam *et al.* [16] extended the foregoing results for generalized φ-contractions due to Boyd and Wong [17].

The aim of this paper is to present some existence and uniqueness results on coincidence points involving a pair of self-mappings f and g defined on ordered metric space X such that f is g-increasing Boyd-Wong type nonlinear contraction (*cf.* [17]) employing our newly introduced notions such as: O-completeness, O-continuity, (g, O)-continuity, O-compatibility, *MCC* property, $\prec\succ$-chain *etc.*

2 Preliminaries

In this section, to make our exposition self-contained, we recall some basic definitions, relevant notions and auxiliary results. Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{N}_0$ for the set of whole numbers (*i.e.* $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$).

Definition 1 [18] A set X together with a partial order $\preceq$ (often denoted by $(X, \preceq)$) is called an ordered set. As expected, $\succeq$ denotes the dual order of $\preceq$ (*i.e.* $x \succeq y$ means $y \preceq x$).

Definition 2 [18] Two elements x and y of an ordered set $(X, \preceq)$ are called comparable if either $x \preceq y$ or $x \succeq y$. For brevity, we denote it by $x \prec\succ y$.

Clearly, the relation $\prec\succ$ is reflexive and symmetric, but not transitive in general (*cf.* [19]).

Definition 3 [18] A subset E of an ordered set $(X, \preceq)$ is called totally or linearly ordered if every pair of elements of E are comparable, *i.e.*,

$$x \prec\succ y \quad \forall x, y \in E.$$

Definition 4 [1] A sequence $\{x_n\}$ in an ordered set $(X, \preceq)$ is said to be

(i) increasing or ascending if for any $m, n \in \mathbb{N}_0$,

$$m \leq n \quad \Rightarrow \quad x_m \preceq x_n,$$

(ii) decreasing or descending if for any $m, n \in \mathbb{N}_0$,

$$m \leq n \quad \Rightarrow \quad x_m \succeq x_n,$$

(iii) monotone if it is either increasing or decreasing,

(iv) bounded above if there is an element $u \in X$ such that

$$x_n \preceq u \quad \forall n \in \mathbb{N}_0$$

so that u is an upper bound of $\{x_n\}$ and

(v) bounded below if there is an element $l \in X$ such that

$$x_n \succeq l \quad \forall n \in \mathbb{N}_0$$

so that l is a lower bound of $\{x_n\}$.

Definition 5 [7] Let f and g be two self-mappings defined on an ordered set $(X, \preceq)$. We say that f is g-increasing (resp. g-decreasing) if for any $x, y \in X$, $g(x) \preceq g(y) \Rightarrow f(x) \preceq f(y)$ (resp. $f(x) \succeq f(y)$). In all, f is called g-monotone if f is either g-increasing or g-decreasing.

Notice that under the restriction $g = I$, the identity mapping on X, the notions of g-increasing, g-decreasing and g-monotone mappings reduce to increasing, decreasing and monotone mappings, respectively.

Definition 6 [20, 21] Let f and g be two self-mappings on a nonempty set X. Then

(i) an element $x \in X$ is called a coincidence point of f and g if

$$g(x) = f(x),$$

(ii) an element $\overline{x} \in X$ with $\overline{x} = g(x) = f(x)$, for some $x \in X$, is called a point of coincidence of f and g,

(iii) an element $x \in X$ is called a common fixed point of f and g if $x = g(x) = f(x)$,

(iv) the pair (f, g) is said to be commuting if for all $x \in X$,

$$g(fx) = f(gx) \quad \text{and}$$

(v) the pair (f, g) is said to be weakly compatible (or partially commuting or coincidentally commuting) if the pair (f, g) commutes at their coincidence points, *i.e.*, for any $x \in X$,

$$g(x) = f(x) \quad \Rightarrow \quad g(fx) = f(gx).$$

Definition 7 [22, 23] Let f and g be two self-mappings on a metric space (X, d). Then

(i) the pair (f, g) is said to be weakly commuting if for all $x \in X$,

$$d(gfx, fgx) \leq d(gx, fx) \quad \text{and}$$

(ii) the pair (f, g) is said to be compatible if for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = z \quad \Rightarrow \quad \lim_{n\to\infty} d(gfx_n, fgx_n) = 0.$$

Definition 8 [24] Let f and g be two self-mappings on a metric space (X, d) and $x \in X$. We say that f is g-continuous at x if for any sequence $\{x_n\} \subset X$,

$$g(x_n) \xrightarrow{d} g(x) \quad \Rightarrow \quad f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called g-continuous if it is g-continuous at each point of X.

