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Classical solutions for the Cahn-Hilliard equation with decayed mobility

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Abstract

We consider the Cahn-Hilliard equation with concentration dependent mobility in two dimensions. Global existence and uniqueness of classical solutions are established for a mobility with some delayed structure and general potential including $-u + \gamma u^3$ for both $\gamma > 0$ and $\gamma < 0$.

Keywords: Cahn-Hilliard equation; decayed mobility; classical solutions

1 Introduction

The Cahn-Hilliard equation, as an important continuous model for a phase transition with a conservative order parameter, arises from a continuum model for a phase transition in binary systems such as alloys, glasses, and polymer-mixtures; see for example [1–3]. It can also be used to characterize the case of the zig-zag instability for the interface dynamics in liquid crystals; see Chevallard *et al.* [4]. Due to its important physical background, this type of equations has been the subject of intensive study by mathematical and physical scientists in recent years; see [5–9] and the references therein. Our interest is particularly motivated by the study of a spinodal decomposition, a phenomenon in which rapid cooling of a homogeneously mixed binary alloy causes separation to occur, resolving the mixture into regions in which one component or the other is dominant. In this context, u typically denotes the concentration of one component of the binary alloy. In this paper we consider the following Cahn-Hilliard equation in two spatial dimensions:

$$\frac{\partial u}{\partial t} + \operatorname{div}\big[m(u)\big(k\nabla\Delta u - \nabla\varphi(u)\big)\big] = 0, \quad (x,t) \in Q_T \equiv \Omega \times (0,T), \tag{1.1}$$

supplemented by the zero mass flux boundary condition

$$\vec{J} \cdot \nu = m(u)\big(k\nabla\Delta u - \nabla\varphi(u)\big) \cdot \nu = 0, \quad x \in \partial\Omega, t \in (0,T), \tag{1.2}$$

the natural boundary value condition

$$\nabla u \cdot \nu = 0, \quad x \in \partial\Omega, t \in (0,T), \tag{1.3}$$

and the initial value condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, ν denotes the unit exterior normal to the boundary $\partial\Omega$, $u = u(x,t)$ is the unknown function describing the concentration of one of the phases of the system, k is a positive constant, $m(u) > 0$ is the mobility depending on the concentration, $\varphi(u)$ is the intrinsic chemical potential, whose typical example is the so-called double-well potential, namely

$$\varphi(u) = -u + \gamma u^3 \tag{1.5}$$

for $\gamma \neq 0$ being a constant. It follows from the boundary value condition (1.3) that the boundary value condition (1.2) can be replaced by

$$\nabla\Delta u(x,t)\cdot\nu = 0, \quad x \in \partial\Omega, t \in (0,T). \tag{1.6}$$

For the readers' convenience, we sketch the derivation of the Cahn-Hilliard equation (1.1) here. One can also find it in almost the same fashion in [10, 11]. We start with a free energy functional of the form, given by Cahn and Hilliard [1],

$$F[u] = \int_\Omega \left(H(u) + \frac{1}{2}k|\nabla u|^2\right) dx, \tag{1.7}$$

where $H'(u) = \varphi(u)$. The Cahn-Hilliard equation arises from the conservation law

$$\frac{\partial u}{\partial t} + \operatorname{div}\vec{J} = 0, \tag{1.8}$$

where $\vec{J}$ is the flux of the order parameter u. A standard phenomenological and well-accepted law for the flux $\vec{J}$ is given by

$$\vec{J} = -m(u)\nabla\mu,$$

where $m(u)$ denotes the mobility associated with concentration u, and is typically assumed to be positive. That is, the composition of the alloy tends to change from configurations for which a small change in concentration is accompanied by a large change in total free energy into configurations to ones in which a small change in concentration is accompanied by a small change in total free energy. Here μ is the chemical potential. Usually, the chemical potential is the derivative of the free energy with respect to the order parameter u. But, since the term ∇u occurs in (1.7), this is no longer valid. Instead μ is now defined as the variational derivative of (1.7) with respect to u. That is, $\mu = \frac{\delta F}{\delta u}$. Then we have

$$\vec{J} = -m(u)\nabla\frac{\delta F}{\delta u}. \tag{1.9}$$

Combining (1.7), (1.8), and (1.9), and by a simple calculation, we obtain the desired Cahn-Hilliard equation (1.1).

The Cahn-Hilliard equation with constant mobility, *i.e.* $m(u) \equiv$ const., has been intensively studied. In one spatial dimension, a well-known work is by to Elliott and Zheng [6], who showed that the sign of γ is crucial to the global existence of solutions. Exactly speaking, if $\gamma > 0$ then solutions exist always globally in time; while if $\gamma < 0$, then solutions must blow up in a finite time for large initial data. From the physical point of view, the mobility should depend on the concentration. In general, for $m(u)$, there is no restriction with positive lower bound, but there is a possibility with degeneracy, see for example [3, 12–14], where, for $\gamma > 0$, the existence of weak solutions for the degenerate case and classical solutions for the uniformly parabolic case are established, respectively. Our interest lies in the case that the mobility $m(s) > 0$ but decays as $s \to \infty$. Some discussion in this topic for one spatial dimension was given in our previous work [8], while the present paper is focused on the discussion for two dimensions. The case with decayed mobility $m(u)$ not only gets rid of some properties for the case $m(u) \equiv$ const., but it also exhibits some new features compared to the case $m(u) \equiv$ const. Indeed, the existence result can be established for both $\gamma < 0$ and $\gamma > 0$ without essential restrictions on the initial data, see [15] for some information in one spatial dimension. However, compared to the one-dimensional case [15], the present work will encounter more difficulties in the arguments of the regularity of solutions. For this reason, we employ the framework based on Campanato spaces to obtain the Hölder continuity of higher order derivatives of solutions.

The main result of this paper is the following theorem.

Theorem 1.1 *Assume that $m(s)$ belongs to $C^{1+\alpha}(\mathbb{R})$ for some $\alpha \in (0,1)$, and there exist positive constants C and p, such that for any $s \in \mathbb{R}$,*

$$0 < m(s) \le C\bigl(1+s^2\bigr)^{-(p+1)}, \qquad \bigl|m'(s)\bigr|^2 \le Cm(s), \tag{1.10}$$

$$\bigl|\varphi'(s)\bigr| \le C\bigl(|s|^{p+1}+1\bigr), \qquad \bigl|\varphi''(s)\bigr| \le C\bigl(|s|^{p}+1\bigr). \tag{1.11}$$

Then the problem (1.1)-(1.4) *admits a unique classical solution for a small smooth initial value $u_0(x)$.*

This paper is organized as follows. Section 2 is devoted to the *a priori* Hölder norm estimates; the gradient Hölder norm estimates are given subsequently in Section 3. Finally, in the last section we prove the existence and uniqueness of classical solutions to the problem (1.1)-(1.4).

2 Hölder norm estimates

In this section, by means of energy estimates, we establish the *a priori* Hölder norm estimates on solutions to the problem (1.1)-(1.4).