Notice that particularly with $g = I$, the identity mapping on X, Definition 8 reduces to the definition of continuity.

Definition 9 [6] A triplet $(X, d, \preceq)$ is called an ordered metric space if (X, d) is a metric space and $(X, \preceq)$ is an ordered set.

Let $(X, d, \preceq)$ be an ordered metric space and $\{x_n\}$ a sequence in X. We adopt the following notations.

(i) If $\{x_n\}$ is increasing and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \uparrow x$.

(ii) If $\{x_n\}$ is decreasing and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \downarrow x$.

(iii) If $\{x_n\}$ is monotone and $x_n \xrightarrow{d} x$, then we denote it symbolically by $x_n \uparrow\downarrow x$.

In order to avoid the continuity requirement of underlying mapping, the following notions are formulated using suitable properties of ordered metric spaces utilized by earlier authors especially those contained in [4, 7, 25, 26] besides some other ones.

Definition 10 [16] Let $(X, d, \preceq)$ be an ordered metric space and g a self-mapping on X. We say that

(i) $(X, d, \preceq)$ has the *g-ICU* (increasing-convergence-upper bound) property if g-image of every increasing convergent sequence $\{x_n\}$ in X is bounded above by g-image of its limit (as an upper bound), *i.e.*,

$$x_n \uparrow x \quad \Rightarrow \quad g(x_n) \preceq g(x) \quad \forall n \in \mathbb{N}_0,$$

(ii) $(X, d, \preceq)$ has the *g-DCL* (decreasing-convergence-lower bound) property if g-image of every decreasing convergent sequence $\{x_n\}$ in X is bounded below by g-image of its limit (as a lower bound), *i.e.*,

$$x_n \downarrow x \quad \Rightarrow \quad g(x_n) \succeq g(x) \quad \forall n \in \mathbb{N}_0 \quad \text{and}$$

(iii) $(X, d, \preceq)$ has the *g-MCB* (monotone-convergence-boundedness) property if it has both the *g-ICU* and the *g-DCL* properties.

Notice that under the restriction $g = I$, the identity mapping on X, the notions of *g-ICU* property, *g-DCL* property, and *g-MCB* property reduce to *ICU* property, *DCL* property, and *MCB* property, respectively.

Inspired by Jleli *et al.* [12], Alam and Imdad [27] defined the following.

Definition 11 [27] Let $(X, \preceq)$ be an ordered set and f and g two self-mappings on X. We say that $(X, \preceq)$ is (f, g)-directed if for every pair $x, y \in X$, $\exists z \in X$ such that $f(x) \prec\succ g(z)$ and $f(y) \prec\succ g(z)$.

In the cases $g = I$ and $f = g = I$ (where I denotes the identity mapping on X), $(X, \preceq)$ is called f-directed and directed, respectively.

Inspired by Turinici [19], Alam and Imdad [27] defined the following.

Definition 12 [27] Let $(X, \preceq)$ be an ordered set, $E \subseteq X$ and $a, b \in E$. A finite subset $\{e_1, e_2, \ldots, e_k\}$ of E is called a $\prec\succ$-chain between a and b in E if

(i) $k \geq 2$,

(ii) $e_1 = a$ and $e_k = b$,

(iii) $e_i \prec\succ e_{i+1}$ for each i $(1 \leq i \leq k-1)$.

We denote by $\mathrm{C}(a, b, \prec\succ, E)$ the class of all $\prec\succ$-chains between a and b in E. In particular for $E = X$, we write $\mathrm{C}(x, y, \prec\succ)$ instead of $\mathrm{C}(x, y, \prec\succ, X)$.

Definition 13 [17, 28] We denote by Ω the family of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(a) $\varphi(t) < t$ for each $t > 0$,

(b) $\limsup_{r \to t^+} \varphi(r) < t$ for each $t > 0$.

We need the following well-known results in the proof of our main results.

Lemma 1 [16] *Let f and g be two self-mappings defined on an ordered set $(X, \preceq)$. If f is g-monotone and $g(x) = g(y)$, then $f(x) = f(y)$.*

Lemma 2 [16] *Let $\varphi \in \Omega$. If $\{a_n\} \subset (0,\infty)$ is a sequence such that $a_{n+1} \leq \varphi(a_n)$ $\forall n \in \mathbb{N}_0$, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 3 [16] *Let f and g be two self-mappings defined on a nonempty set X. If the pair (f,g) is weakly compatible, then every point of coincidence of f and g is also a coincidence point of f and g.*

3 Order-theoretic metrical notions

Firstly, we adopt several well-known metrical notions to order-theoretic metric setting.