Proposition 2.1 *If u is a solution of the problem* (1.1)-(1.4), *and all the assumptions in Theorem* 1.1 *hold, then there exists a constant* $0 < \alpha < 1$, *such that*

$$\bigl|u(x_1,t_1)-u(x_2,t_2)\bigr| \le C\bigl(|x_1-x_2|^{\alpha}+|t_1-t_2|^{\alpha/4}\bigr), \quad \forall (x_1,t_1),(x_2,t_2)\in Q_T,$$

where C is a positive constant.

Proof Multiplying both sides of (1.1) by Δu and integrating the resulting relation with respect to x over Ω, we have

$$\int_\Omega u_t \Delta u\,dx + \int_\Omega \operatorname{div}\bigl[m(u)\bigl(k\nabla\Delta u - \nabla\varphi(u)\bigr)\bigr]\Delta u\,dx = 0.$$

By the assumptions (1.10), (1.11), we have $m(s)|\varphi'(s)|^2 \le C$, $\forall s \in \mathbb{R}$. Noticing this fact and integrating the above equality by parts, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + 2k\int_\Omega m(u)|\nabla\Delta u|^2\,dx \\
&\quad = 2\int_\Omega m(u)\varphi'(u)\nabla u\nabla\Delta u\,dx \\
&\quad \le k\int_\Omega m(u)|\nabla\Delta u|^2\,dx + \frac{1}{k}\int_\Omega m(u)\big|\varphi'(u)\big|^2|\nabla u|^2\,dx \\
&\quad \le k\int_\Omega m(u)|\nabla\Delta u|^2\,dx + C\int_\Omega |\nabla u|^2\,dx,
\end{aligned}$$

namely,

$$\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx + k\int_\Omega m(u)|\nabla\Delta u|^2\,dx \le C\int_\Omega |\nabla u|^2\,dx.$$

It follows from the Gronwall inequality that

$$\begin{aligned}
&\int_\Omega |\nabla u|^2\,dx \le C\int_\Omega |\nabla u_0|^2\,dx, \quad \forall 0 < t < T, \qquad (2.1)\\
&\iint_{Q_T} m(u)|\nabla\Delta u|^2\,dx\,dt \le C\int_\Omega |\nabla u_0|^2\,dx.
\end{aligned}$$

Multiplying both sides of (1.1) by $\Delta^2 u$ and integrating the resulting relation with respect to x over Ω, we have

$$\int_\Omega u_t\Delta^2 u\,dx + \int_\Omega \operatorname{div}\big[m(u)\big(k\nabla\Delta u - \nabla\varphi(u)\big)\big]\Delta^2 u\,dx = 0.$$

By the boundary conditions (1.3) and (1.6), we integrate by parts to conclude

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2\,dx + k\int_\Omega m(u)\big|\Delta^2 u\big|^2\,dx \\
&\quad = -k\int_\Omega m'(u)\nabla u\nabla\Delta u\Delta^2 u\,dx + \int_\Omega m(u)\varphi'(u)\Delta u\Delta^2 u\,dx \\
&\qquad + \int_\Omega m(u)\varphi''(u)|\nabla u|^2\Delta^2 u\,dx \\
&\qquad + \int_\Omega m'(u)\varphi'(u)|\nabla u|^2\Delta^2 u\,dx.
\end{aligned}$$

It follows from (1.10), (1.11), and the Hölder inequality that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |\Delta u|^2\,dx + k\int_\Omega m(u)\big|\Delta^2 u\big|^2\,dx \\
&\quad \le \frac{k}{4}\int_\Omega m(u)\big|\Delta^2 u\big|^2\,dx + C\int_\Omega |\nabla u|^2|\nabla\Delta u|^2\,dx \\
&\qquad + C\int_\Omega |\Delta u|^2\,dx + C\int_\Omega \big(\big|\varphi''(u)\big|^2 + \big|\varphi'(u)\big|^2\big)|\nabla u|^4\,dx
\end{aligned}$$

$$\le \frac{k}{4}\int_\Omega m(u)\left|\Delta^2 u\right|^2 dx + C\left(\int_\Omega |\nabla u|^8\,dx\right)^{1/4}\left(\int_\Omega |\nabla\Delta u|^{8/3}\,dx\right)^{3/4}$$
$$+ C\int_\Omega |\Delta u|^2\,dx + C\sup\left(\left|\varphi''(u)\right|^2 + \left|\varphi'(u)\right|^2\right)\int_\Omega |\nabla u|^4\,dx. \qquad (2.2)$$

Next, we will estimate the terms in the right-hand side of the above inequality (2.2). It follows from the Cagliardo-Nirenberg inequality that

$$\left(\int_\Omega |\nabla u|^8\,dx\right)^{1/8}$$
$$\le C\left(\int_\Omega |\nabla u|^2\,dx\right)^{3/8}\left(\left(\int_\Omega \left|\Delta^2 u\right|^2 dx\right)^{1/8} + \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/8}\right)$$

and

$$\left(\int_\Omega |\nabla\Delta u|^{8/3}\,dx\right)^{3/8}$$
$$\le C\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/8}\left(\left(\int_\Omega \left|\Delta^2 u\right|^2 dx\right)^{3/8} + \left(\int_\Omega |\nabla u|^2\,dx\right)^{3/8}\right).$$

Combining the above inequality with (2.1) and the Young inequality, we have

$$\left(\int_\Omega |\nabla u|^8\,dx\right)^{1/4}\left(\int_\Omega |\nabla\Delta u|^{8/3}\,dx\right)^{3/4}$$
$$\le C\int_\Omega |\nabla u|^2\,dx\left(\left(\int_\Omega \left|\Delta^2 u\right|^2 dx\right)^{1/4} + \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/4}\right)$$
$$\cdot\left(\left(\int_\Omega \left|\Delta^2 u\right|^2 dx\right)^{3/4} + \left(\int_\Omega |\nabla u|^2\,dx\right)^{3/4}\right)$$
$$\le C\int_\Omega |\nabla u|^2\,dx\left(\int_\Omega \left|\Delta^2 u\right|^2 dx + \int_\Omega |\nabla u|^2\,dx\right.$$
$$+ \left(\int_\Omega \left|\Delta^2 u\right|^2 dx\right)^{1/4}\cdot\left(\int_\Omega |\nabla u|^2\,dx\right)^{3/4}$$
$$\left. + \left(\int_\Omega \left|\Delta^2 u\right|^2 dx\right)^{3/4}\cdot\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/4}\right)$$
$$\le C\int_\Omega |\nabla u|^2\,dx\left(\int_\Omega \left|\Delta^2 u\right|^2 dx + \int_\Omega |\nabla u|^2\,dx\right)$$
$$\le C\int_\Omega |\nabla u_0|^2\,dx\left(\int_\Omega \left|\Delta^2 u\right|^2 dx + \int_\Omega |\nabla u_0|^2\,dx\right). \qquad (2.3)$$

For the third term in the right-hand side of (2.2), we first notice that

$$\int_\Omega |\Delta u|^2\,dx = -\int_\Omega \nabla u\nabla\Delta u\,dx$$
$$\le \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega |\nabla\Delta u|^2\,dx\right)^{1/2}$$
$$= \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}\left(-\int_\Omega \Delta u\Delta^2 u\,dx\right)^{1/2}$$

$$\le \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}\left(\int_\Omega |\Delta u|^2\,dx\right)^{1/4}\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{1/4}$$
$$\le C\left(\int_\Omega |\nabla u_0|^2\,dx\right)^{1/2}\left(\int_\Omega |\Delta u|^2\,dx\right)^{1/4}\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{1/4}.$$

Then we have

$$\int_\Omega |\Delta u|^2\,dx \le C\left(\int_\Omega |\nabla u_0|^2\,dx\right)^{2/3}\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{1/3}$$
$$\le C\int_\Omega |\nabla u_0|^2\,dx\int_\Omega \left|\Delta^2 u\right|^2\,dx + C\left(\int_\Omega |\nabla u_0|^2\,dx\right)^{1/2}. \tag{2.4}$$

For the fourth term in the right-hand side of (2.2), we use the Cagliardo-Nirenberg inequality to conclude that

$$\left(\int_\Omega |\nabla u|^4\,dx\right)^{1/4}$$
$$\le C\left(\int_\Omega |\nabla u|^2\,dx\right)^{5/12}\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{1/12} + C\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}.$$

It follows from (2.1) that

$$\int_\Omega |\nabla u|^4\,dx \le C\left(\int_\Omega |\nabla u|^2\,dx\right)^{5/3}\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{1/3} + C\left(\int_\Omega |\nabla u|^2\,dx\right)^{2}$$
$$\le C\left(\int_\Omega |\nabla u_0|^2\,dx\right)^{5/3}\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{1/3} + C\left(\int_\Omega |\nabla u_0|^2\,dx\right)^{2}. \tag{2.5}$$

On the other hand, for small $\delta > 0$, by the embedding theorem and the Poincaré inequality, we have

$$\sup|u| \le C\left(\int_\Omega |\nabla u|^{(2+\delta)}\,dx\right)^{1/(2+\delta)} + C\left(\int_\Omega |u|^{(2+\delta)}\,dx\right)^{1/(2+\delta)}$$
$$\le C\sup|\nabla u|^{\delta/(2+\delta)}\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/(2+\delta)}$$
$$+ C\sup|u|^{\delta/(2+\delta)}\left(\int_\Omega u^2\,dx\right)^{1/(2+\delta)}$$
$$\le C\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{\delta/(2+\delta)}\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/(2+\delta)}$$
$$+ C\sup|u|^{\delta/(2+\delta)}\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/(2+\delta)}.$$

Then it follows from the Young inequality and (2.1) that

$$\sup|u| \le C\left(\int_\Omega \left|\Delta^2 u\right|^2\,dx\right)^{\delta/(2+\delta)}\left(\int_\Omega |\nabla u_0|^2\,dx\right)^{1/(2+\delta)} + C\left(\int_\Omega |\nabla u_0|^2\,dx\right)^{1/2}.$$

Combining the above inequality with (2.5), by Young's inequality, we have

$$\begin{aligned}
&\sup\left(\left|\varphi''(u)\right|^2+\left|\varphi'(u)\right|^2\right)\int_\Omega|\nabla u|^4\,dx\\
&\quad\le C\left(\left(\int_\Omega\left|\Delta^2u\right|^2dx\right)^{q\delta/(2+\delta)}+1\right)\\
&\qquad\cdot\left(\left(\int_\Omega|\nabla u_0|^2\,dx\right)^{5/3}\left(\int_\Omega\left|\Delta^2u\right|^2dx\right)^{1/3}+C\left(\int_\Omega|\nabla u_0|^2\,dx\right)^2\right)\\
&\quad\le C\left(\left(\int_\Omega|\nabla u_0|^2\,dx\right)^{5/3}\int_\Omega\left|\Delta^2u\right|^2dx+1\right),
\end{aligned}\tag{2.6}$$

where q is a positive constant depending on p, and here we used the smallness of δ to conclude that $q\delta/(2+\delta)\le 2/3$.

It follows from (2.2), (2.3), (2.4), (2.6), and the smallness of $\|u_0(x)\|_{H^1(\Omega)}$ that

$$\frac{d}{dt}\int_\Omega|\Delta u|^2\,dx+k\int_\Omega m(u)\left|\Delta^2u\right|^2dx\le C.$$

Then we have

$$\sup_{0<t<T}\int_\Omega\left|\Delta u(x,t)\right|^2dx\le C.\tag{2.7}$$

By (2.7) and the embedding theorem, we know that there exists a constant $0<\alpha<1$, such that

$$\left|u(x_1,t)-u(x_2,t)\right|\le C|x_1-x_2|^\alpha,\quad\forall(x_1,t),(x_2,t)\in Q_T.$$

Then, by (1.1) itself, we can conclude that

$$\left|u(x,t_1)-u(x,t_2)\right|\le C|t_1-t_2|^{\alpha/4}$$

holds for any given $(x,t_1),(x,t_2)\in Q_T$. The proof of this proposition is complete. □

3 Gradient Hölder norm estimates

In this section, we establish the gradient Hölder norm estimates on solutions to the problem (1.1)-(1.4).

Proposition 3.1 *If u is a solution of the problem* (1.1)-(1.4), *and all the assumptions in Theorem* 1.1 *hold, then there exists a constant* $0<\alpha<1$, *such that*

$$\left|\nabla u(x_1,t_1)-\nabla u(x_2,t_2)\right|\le C\left(|x_1-x_2|^\alpha+|t_1-t_2|^{\alpha/4}\right),\quad\forall(x_1,t_1),(x_2,t_2)\in Q_T,$$

where C is a positive constant.

We will employ the theory of Campanato spaces to prove Proposition 3.1. That is to say, we use the Campanato spaces to describe the integral characteristic of the Hölder continuous functions. To shorten the length of this paper, we omit the definition and properties of the Campanato spaces, which can be found in [14, 16–18]. In order to obtain the *a priori* estimate on the solutions in a suitable Campanato space, we first rewrite (1.1) into the following form:

$$\frac{\partial u}{\partial t} + \operatorname{div}\big(a(x,t)\nabla\Delta u\big) = \operatorname{div}\vec{f}, \tag{3.1}$$

where

$$a(x,t) = km\big(u(x,t)\big), \qquad \vec{f} = m\big(u(x,t)\big)\nabla\varphi\big(u(x,t)\big).$$

Since the Hölder norm estimate of u has been already established in the previous section, we may assume that $a(x,t)$ is a known Hölder continuous function. For a qualitative calculation, without loss of generality, we may also assume that $a(x,t)$ and $\vec{f}$ are sufficiently smooth, otherwise we replace them by their approximation functions.