Definition 14 An ordered metric space $(X,d,\preceq)$ is called
(i) $\overline{\mathrm{O}}$-complete if every increasing Cauchy sequence in X converges,
(ii) $\underline{\mathrm{O}}$-complete if every decreasing Cauchy sequence in X converges, and
(iii) O-complete if every monotone Cauchy sequence in X converges.
Here it can be pointed out that the notion of $\overline{\mathrm{O}}$-completeness was already defined by Turinici [29] stating that d is $(\preceq)$-complete.

Remark 1 In an ordered metric space, completeness $\Rightarrow$ O-completeness $\Rightarrow$ $\overline{\mathrm{O}}$-completeness as well as $\underline{\mathrm{O}}$-completeness.

Definition 15 Let $(X,d,\preceq)$ be an ordered metric space, $f: X \to X$ a mapping and $x \in X$. Then f is called:
(i) $\overline{\mathrm{O}}$-continuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,

$$x_n \uparrow x \quad \Rightarrow \quad f(x_n) \xrightarrow{d} f(x),$$

(ii) $\underline{\mathrm{O}}$-continuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,

$$x_n \downarrow x \quad \Rightarrow \quad f(x_n) \xrightarrow{d} f(x) \quad \text{and}$$

(iii) O-continuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,

$$x_n \uparrow\downarrow x \quad \Rightarrow \quad f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called O-continuous (resp. $\overline{\mathrm{O}}$-continuous, $\underline{\mathrm{O}}$-continuous) if it is O-continuous (resp. $\overline{\mathrm{O}}$-continuous, $\underline{\mathrm{O}}$-continuous) at each point of X.

Here it can be pointed out that the notion of $\overline{\mathrm{O}}$-continuity was earlier defined by Turinici [29] wherein he said that f is $(d,\preceq)$-continuous.

Remark 2 In an ordered metric space, continuity $\Rightarrow$ O-continuity $\Rightarrow$ $\overline{\mathrm{O}}$-continuity as well as $\underline{\mathrm{O}}$-continuity.

Definition 16 Let $(X,d,\preceq)$ be an ordered metric space, f and g two self-mappings on X and $x \in X$. Then f is called:
(i) $(g,\overline{\mathrm{O}})$-continuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,

$$g(x_n) \uparrow g(x) \quad \Rightarrow \quad f(x_n) \xrightarrow{d} f(x),$$

(ii) $(g, \underline{\mathrm{O}})$-continuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,

$$g(x_n) \downarrow g(x) \quad \Rightarrow \quad f(x_n) \xrightarrow{d} f(x) \quad \text{and}$$

(iii) (g, O)-continuous at $x \in X$ if for any sequence $\{x_n\} \subset X$,

$$g(x_n) \uparrow\downarrow g(x) \quad \Rightarrow \quad f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called (g, O)-continuous (resp. $(g, \overline{\mathrm{O}})$-continuous, $(g, \underline{\mathrm{O}})$-continuous) if it is (g, O)-continuous (resp. $(g, \overline{\mathrm{O}})$-continuous, $(g, \underline{\mathrm{O}})$-continuous) at each point of X.

Notice that on setting $g = I$ (the identity mapping on X), Definition 16 reduces to Definition 15.

Remark 3 In an ordered metric space, g-continuity $\Rightarrow$ (g, O)-continuity $\Rightarrow$ $(g, \overline{\mathrm{O}})$-continuity as well as $(g, \underline{\mathrm{O}})$-continuity.

Definition 17 Let $(X, d, \preceq)$ be an ordered metric space and f and g two self-mappings on X. We say that the pair (f, g) is

(i) $\overline{\mathrm{O}}$-compatible if for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$g(x_n) \uparrow z \quad \text{and} \quad f(x_n) \uparrow z \quad \Rightarrow \quad \lim_{n\to\infty} d(gfx_n, fgx_n) = 0,$$

(ii) $\underline{\mathrm{O}}$-compatible if for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$g(x_n) \downarrow z \quad \text{and} \quad f(x_n) \downarrow z \quad \Rightarrow \quad \lim_{n\to\infty} d(gfx_n, fgx_n) = 0 \quad \text{and}$$

(iii) O-compatible if for any sequence $\{x_n\} \subset X$ and for any $z \in X$,

$$g(x_n) \uparrow\downarrow z \quad \text{and} \quad f(x_n) \uparrow\downarrow z \quad \Rightarrow \quad \lim_{n\to\infty} d(gfx_n, fgx_n) = 0.$$

Here it can be pointed out that the notion of O-compatibility is slightly weaker than the notion of O-compatibility defined by Luong and Thuan [30]. Luong and Thuan [30] assumed that only the sequence $\{gx_n\}$ is monotone but we assume that both $\{gx_n\}$ and $\{fx_n\}$ are monotone.