Let $(x_0,t_0) \in \Omega \times (0,T)$ be fixed and define

$$\theta(u,\rho) = \iint_{S_\rho} \big(\big|\nabla u - (\nabla u)_\rho\big|^2 + \rho^4|\nabla\Delta u|^2\big)\,dx\,dt,$$

where

$$S_\rho = \big(t_0 - \rho^4, t_0 + \rho^4\big) \times B_\rho(x_0), \qquad (\nabla u)_\rho = \frac{1}{|S_\rho|}\iint_{S_\rho} \nabla u\,dx\,dt$$

and $B_\rho(x_0)$ is the ball centered at x_0 with radius ρ.

We split the solution u of the problem (1.1)-(1.4) on S_R as $u = u_1 + u_2$, where u_1 is the solution of the problem

$$\frac{\partial u_1}{\partial t} + a(x_0,t_0)\Delta^2 u_1 = 0, \quad (x,t) \in S_R, \tag{3.2}$$

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u}{\partial \nu}, \qquad \frac{\partial \Delta u_1}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu}, \quad (x,t) \in \big(t_0 - R^4, t_0 + R^4\big) \times \partial B_R(x_0), \tag{3.3}$$

$$u_1 = u, \qquad t = t_0 - R^4, \quad x \in B_R(x_0), \tag{3.4}$$

and u_2 solves the problem

$$\frac{\partial u_2}{\partial t} + a(x_0,t_0)\Delta^2 u_2 = \operatorname{div}\vec{f} + \operatorname{div}\big[\big(a(x_0,t_0) - a(x,t)\big)\nabla\Delta u\big], \quad (x,t) \in S_R, \tag{3.5}$$

$$\frac{\partial u_2}{\partial \nu} = 0, \qquad \frac{\partial \Delta u_2}{\partial \nu} = 0, \quad (x,t) \in \big(t_0 - R^4, t_0 + R^4\big) \times \partial B_R(x_0), \tag{3.6}$$

$$u_1 = 0, \qquad t = t_0 - R^4, \quad x \in B_R(x_0). \tag{3.7}$$

By the classical linear theory, the above decomposition is uniquely determined by u. The following lemmas will be used to establish the *a priori* estimates of the solutions in the Campanato space.

Lemma 3.1 *Assume there exists a constant* $0 < \sigma < 1$ *such that*

$$\big|a(x,t) - a(x_0,t_0)\big| \le C\big(|x - x_0|^\sigma + |t - t_0|^{\sigma/4}\big)$$

holds for any given $x \in B_R(x_0)$ *and* $t \in (t_0 - R^4, t_0 + R^4)$. *Then*

$$\sup_{(t_0-R^4,t_0+R^4)} \int_{B_R(x_0)} \left|\nabla u_2(x,t)\right|^2 dx + \iint_{S_R} |\nabla\Delta u_2|^2\,dx\,dt$$
$$\le C\sup_{S_R} |\vec{f}|^2 R^6 + CR^{2\sigma} \int_{t_0-R^4}^{t} \int_{B_R(x_0)} |\nabla\Delta u|^2\,dx\,dt.$$

Proof Multiplying both sides of (3.5) by Δu_2 and integrating the resulting relation over $(t_0 - R^4, t) \times B_R(x_0)$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{B_R(x_0)} \left|\nabla u_2(x,t)\right|^2 dx + a(x_0,t_0)\int_{t_0-R^4}^{t}\int_{B_R(x_0)} |\nabla\Delta u_2|^2\,dx\,dt \\
&\quad= \int_{t_0-R^4}^{t}\int_{B_R(x_0)} \left[\left(a(x_0,t_0) - a(x,t)\right)\nabla\Delta u\right]\cdot\nabla\Delta u_2\,dx\,dt \\
&\qquad+ \int_{t_0-R^4}^{t}\int_{B_R(x_0)} \vec{f}\cdot\nabla\Delta u_2\,dx\,dt \\
&\quad\le \frac{1}{2}a(x_0,t_0)\int_{t_0-R^4}^{t}\int_{B_R(x_0)} |\nabla\Delta u_2|^2\,dx\,dt + C\int_{t_0-R^4}^{t}\int_{B_R(x_0)} |\vec{f}|^2\,dx\,dt \\
&\qquad+ C\int_{t_0-R^4}^{t}\int_{B_R(x_0)} \left|\left(a(x_0,t_0) - a(x,t)\right)\nabla\Delta u\right|^2\,dx\,dt \\
&\quad\le \frac{1}{2}a(x_0,t_0)\int_{t_0-R^4}^{t}\int_{B_R(x_0)} |\nabla\Delta u_2|^2\,dx\,dt + C\sup_{S_R}|\vec{f}|^2 R^6 \\
&\qquad+ CR^{2\sigma}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} |\nabla\Delta u|^2\,dx\,dt.
\end{aligned}$$

The above inequality implies the desired result of this lemma. The proof of this lemma is complete. □

Lemma 3.2 *There exists a positive constant C such that*

$$\begin{aligned}
&\frac{|\nabla u_1(x_1,t_1) - \nabla u_1(x_2,t_2)|^2}{|x_1 - x_2| + |t_1 - t_2|^{1/4}} \\
&\quad\le C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)} \left(\rho^{-3}\left|\nabla u_1 - (\nabla u_1)_\rho\right|^2 + \rho|\nabla\Delta u_1|^2\right)dx \\
&\qquad+ C\iint_{S_\rho} \left(\rho^{-3}|\nabla\Delta u_1|^2 + \rho\left|\nabla\Delta^2 u_1\right|\right)dx\,dt
\end{aligned}$$

holds for any given $(x_1,t_1), (x_2,t_2) \in S_\rho$.

Proof From the Sobolev embedding theorem, we get, for any $(x_1,t_1), (x_2,t_2) \in S_\rho$,

$$\begin{aligned}
&\frac{|\nabla u_1(x_1,t_1) - \nabla u_1(x_2,t_2)|^2}{|x_1 - x_2| + |t_1 - t_2|^{1/4}} \\
&\quad\le C\sup_{(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)} \left(\rho^{-3}\left|\nabla u_1 - (\nabla u_1)_\rho\right|^2 + \rho|\nabla\Delta u_1|^2\right)dx.
\end{aligned}$$

Then by using (3.2) itself we can obtain the desired estimate at once. The proof of this lemma is complete. □

Lemma 3.3 (Caccioppoli-type inequalities) *We have*

$$\begin{aligned}
&\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}\left|\nabla u_1-(\nabla u_1)_R\right|^2dx+\iint_{S_{R/2}}|\nabla\Delta u_1|^2\,dx\,dt\\
&\quad\le\frac{C}{R^4}\iint_{S_R}\left|\nabla u_1-(\nabla u_1)_R\right|^2dx\,dt,\\
&\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|\Delta u_1|^2\,dx+\iint_{S_{R/2}}\left|\Delta^2u_1\right|^2dx\,dt\\
&\quad\le\frac{C}{R^4}\iint_{S_R}|\Delta u_1|^2\,dx\,dt\le\frac{C}{R^6}\iint_{S_{2R}}\left|\nabla u_1-(\nabla u_1)_R\right|^2dx\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|\nabla\Delta u_1|^2\,dx+\iint_{S_{R/2}}\left|\nabla\Delta^2u_1\right|^2dx\,dt\\
&\quad\le\frac{C}{R^4}\iint_{S_R}|\nabla\Delta u_1|^2\,dx\,dt.
\end{aligned}$$