Remark 4 In an ordered metric space, commutativity $\Rightarrow$ weak commutativity $\Rightarrow$ compatibility $\Rightarrow$ O-compatibility $\Rightarrow$ $\overline{\mathrm{O}}$-compatibility as well as $\underline{\mathrm{O}}$-compatibility $\Rightarrow$ weak compatibility.

Now, we define the following notions, which are weaker than those of Definition 10.

Definition 18 Let $(X, d, \preceq)$ be an ordered metric space. We say that:

(i) $(X, d, \preceq)$ has the *ICC* (increasing-convergence-comparable) property if every increasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{x_{n_k}\}$ is comparable with the limit of $\{x_n\}$, *i.e.*,

$$x_n \uparrow x \quad \Rightarrow \quad \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } x_{n_k} \prec\succ x \ \forall k \in \mathbb{N}_0,$$

(ii) $(X,d,\preceq)$ has the *DCC* (decreasing-convergence-comparable) property if every decreasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{x_{n_k}\}$ is comparable with the limit of $\{x_n\}$, *i.e.*,

$$x_n \downarrow x \quad \Rightarrow \quad \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } x_{n_k} \prec\succ x \ \forall k \in \mathbb{N}_0 \quad \text{and}$$

(iii) $(X,d,\preceq)$ has the *MCC* (monotone-convergence-comparable) property if every monotone convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{x_{n_k}\}$ is comparable with the limit of $\{x_n\}$, *i.e.*,

$$x_n \uparrow\downarrow x \quad \Rightarrow \quad \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } x_{n_k} \prec\succ x \ \forall k \in \mathbb{N}_0.$$

Remark 5 For an ordered metric space:
ICU property $\Rightarrow$ *ICC* property.
DCL property $\Rightarrow$ *DCC* property.
MCB property $\Rightarrow$ *MCC* property $\Rightarrow$ *ICC* property as well as *DCC* property.

Definition 19 Let $(X,d,\preceq)$ be an ordered metric space and g a self-mapping on X. We say that:

(i) $(X,d,\preceq)$ has the *g-ICC* property if every increasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g-image of the limit of $\{x_n\}$, *i.e.*,

$$x_n \uparrow x \quad \Rightarrow \quad \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } g(x_{n_k}) \prec\succ g(x) \ \forall k \in \mathbb{N}_0,$$

(ii) $(X,d,\preceq)$ has the *g-DCC* property if each decreasing convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g-image of the limit of $\{x_n\}$, *i.e.*,

$$x_n \downarrow x \quad \Rightarrow \quad \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } g(x_{n_k}) \prec\succ g(x) \ \forall k \in \mathbb{N}_0 \quad \text{and}$$

(iii) $(X,d,\preceq)$ has the *g-MCC* property if each monotone convergent sequence $\{x_n\}$ in X has a subsequence $\{x_{n_k}\}$ such that every term of $\{gx_{n_k}\}$ is comparable with g-image of the limit of $\{x_n\}$, *i.e.*,

$$x_n \uparrow\downarrow x \quad \Rightarrow \quad \exists \text{ a subsequence } \{x_{n_k}\} \text{ of } \{x_n\} \text{ with } g(x_{n_k}) \prec\succ g(x) \ \forall k \in \mathbb{N}_0.$$

Notice that on setting $g = I$ (the identity mapping on X), Definition 19 reduces to Definition 18.

Remark 6 For an ordered metric space:
g-ICU property $\Rightarrow$ *g-ICC* property.
g-DCL property $\Rightarrow$ *g-DCC* property.
g-MCB property $\Rightarrow$ *g-MCC* property $\Rightarrow$ *g-ICC* property as well as *g-DCC* property.

4 Main results

Firstly, we prove some results which ensure the existence of coincidence points.

Theorem 1 *Let $(X,d,\preceq)$ be an ordered metric space and f and g two self-mappings on X. Suppose that the following conditions hold*:

(a) $f(X)\subseteq g(X)$,
(b) *f is g-increasing*,
(c) *there exists $x_0\in X$ such that $g(x_0)\preceq f(x_0)$*,
(d) *there exists $\varphi\in\Omega$ such that*

$$d(fx,fy)\leq\varphi\bigl(d(gx,gy)\bigr)\quad\forall x,y\in X \text{ with } g(x)\prec\succ g(y),$$

(e) (e1) *$(X,d,\preceq)$ is $\overline{\mathrm{O}}$-complete*,
    (e2) *(f,g) is $\overline{\mathrm{O}}$-compatible pair*,
    (e3) *g is $\overline{\mathrm{O}}$-continuous*,
    (e4) *either f is $\overline{\mathrm{O}}$-continuous or $(X,d,\preceq)$ has the g-ICC property*,
    or alternately
(e′) (e′1) *there exists a subset Y of X such that $f(X)\subseteq Y\subseteq g(X)$ and $(Y,d,\preceq)$ is $\overline{\mathrm{O}}$-complete*,
    (e′2) *either f is $(g,\overline{\mathrm{O}})$-continuous or f and g are continuous or $(Y,d,\preceq)$ has the ICC property.*

Then f and g have a coincidence point.