Proof As an example, we only prove the first inequality, since the other two can be shown similarly. Choose a cut-off function $\chi(x)$ defined on $B_R(x_0)$ such that $\chi(x)=1$ in $B_{R/2}(x_0)$ and

$$\begin{aligned}
&|\nabla\chi|\le\frac{C}{R},\qquad|\Delta\chi|\le\frac{C}{R^2},\\
&|\nabla\Delta\chi|\le\frac{C}{R^3},\qquad\left|\Delta^2\chi\right|\le\frac{C}{R^4}.
\end{aligned}$$

Let $g(t)\in C_0^\infty(\mathbb{R})$ with $0\le g(t)\le1$, $0\le g'(t)\le\frac{C}{R^4}$ for all $t\in\mathbb{R}$, $g(t)=1$ for $t\ge t_0-(R/2)^4$ and $g(t)=0$ for $t\le t_0-R^4$. Multiplying both sides of (3.2) by $g(t)\nabla\cdot[\chi^4(\nabla u_1-(\nabla u_1)_R)]$ and integrating the resulting relation over $(t_0-R^4,t)\times B_R(x_0)$, we have

$$\begin{aligned}
&\int_{t_0-R^4}^t g(t)\,dt\int_{B_R(x_0)}\frac{\partial u_1}{\partial t}\nabla\cdot\left[\chi^4\left(\nabla u_1-(\nabla u_1)_R\right)\right]dx\\
&\quad+a(x_0,t_0)\int_{t_0-R^4}^t g(t)\,dt\int_{B_R(x_0)}\Delta^2u_1\nabla\cdot\left[\chi^4\left(\nabla u_1-(\nabla u_1)_R\right)\right]dx=0.
\end{aligned}\tag{3.8}$$

The first term of the left-hand side in the above equality can be written

$$\begin{aligned}
&\int_{t_0-R^4}^t g(t)\,dt\int_{B_R(x_0)}\frac{\partial u_1}{\partial t}\nabla\cdot\left[\chi^4\left(\nabla u_1-(\nabla u_1)_R\right)\right]dx\\
&\quad=-\int_{t_0-R^4}^t g(t)\,dt\int_{B_R(x_0)}\frac{\partial\nabla u_1}{\partial t}\chi^4\left(\nabla u_1-(\nabla u_1)_R\right)dx\\
&\quad=-\frac12\int_{t_0-R^4}^t g(t)\,dt\int_{B_R(x_0)}\chi^4\frac{\partial}{\partial t}\left|\nabla u_1-(\nabla u_1)_R\right|^2dx
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_{t_0-R^4}^{t} \frac{d}{dt}\int_{B_R(x_0)} g(t)\chi^4 \big|\nabla u_1 - (\nabla u_1)_R\big|^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g'(t)\chi^4 \big|\nabla u_1 - (\nabla u_1)_R\big|^2\,dx\,dt \\
&= -\frac{1}{2}\int_{B_R(x_0)} g(t)\chi^4 \big|\nabla u_1 - (\nabla u_1)_R\big|^2\,dx \\
&\quad + \frac{1}{2}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g'(t)\chi^4 \big|\nabla u_1 - (\nabla u_1)_R\big|^2\,dx\,dt.
\end{aligned}
$$

For the second term of (3.8), we just notice that

$$
\begin{aligned}
&\int_{B_R(x_0)} \Delta^2 u_1 \nabla\cdot\big[\chi^4\big(\nabla u_1 - (\nabla u_1)_R\big)\big]\,dx \\
&= -\int_{B_R(x_0)} \nabla\Delta u_1 \Delta\big[\chi^4\big(\nabla u_1 - (\nabla u_1)_R\big)\big]\,dx \\
&= -\int_{B_R(x_0)} \chi^4|\nabla\Delta u_1|^4\,dx - 2\int_{B_R(x_0)} \nabla\chi^4\nabla\Delta u_1\Delta u_1\,dx \\
&\quad - \int_{B_R(x_0)} \nabla\Delta u_1\big(\nabla u_1 - (\nabla u_1)_R\big)\Delta\chi^4\,dx \\
&\equiv -I_1 - I_2 - I_3,
\end{aligned}
$$

where

$$
\begin{aligned}
I_2 &= 2\int_{B_R(x_0)} \nabla\chi^4\nabla\Delta u_1\Delta u_1\,dx = 8\int_{B_R(x_0)} \chi\nabla\chi\nabla\Delta u_1\Delta u_1\,dx \\
&\ge -\frac{1}{8}\int_{B_R(x_0)} \chi^4|\nabla\Delta u_1|^2\,dx - 128\int_{B_R(x_0)} |\chi\nabla\chi|^2|\Delta u_1|^2\,dx \\
&\equiv -\frac{1}{8}I_1 + I_4
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= -128\int_{B_R(x_0)} |\chi\nabla\chi|^2|\Delta u_1|^2\,dx \\
&\ge -\frac{C}{R^2}\int_{B_R(x_0)} \chi^2|\Delta u_1|^2\,dx \\
&= -\frac{C}{R^2}\int_{B_R(x_0)} \chi^2\Delta u_1\nabla\cdot\big(\nabla u_1 - (\nabla u_1)_R\big)\,dx \\
&= \frac{C}{R^2}\int_{B_R(x_0)} \big(\nabla u_1 - (\nabla u_1)_R\big)\nabla\big(\chi^2\Delta u_1\big)\,dx \\
&= \frac{C}{R^2}\int_{B_R(x_0)} \big(\nabla u_1 - (\nabla u_1)_R\big)\chi^2\nabla\Delta u_1\,dx \\
&\quad + \frac{C}{R^2}\int_{B_R(x_0)} \chi\nabla\chi\Delta u_1\big(\nabla u_1 - (\nabla u_1)_R\big)\,dx \\
&\ge -\frac{1}{16}\int_{B_R(x_0)} \chi^4|\nabla\Delta u_1|^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{C}{R^4} \int_{B_R(x_0)} \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx \\
&- 64 \int_{B_R(x_0)} |\chi \nabla \chi|^2 |\Delta u_1|^2 dx.
\end{aligned}
$$

Then we have

$$
I_4 \geq -\frac{1}{8} I_1 - \frac{C}{R^4} \int_{B_R(x_0)} \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx.
$$

Thus

$$
I_2 \geq -\frac{1}{4} I_1 - \frac{C}{R^4} \int_{B_R(x_0)} \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx
$$

and

$$
\begin{aligned}
I_3 &= \int_{B_R(x_0)} \nabla \Delta u_1 \bigl(\nabla u_1 - (\nabla u_1)_R\bigr) \Delta \chi^4 \, dx \\
&= \int_{B_R(x_0)} \nabla \Delta u_1 \bigl(\nabla u_1 - (\nabla u_1)_R\bigr) \bigl(4\chi^3 \Delta \chi + 12 \chi^2 |\nabla \chi|^2\bigr) \, dx \\
&\geq -\frac{1}{4} \int_{B_R(x_0)} \chi^4 |\nabla \Delta u_1|^2 dx - 32 \int_{B_R(x_0)} |\chi \Delta \chi|^2 \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx \\
&\quad - 288 \int_{B_R(x_0)} |\nabla \chi|^4 \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx \\
&\geq -\frac{1}{4} I_1 - \frac{C}{R^4} \int_{B_R(x_0)} \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx,
\end{aligned}
$$

and hence

$$
I_1 + I_2 + I_3 \geq \frac{1}{2} I_1 - \frac{C}{R^4} \int_{B_R(x_0)} \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx.
$$

Then we can obtain the following estimate on the second term of (3.8):

$$
\begin{aligned}
&a(x_0, t_0) \int_{t_0 - R^4}^{t} g(t) \, dt \int_{B_R(x_0)} \Delta^2 u_1 \nabla \cdot \bigl[\chi^4 \bigl(\nabla u_1 - (\nabla u_1)_R\bigr)\bigr] \, dx \\
&\quad = -a(x_0, t_0) \int_{t_0 - R^4}^{t} g(t)(I_1 + I_2 + I_3) \, dt \\
&\quad \leq -\frac{1}{2} a(x_0, t_0) \int_{t_0 - R^4}^{t} g(t) \, dt \int_{B_R(x_0)} \chi^4 |\nabla \Delta u_1|^2 dx \\
&\qquad + \frac{C}{R^4} \int_{t_0 - R^4}^{t} g(t) \, dt \int_{B_R(x_0)} \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx,
\end{aligned}
$$

which, together with the estimate on the first term of (3.8), implies that

$$
\begin{aligned}
&\frac{1}{2} \int_{B_R(x_0)} g(t) \chi^4 \left|\nabla u_1 - (\nabla u_1)_R\right|^2 dx \\
&\quad + \frac{a(x_0, t_0)}{2} \int_{t_0 - R^4}^{t} g(t) \, dt \int_{B_R(x_0)} \chi^4 |\nabla \Delta u_1|^2 dx
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1}{2}\int_{t_0-R^4}^{t}\int_{B_R(x_0)} g'(t)\chi^4\left|\nabla u_1-(\nabla u_1)_R\right|^2\,dx\,dt \\
&\quad + \frac{C}{R^4}\int_{t_0-R^4}^{t} g(t)\,dt\int_{B_R(x_0)}\left|\nabla u_1-(\nabla u_1)_R\right|^2\,dx \\
&\le \frac{C}{R^4}\iint_{S_R}\left|\nabla u_1-(\nabla u_1)_R\right|^2\,dx\,dt.
\end{aligned}$$

By the definition of $g(t)$ and χ, we immediately obtain the desired first inequality of this lemma, and thus we complete the proof. □

Lemma 3.4 *For any* $\rho \in (0,R)$, *we have*

$$\theta(u_1,\rho) \le C\left(\frac{\rho}{R}\right)^7\theta(u_1,R).$$

Proof It is sufficient to show the inequality for $\rho \le R/2$. By the mean value theorem, there exists a point $(x_*,t_*) \in S_\rho$ such that

$$(\nabla u_1)_\rho = \nabla u_1(x_*,t_*).$$

Then, by Lemma 3.2 and Lemma 3.3, one has

$$\begin{aligned}
&\iint_{S_\rho}\left|\nabla u_1-(\nabla u_1)_\rho\right|^2\,dx\,dt \\
&\quad = \iint_{S_\rho}\left|\nabla u_1-\nabla u_1(x_*,t_*)\right|^2\,dx\,dt \\
&\quad \le C\rho^6\sup_{(x,t)\in S_\rho}\left|\nabla u_1-\nabla u_1(x_*,t_*)\right|^2 \\
&\quad \le C\rho^7\sup_{t\in(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}\left(R^{-3}\left|\nabla u_1-(\nabla u_1)_R\right|^2+R|\nabla\Delta u_1|^2\right)dx \\
&\qquad + C\rho^7\iint_{S_{R/2}}\left(R^{-3}|\nabla\Delta u_1|^2+R\left|\nabla\Delta^2 u_1\right|^2\right)dx\,dt \\
&\quad \le C\left(\frac{\rho}{R}\right)^7\iint_{S_R}\left(\left|\nabla u_1-(\nabla u_1)_R\right|^2+R^4|\nabla\Delta u_1|^2\right)dx\,dt
\end{aligned}$$

and

$$\begin{aligned}
\iint_{S_\rho}\rho^4|\nabla\Delta u_1|^2\,dx\,dt &\le C\rho^8\sup_{t\in(t_0-\rho^4,t_0+\rho^4)}\int_{B_\rho(x_0)}|\nabla\Delta u_1|^2\,dx \\
&\le C\rho^7 R\sup_{t\in(t_0-(R/2)^4,t_0+(R/2)^4)}\int_{B_{R/2}(x_0)}|\nabla\Delta u_1|^2\,dx \\
&\le C\left(\frac{\rho}{R}\right)^7\iint_{S_R}R^4|\nabla\Delta u_1|^2\,dx\,dt.
\end{aligned}$$

The proof of this lemma is complete. □

The following technical lemma is required to estimate the Hölder norm of ∇u. One can find its proof in Giaquinta [19].

Lemma 3.5 *Let $\theta(\rho)$ be a nonnegative and nondecreasing function satisfying*

$$\theta(\rho) \le A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \varepsilon\right]\theta(R) + BR^{\beta}, \quad \forall 0 < \rho \le R \le R_0,$$

where A, B, α, β are positive constants with $\beta < \alpha$. Then there exists a constant $\varepsilon_0 = \varepsilon_0(A, \alpha, \beta)$, such that for all $0 < \varepsilon < \varepsilon_0$, and the following inequality holds:

$$\theta(\rho) \le C\left(\frac{\rho}{R}\right)^{\beta}\left[\theta(R) + BR^{\beta}\right], \quad \forall 0 < \rho \le R \le R_0,$$

where C is a positive constant depending only on α, β, and A.

Lemma 3.6 *For $\lambda \in (6, 7)$, we have*

$$\theta(u, \rho) \le C\Big(1 + \sup_{S_R} |\vec{f}|^2\Big)\rho^{\lambda}, \quad \forall 0 < \rho \le R \le R_0,$$

where $R_0 \triangleq \min\{\operatorname{dist}(x_0, \partial\Omega), t_0^{1/4}\}$.