Proof The proof of this theorem runs along the lines of the proof of Theorem 1 proved in [16]. We define a sequence $\{x_n\}\subset X$ (of joint iterates) such that

$$g(x_{n+1})=f(x_n)\quad\forall n\in\mathbb{N}_0. \tag{1}$$

Following the lines of the proof of Theorem 1 of [16], we can show that the sequence $\{gx_n\}$ (and hence $\{fx_n\}$ also) is increasing and Cauchy.

Assume that (e) holds. Then $\overline{\mathrm{O}}$-completeness of X implies the existence of $z\in X$ such that

$$g(x_n)\uparrow z\quad\text{and}\quad f(x_n)\uparrow z. \tag{2}$$

Owing to (2), we use $\overline{\mathrm{O}}$-continuity and $\overline{\mathrm{O}}$-compatibility instead of continuity and O-compatibility. To prove that $z\in X$ is a coincidence point of f and g, firstly we suppose that f is $\overline{\mathrm{O}}$-continuous, then proceeding along the lines of the proof of Theorem 1 of [16], we show that $f(z)=g(z)$. Otherwise suppose that $(X,d,\preceq)$ has the *g-ICC* property, then owing to (2), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(gx_{n_k})\prec\succ g(z)\quad\forall k\in\mathbb{N}_0. \tag{3}$$

As $g(x_{n_k})\uparrow z$, proceeding on the lines of the proof of Theorem 1 of [16] for the *g-ICU* property, we get $g(z)=f(z)$.

Next, assume that (e′) holds. Then the assumption $f(X) \subseteq Y$ and $\overline{\mathrm{O}}$-completeness of Y implies the existence of $y \in Y$ such that $f(x_n) \uparrow y$. Again owing to assumption $Y \subseteq g(X)$, we can find $u \in X$ such that $y = g(u)$. Hence, on using (1), we get

$$g(x_n) \uparrow g(u). \tag{4}$$

To prove that $u \in X$ is a coincidence point of f and g, firstly we suppose that f is $(g, \overline{\mathrm{O}})$-continuous, then $g(x_{n+1}) = f(x_n) \xrightarrow{d} f(u)$. Using uniqueness of the limit, $g(u) = f(u)$, and hence we are through. Next, suppose that f and g are continuous, then our proof runs on the lines of Theorem 1 of [16]. Finally, suppose that $(Y, d, \preceq)$ has the *ICC* property, then due to (4), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(x_{n_k}) \prec\succ g(u) \quad \forall k \in \mathbb{N}_0. \tag{5}$$

As $g(x_{n_k}) \uparrow g(u)$, proceeding on the lines of the proof of Theorem 1 of [16] for the *ICU* property, the desired result can also be proved. □

Theorem 2 *Theorem* 1 *remains true if certain involved terms namely*: $\overline{\mathrm{O}}$-*complete*, $\overline{\mathrm{O}}$-*compatible pair*, $\overline{\mathrm{O}}$-*continuous*, $(g, \overline{\mathrm{O}})$-*continuous*, *ICC property*, *and g-ICC property are*, *respectively*, *replaced by* $\underline{\mathrm{O}}$-*complete*, $\underline{\mathrm{O}}$-*compatible pair*, $\underline{\mathrm{O}}$-*continuous*, $(g, \underline{\mathrm{O}})$-*continuous*, *DCC property*, *and g-DCC property provided the assumption* (*c*) *is replaced by the following* (*besides retaining the rest of the hypotheses*):

(c)′ *there exists* $x_0 \in X$ *such that* $g(x_0) \succeq f(x_0)$.

Proof The proof is similar to Theorem 2 of [16]. We define a sequence $\{x_n\} \subset X$ (of joint iterates) such that

$$g(x_{n+1}) = f(x_n) \quad \forall n \in \mathbb{N}_0. \tag{6}$$

Following the lines of the proof of Theorem 2 in [16], we show that the sequence $\{gx_n\}$ (and hence also $\{fx_n\}$) is decreasing and Cauchy.