Proof A simple calculation gives

$$(\nabla u)_\rho = (\nabla u_1)_\rho + (\nabla u_2)_\rho$$

and

$$\iint_{S_\rho} \left|\nabla u - (\nabla u)_\rho\right|^2 \,dx\,dt \le \iint_{S_\rho} |\nabla u|^2 \,dx\,dt.$$

Then, by Cauchy's inequality and using Lemmas 3.1 and 3.4, we have

$$\begin{aligned}
\theta(u, \rho) &\le 2\theta(u_1, \rho) + 2\theta(u_2, \rho) \\
&\le C\left(\frac{\rho}{R}\right)^{7}\theta(u_1, R) + 2\theta(u_2, R) \\
&\le C\left(\frac{\rho}{R}\right)^{7}\theta(u, R) + 2\iint_{S_R}\left(|\nabla u_2|^2 + R^4|\nabla\Delta u_2|^2\right)dx\,dt \\
&\le C\left[\left(\frac{\rho}{R}\right)^{7} + R^{2\sigma}\right]\theta(u, R) + C\sup_{S_R}|\vec{f}|^2 R^{10} \\
&\le C\left[\left(\frac{\rho}{R}\right)^{7} + R^{2\sigma}\right]\theta(u, R) + C\sup_{S_R}|\vec{f}|^2 R^{\lambda},
\end{aligned}$$

where $6 < \lambda < 7$ is a constant. For ε_0 in Lemma 3.5, we can choose $R_0 > 0$ such that $R^{2\sigma} < \varepsilon_0$ whenever $R \le R_0$. Then, by Lemma 3.5, one can complete the proof of this lemma immediately. □

Now we can give the proof of the main result in this section.

Proof of Proposition 3.1 From the integral characteristic of the Hölder continuous functions and Lemma 3.6, one has

$$\frac{|\nabla u(x_1,t_1)-\nabla u(x_2,t_2)|}{|x_1-x_2|^{(\lambda-6)/2}+|t_1-t_2|^{(\lambda-6)/8}} \le C\Bigl(1+\sup_{S_R}|\vec{f}|\Bigr) \le C\Bigl(1+\sup_{S_R}|\nabla u|\Bigr).$$

By the interpolation inequality, we have

$$\bigl|\nabla u(x_1,t_1)-\nabla u(x_2,t_2)\bigr| \le C\bigl(|x_1-x_2|^{(\lambda-6)/2}+|t_1-t_2|^{(\lambda-6)/8}\bigr)$$

for any given $(x_1,t_1),(x_2,t_2)\in S_R$.

For the Hölder continuous of ∇u near the boundary of Q_T, we can deal with it in the same way. Let $(x_0,t_0)\in\partial\Omega\times(0,T)$ be fixed and assume that $\partial\Omega$ can be explicitly expressed by a function $y=\phi(x)$ in some neighborhood of x_0. We split u as u_1+u_2 in $\hat{S}_R=(t_0-R^4,t_0+R^4)\times\Omega_R(x_0)$ with $\Omega_R(x_0)=B_R(x_0)\cap\Omega$. u_1 solves the following problem:

$$\begin{aligned}
&\frac{\partial u_1}{\partial t}+a(x_0,t_0)\Delta^2u_1=0, \quad (x,t)\in\hat{S}_R,\\
&\frac{\partial u_1}{\partial\nu}=\frac{\partial u}{\partial\nu}, \qquad \frac{\partial\Delta u_1}{\partial\nu}=\frac{\partial\Delta u}{\partial\nu}, \quad (x,t)\in\bigl(t_0-R^4,t_0+R^4\bigr)\times\partial\Omega_R(x_0),\\
&u_1=u, \qquad t=t_0-R^4, \quad x\in\Omega_R(x_0),
\end{aligned}$$

and u_2 solves the problem

$$\begin{aligned}
&\frac{\partial u_2}{\partial t}+a(x_0,t_0)\Delta^2u_2=\operatorname{div}\bigl[\bigl(a(x_0,t_0)-a(x,t)\bigr)\nabla\Delta u\bigr]+\operatorname{div}\vec{f}, \quad (x,t)\in\hat{S}_R,\\
&\frac{\partial u_2}{\partial\nu}=0, \qquad \frac{\partial\Delta u_2}{\partial\nu}=0, \quad (x,t)\in\bigl(t_0-R^4,t_0+R^4\bigr)\times\partial\Omega_R(x_0),\\
&u_1=0, \qquad t=t_0-R^4, \quad x\in\Omega_R(x_0).
\end{aligned}$$

We can modify the function $\theta(u,\rho)$ as

$$\theta(u,\rho)=\iint_{S_\rho}\bigl(|\partial_n u|^2+\bigl|\partial_\tau u-(\partial_\tau u)_\rho\bigr|^2+\rho^4|\nabla\Delta u|^2\bigr)\,dx\,dt,$$

where

$$\partial_n=\phi'(x)\frac{\partial}{\partial x^1}-\frac{\partial}{\partial x^2}, \qquad \partial_\tau=\frac{\partial}{\partial x^1}+\phi'(x)\frac{\partial}{\partial x^2}$$

denote the normal and tangential derivatives, respectively. The remaining part of the proof is similar to that in the proof of the previous lemmas, and we omit the details here. The proof of this theorem is complete. □

4 Existence and uniqueness

In this section, we give the proof of the existence and uniqueness of classical solutions to the problem (1.1)-(1.4).

Proof of Theorem 1.1 Equation (1.1) can be rewritten

$$\frac{\partial u}{\partial t} + a_1(x,t)\Delta^2 u + \vec{b}_1(x,t)\nabla\Delta u + a_2(x,t)\Delta u + \vec{b}_2(x,t)\nabla u = 0, \tag{4.1}$$

where

$$a_1(x,t) = km\big(u(x,t)\big), \qquad \vec{b}_1(x,t) = km'\big(u(x,t)\big)\nabla u(x,t),$$
$$a_2(x,t) = -m\big(u(x,t)\big)\varphi'\big(u(x,t)\big), \qquad \vec{b}_2(x,t) = -\nabla\big(m\big(u(x,t)\big)\varphi'(u)\big).$$

By the *a priori* Hölder norm estimates on u and ∇u, we see that the Hölder norm of $a_1(x,t)$, $a_2(x,t)$, $\vec{b}_1(x,t)$, and $\vec{b}_2(x,t)$ can be estimated by known quantities. Define a linear space

$$X = \big\{u \in C^{1+\alpha,\frac{1+\alpha}{4}}(\overline{Q}_T); \nabla u\cdot\nu|_{\partial\Omega} = 0, u(x,0) = u_0(x)\big\}$$

and an associated operator T on X,

$$T : X \to X, \qquad u \mapsto w,$$

where w is determined by the following linear problem:

$$\frac{\partial w}{\partial t} + a_1(x,t)\Delta^2 w + \vec{b}_1(x,t)\nabla\Delta w + a_2(x,t)\Delta w + \vec{b}_2(x,t)\nabla w = 0, \quad (x,t)\in Q_T,$$
$$\nabla w\cdot\nu = \nabla\Delta w\cdot\nu = 0, \quad x\in\partial\Omega, t\in(0,T),$$
$$w(x,0) = u_0(x).$$

By classical linear theory, the above problem admits a unique solution in the space $C^{4+\beta,\frac{4+\beta}{4}}(\overline{Q}_T)$. So the operator T is well defined and compact. Moreover, if $u = \sigma Tu$ for some $\sigma\in(0,1]$, then u satisfies (4.1), (1.3), (1.6), and $u(x,0) = \sigma u_0(x)$. Thus, from above discussion, the norm of u in the space $C^{4+\alpha,\frac{4+\alpha}{4}}(\overline{Q}_T)$ can be determined by some constant C depending only on the known quantities. By the Leray-Schauder fixed point theorem, the operator T has a fixed point u, which is the desired classical solution of the problem (1.1)-(1.4).

Next, we prove the uniqueness of the classical solution of the problem (1.1)-(1.4). Suppose u_1 and u_2 are two solutions of the problem (1.1)-(1.4). Then, for any smooth function $\psi(x,t)$ satisfying

$$\nabla\psi(x,t)\cdot\nu|_{\partial\Omega} = \nabla\Delta\psi(x,t)\cdot\nu|_{\partial\Omega} = \psi(x,T) = 0,$$

we have

$$\iint_{Q_T}(u_1-u_2)\frac{\partial\psi}{\partial t}\,dx\,dt + k\iint_{Q_T}\big(m(u_1)\nabla\Delta u_1 - m(u_2)\nabla\Delta u_2\big)\nabla\psi\,dx\,dt$$
$$-\iint_{Q_T}\big(m(u_1)\nabla\varphi(u_1) - m(u_2)\nabla\varphi(u_2)\big)\nabla\psi\,dx\,dt = 0. \tag{4.2}$$

Since the second term of the left-hand side can be rewritten

$$\begin{aligned}
&k\iint_{Q_T}\bigl(m(u_1)\nabla\Delta u_1-m(u_2)\nabla\Delta u_2\bigr)\nabla\psi\,dx\,dt\\
&\quad=k\iint_{Q_T}\bigl(m(u_1)-m(u_2)\bigr)\nabla\Delta u_1\nabla\psi\,dx\,dt\\
&\qquad+k\iint_{Q_T}m(u_2)(\nabla\Delta u_1-\nabla\Delta u_2)\nabla\psi\,dx\,dt\\
&\quad=k\iint_{Q_T}(u_1-u_2)\int_0^1 m'\bigl(\lambda u_1+(1-\lambda)u_2\bigr)\,d\lambda\cdot\nabla\Delta u_1\nabla\psi\,dx\,dt\\
&\qquad-k\iint_{Q_T}(u_1-u_2)\Delta\bigl[\operatorname{div}\bigl(m(u_2)\nabla\psi\bigr)\bigr]\,dx\,dt
\end{aligned}$$

and the third term of the left-hand side of (4.2) can be rewritten

$$\begin{aligned}
&-\iint_{Q_T}\bigl(m(u_1)\nabla\varphi(u_1)-m(u_2)\nabla\varphi(u_2)\bigr)\nabla\psi\,dx\,dt\\
&\quad=-\iint_{Q_T}\bigl(m(u_1)-m(u_2)\bigr)\nabla\varphi(u_1)\nabla\psi\,dx\,dt\\
&\qquad-\iint_{Q_T}\bigl(\nabla\varphi(u_1)-\nabla\varphi(u_2)\bigr)m(u_2)\nabla\psi\,dx\,dt\\
&\quad=-\iint_{Q_T}\bigl(m(u_1)-m(u_2)\bigr)\nabla\varphi(u_1)\nabla\psi\,dx\,dt\\
&\qquad+\iint_{Q_T}\bigl(\varphi(u_1)-\varphi(u_2)\bigr)\operatorname{div}\bigl(m(u_2)\nabla\psi\bigr)\,dx\,dt\\
&\quad=-\iint_{Q_T}(u_1-u_2)\int_0^1 m'\bigl(\lambda u_1+(1-\lambda)u_2\bigr)\,d\lambda\cdot\nabla\varphi(u_1)\nabla\psi\,dx\,dt\\
&\qquad+\iint_{Q_T}(u_1-u_2)\int_0^1 \varphi'\bigl(\lambda u_1+(1-\lambda)u_2\bigr)\,d\lambda\cdot\operatorname{div}\bigl(m(u_2)\nabla\psi\bigr)\,dx\,dt,
\end{aligned}$$

(4.2) becomes

$$\begin{aligned}
&\iint_{Q_T}(u_1-u_2)\frac{\partial\psi}{\partial t}\,dx\,dt-k\iint_{Q_T}(u_1-u_2)\Delta\bigl[\operatorname{div}\bigl(\hat{a}(x,t)\nabla\psi\bigr)\bigr]\,dx\,dt\\
&\qquad+\iint_{Q_T}(u_1-u_2)\hat{b}(x,t)\Delta\psi\,dx\,dt+\iint_{Q_T}(u_1-u_2)\hat{c}(x,t)\nabla\psi\,dx\,dt=0,
\end{aligned}$$

where

$$\begin{aligned}
&\hat{a}(x,t)=m\bigl(u_2(x,t)\bigr),\\
&\hat{b}(x,t)=m(u_2)\int_0^1\varphi'\bigl(\lambda u_1+(1-\lambda)u_2\bigr)\,d\lambda,\\
&\hat{c}(x,t)=\int_0^1 m'\bigl(\lambda u_1+(1-\lambda)u_2\bigr)\,d\lambda\cdot\bigl(k\nabla\Delta u_1-\nabla\varphi(u_1)\bigr)\\
&\qquad\quad+\int_0^1\varphi'\bigl(\lambda u_1+(1-\lambda)u_2\bigr)\,d\lambda\cdot\nabla m(u_2).
\end{aligned}$$

For any given $f \in C_0^\infty(Q_T)$, we consider the following linear problem:

$$\frac{\partial\psi}{\partial t} - k\Delta\big[\operatorname{div}\big(\hat{a}(x,t)\nabla\psi\big)\big] + \hat{b}(x,t)\Delta\psi + \hat{c}(x,t)\nabla\psi = f(x,t),$$

$$\nabla\psi(x,t)\cdot\nu|_{\partial\Omega} = \nabla\Delta\psi(x,t)\cdot\nu|_{\partial\Omega} = 0,$$

$$\psi(x,T) = 0.$$

Since $\hat{a}(x,t) \in C^{3+\alpha,\frac{3+\alpha}{4}}(Q_T)$, $\hat{b}(x,t) \in C^{\alpha,\frac{\alpha}{4}}(Q_T)$, $\hat{c}(x,t) \in C^{\alpha,\frac{\alpha}{4}}(Q_T)$ for some $0 < \alpha < 1$, we know from the classical parabolic theory that the above linear problem admits a unique solution $\psi \in C^{4+\alpha,1+\frac{\alpha}{4}}(Q_T)$. Then we have

$$\iint_{Q_T} (u_1 - u_2) f\,dx\,dt = 0.$$

It follows from the arbitrariness of the function f that $u_1 = u_2$ a.e. in Q_T. Then, by the continuity of u_1 and u_2, we have $u_1 = u_2$ in Q_T. The proof of this theorem is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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