Assume that (e) holds. The $\underline{\mathrm{O}}$-completeness of X implies the existence of $z \in X$ such that

$$g(x_n) \downarrow z \quad \text{and} \quad f(x_n) \downarrow z. \tag{7}$$

In view of (7), we use $\underline{\mathrm{O}}$-continuity and $\underline{\mathrm{O}}$-compatibility instead of continuity and O-compatibility. To prove that $z \in X$ is a coincidence point of f and g, firstly we suppose that f is $\underline{\mathrm{O}}$-continuous, then proceeding on the lines of the proof of Theorem 2 of [16], we show that $f(z) = g(z)$. Otherwise suppose that $(X, d, \preceq)$ has the *g-DCC* property, then owing to (7), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(gx_{n_k}) \prec\succ g(z) \quad \forall k \in \mathbb{N}_0. \tag{8}$$

As $g(x_{n_k}) \downarrow z$, proceeding on the lines of the proof of Theorem 2 of [16] for the *g-DCL* property, we get $g(z) = f(z)$.

On the other hand, assume that (e′) holds. Then due to availability of an analogous to (4), the $\underline{\mathrm{O}}$-completeness of Y implies the existence of $u \in X$ such that

$$g(x_n) \downarrow g(u). \tag{9}$$

To prove that $u \in X$ is a coincidence point of f and g, firstly we suppose that f is $(g, \underline{\mathrm{O}})$-continuous, then $g(x_{n+1}) = f(x_n) \xrightarrow{d} f(u)$. Using the uniqueness of the limit, $g(u) = f(u)$, and hence we are done. Next, suppose that f and g are continuous, then a proof can be completed along the lines of the proof of Theorem 2 of [16]. Finally, suppose that $(Y, d, \preceq)$ has the *DCC* property, then, due to (9), there exists a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$ such that

$$g(x_{n_k}) \prec\succ g(u) \quad \forall k \in \mathbb{N}_0. \tag{10}$$

As $g(x_{n_k}) \downarrow g(u)$, proceeding on the lines of the proof of Theorem 2 of [16] for the *DCL* property, this result can be proved. □

Now, combining Theorems 1 and 2 and making use of Remarks 1-6, we obtain the following result.

Theorem 3 *Theorem* 1 *remains true if certain involved terms namely*: $\overline{\mathrm{O}}$*-complete*, $\overline{\mathrm{O}}$*-compatible pair*, $\overline{\mathrm{O}}$*-continuous*, $(g, \overline{\mathrm{O}})$*-continuous*, *ICC property*, *and g-ICC property are*, *respectively*, *replaced by O-complete*, *O-compatible pair*, *O-continuous*, (g, O)*-continuous*, *MCC property*, *and g-MCC property provided the assumption* (c) *is replaced by the following* (*besides retaining the rest*):

(c)″ *there exists* $x_0 \in X$ *such that* $g(x_0) \prec\succ f(x_0)$.

Remark 7 In view of Remarks 1-6, it is clear that Theorems 1, 2 and 3 enrich, respectively, Theorems 1, 2, and 3 of Alam *et al.* [16].

Taking $\varphi(t) = \alpha t$ with $\alpha \in [0,1)$, in Theorem 1 (resp. in Theorem 2 or Theorem 3), we get the corresponding results for linear contractions as follows.

Corollary 1 *Theorem* 1 (*resp. Theorem* 2 *or Theorem* 3) *remains true if we replace condition* (d) *by the following condition* (*besides retaining the rest of the hypotheses*):

(d)′ *there exists* $\alpha \in [0,1)$ *such that*

$$d(fx, fy) \le \alpha d(gx, gy) \quad \forall x, y \in X \text{ with } g(x) \prec\succ g(y).$$

Now, we prove certain results ensuring the uniqueness of coincidence point, point of coincidence, and common fixed point corresponding to some earlier results. For a pair f and g of self-mappings on a nonempty set X, we adopt the following notations:

$$\mathrm{C}(f,g) = \{x \in X : gx = fx\}, \quad \textit{i.e.}, \text{ the set of all coincidence points of } f \text{ and } g,$$

$$\overline{\mathrm{C}}(f,g) = \{\overline{x} \in X : \text{there exists an } x \in X \text{ such that } \overline{x} = gx = fx\},$$

i.e., the set of all points of coincidence of f and g.

Theorem 4 *In addition to the hypotheses* (a)-(d) *along with* (e′) *of Theorem* 1 (*resp. Theorem* 2 *or Theorem* 3), *suppose that the following condition* (*see Definition* 12) *holds*:

(u_0) $C(fx, fy, \prec\succ, gX)$ *is nonempty, for each* $x, y \in X$.

Then f *and* g *have a unique point of coincidence.*

Proof In view of Theorem 1 (resp. Theorem 2 or Theorem 3), $\overline{C}(f,g) \neq \emptyset$. Take $\overline{x}, \overline{y} \in \overline{C}(f,g)$, then $\exists x, y \in X$ such that

$$\overline{x} = g(x) = f(x) \quad \text{and} \quad \overline{y} = g(y) = f(y). \tag{11}$$

Now, we show that $\overline{x} = \overline{y}$. As $f(x), f(y) \in f(X) \subseteq g(X)$, by ($u_0$), there exists a $\prec\succ$-chain $\{gz_1, gz_2, \ldots, gz_k\}$ between $f(x)$ and $f(y)$ in $g(X)$, where $z_1, z_2, \ldots, z_k \in X$. Owing to (11), without loss of generality, we can choose $z_1 = x$ and $z_k = y$. We have

$$g(z_i) \prec\succ g(z_{i+1}) \quad \text{for each } i\ (1 \leq i \leq k-1). \tag{12}$$

Define the constant sequences $z_n^1 = z_1 = x$ and $z_n^k = z_k = y$, then using (11), we have $g(z_{n+1}^1) = f(z_n^1)$ and $g(z_{n+1}^k) = f(z_n^k)$ $\forall n \in \mathbb{N}_0$. Put $z_0^2 = z_2, z_0^3 = z_3, \ldots, z_0^{k-1} = z_{k-1}$. Since $f(X) \subseteq g(X)$, we can define sequences $\{z_n^2\}, \{z_n^3\}, \ldots, \{z_n^{k-1}\}$ in X such that $g(z_{n+1}^2) = f(z_n^2), g(z_{n+1}^3) = f(z_n^3), \ldots, g(z_{n+1}^{k-1}) = f(z_n^{k-1})$ $\forall n \in \mathbb{N}_0$. Hence, we have

$$g\bigl(z_{n+1}^i\bigr) = f\bigl(z_n^i\bigr) \quad \forall n \in \mathbb{N}_0 \text{ and for each } i\ (1 \leq i \leq k). \tag{13}$$

Now, we claim that

$$g\bigl(z_n^i\bigr) \prec\succ g\bigl(z_n^{i+1}\bigr) \quad \forall n \in \mathbb{N}_0 \text{ and for each } i\ (1 \leq i \leq k-1). \tag{14}$$

We prove this fact by induction. It follows from (12) that (14) holds for $n = 0$. Suppose that (14) holds for $n = r > 0$, *i.e.*,

$$g\bigl(z_r^i\bigr) \prec\succ g\bigl(z_r^{i+1}\bigr) \quad \text{for each } i\ (1 \leq i \leq k-1).$$

As f is g-increasing, we obtain

$$f\bigl(z_r^i\bigr) \prec\succ f\bigl(z_r^{i+1}\bigr) \quad \text{for each } i\ (1 \leq i \leq k-1),$$

which on using (13), gives rise to

$$g\bigl(z_{r+1}^i\bigr) \prec\succ g\bigl(z_{r+1}^{i+1}\bigr) \quad \text{for each } i\ (1 \leq i \leq k-1).$$

It follows that (14) holds for $n = r + 1$. Thus, by induction, (14) holds for all $n \in \mathbb{N}_0$. Now, for each $n \in \mathbb{N}_0$ and for each i ($1 \leq i \leq k-1$), define $t_n^i := d(gz_n^i, gz_n^{i+1})$. We claim that

$$\lim_{n\to\infty} t_n^i = 0 \quad \text{for each } i\ (1 \leq i \leq k-1). \tag{15}$$

On fixing i, two cases arise. Firstly, suppose that $t^i_{n_0} = d(gz^i_{n_0}, gz^{i+1}_{n_0}) = 0$ for some $n_0 \in \mathbb{N}_0$, then by Lemma 1, we obtain $d(fz^i_{n_0}, fz^{i+1}_{n_0}) = 0$. Consequently on using (13), we get $t^i_{n_0+1} = d(gz^i_{n_0+1}, gz^{i+1}_{n_0+1}) = d(fz^i_{n_0}, fz^{i+1}_{n_0}) = 0$. Thus by induction, we get $t^i_n = 0$ $\forall n \geq n_0$, yielding thereby $\lim_{n\to\infty} t^i_n = 0$. Secondly, suppose that $t_n > 0$ $\forall n \in \mathbb{N}_0$, then on using (13), (14), and assumption (d), we have

$$\begin{aligned} t^i_{n+1} &= d\bigl(gz^i_{n+1}, gz^{i+1}_{n+1}\bigr) \\ &= d\bigl(fz^i_n, fz^{i+1}_n\bigr) \\ &\leq \varphi\bigl(d\bigl(gz^i_n, z^{i+1}_n\bigr)\bigr) \\ &= \varphi\bigl(t^i_n\bigr), \end{aligned}$$

so that

$$t^i_{n+1} \leq \varphi\bigl(t^i_n\bigr).$$

Now, on applying Lemma 2, we obtain $\lim_{n\to\infty} t^i_n = 0$. Thus, in both cases, (15) is proved for each i $(1 \leq i \leq k-1)$. On using the triangular inequality and (15), we obtain

$$d(\overline{x}, \overline{y}) \leq t^1_n + t^2_n + \cdots + t^{k-1}_n \to 0 \quad \text{as } n \to \infty$$

so that

$$\overline{x} = \overline{y}. \qquad \square$$

Theorem 5 *In addition to the hypotheses of Theorem* 4, *suppose that the following condition holds*:

(u_1) *one of f and g is one-one.*

Then f and g have a unique coincidence point.

Proof In view of Theorem 1 (or Theorem 2 or Theorem 3), $C(f,g) \neq \emptyset$. Take $x, y \in C(f,g)$, then using Theorem 4, we can write

$$g(x) = f(x) = f(y) = g(y).$$

As f or g is one-one, we have

$$x = y. \qquad \square$$

Theorem 6 *In addition to the hypotheses of Theorem* 4, *suppose that the following condition holds*:

(u_2) *(f,g) is weakly compatible pair.*

Then f and g have a unique common fixed point.

Proof Let x be a coincidence point of f and g. Write $g(x) = f(x) = \overline{x}$. In view of Lemma 3 and (u_2), $\overline{x}$ is also a coincidence point of f and g. It follows from Theorem 4 with $y = \overline{x}$ that $g(x) = g(\overline{x})$, *i.e.*, $\overline{x} = g(\overline{x})$, which shows

$$\overline{x} = g(\overline{x}) = f(\overline{x}).$$

Hence, $\overline{x}$ is a common fixed point of f and g. To prove uniqueness, assume that x^* is another common fixed point of f and g. Then again from Theorem 4, we have

$$x^* = g(x^*) = g(\overline{x}) = \overline{x}.$$

This completes the proof. □

Theorem 7 *In addition to the hypotheses* (a)-(e) *of Theorem* 1 (*resp. Theorem* 2 *or Theorem* 3), *suppose that the condition* (u_0) (*of Theorem* 4) *holds. Then f and g have a unique common fixed point.*

Proof We know that in an ordered metric space, each of an O-compatible pair, an $\overline{\text{O}}$-compatible pair, and an $\underline{\text{O}}$-compatible pair is weakly compatible so that (u_2) is trivially satisfied. Hence proceeding along the lines of the proofs of Theorems 4 and 6 our result follows. □

Corollary 2 *Theorem* 4 (*resp. Theorem* 7) *remains true if we replace the condition* (u_0) *by one of the following conditions* (*besides retaining rest of the hypotheses*):

(u_0^1) $(fX, \preceq)$ *is totally ordered,*
(u_0^2) $(X, \preceq)$ *is* (f,g)*-directed.*

Proof Suppose that (u_0^1) holds, then for each pair $x, y \in X$, we have

$$f(x) \prec\succ f(y),$$

which implies that $\{fx, fy\}$ is a $\prec\succ$-chain between $f(x)$ and $f(y)$ in $g(X)$. It follows that $C(fx, fy, \prec\succ, gX)$ is nonempty for each $x, y \in X$, *i.e.*, (u_0) holds and hence Theorem 4 (resp. Theorem 7) is applicable.

Next, assume that (u_0^2) holds, then for each pair $x, y \in X$, $\exists z \in X$ such that

$$f(x) \prec\succ g(z) \prec\succ f(y),$$

which implies that $\{fx, gz, fy\}$ is a $\prec\succ$-chain between $f(x)$ and $f(y)$ in $g(X)$. It follows that $C(fx, fy, \prec\succ, gX)$ is nonempty for each $x, y \in X$, *i.e.*, (u_0) holds and hence Theorem 4 (resp. Theorem 7) is applicable. □

Remark 8 Notice that Alam *et al.* [16] used condition (u_0^2) to prove uniqueness results (see Theorem 5 [16] along with comments). Here, we use condition (u_0), which is relatively weak in view of Corollary 2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed equally. Thus formally, all the authors read and approved the final manuscript.

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