Lin Boundary Value Problems 2012, 2012:118
http://www.boundaryvalueproblems.com/content/2012/1/118
Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Multiple positive solutions for semilinear elliptic systems involving subcritical nonlinearities in $\mathbb{R}^N$

Huei-li Lin*

*Correspondence: hlin@mail.cgu.edu.tw
Department of Natural Sciences in the Center for General Education, Chang Gung University, Tao-Yuan, 333, Taiwan, R.O.C.

Abstract

In this paper, we investigate the effect of the coefficient $f(x)$ of the subcritical nonlinearity. Under some assumptions, for sufficiently small $\varepsilon, \lambda, \mu > 0$, there are at least k (≥ 1) positive solutions of the semilinear elliptic systems

$$\begin{cases} -\varepsilon^2 \Delta \overline{u} + \overline{u} = \lambda g(x)|\overline{u}|^{q-2}\overline{u} + \frac{\alpha}{\alpha+\beta} f(x)|\overline{u}|^{\alpha-2}\overline{u}|\overline{v}|^{\beta} & \text{in } \mathbb{R}^N; \\ -\varepsilon^2 \Delta \overline{v} + \overline{v} = \mu h(x)|\overline{v}|^{q-2}\overline{v} + \frac{\beta}{\alpha+\beta} f(x)|\overline{u}|^{\alpha}|\overline{v}|^{\beta-2}\overline{v} & \text{in } \mathbb{R}^N; \\ \overline{u}, \overline{v} \in H^1(\mathbb{R}^N), \end{cases}$$

where $\alpha > 1$, $\beta > 1$, $2 < q < p = \alpha + \beta < 2^* = 2N/(N-2)$ for $N \geq 3$.

MSC: 35J20; 35J25; 35J65

Keywords: semilinear elliptic systems; subcritical exponents; Nehari manifold

1 Introduction

For $N \geq 3$, $\alpha > 1$, $\beta > 1$ and $2 < q < p = \alpha + \beta < 2^* = 2N/(N-2)$, we consider the semilinear elliptic systems

$$\begin{cases} -\varepsilon^2 \Delta \overline{u} + \overline{u} = \lambda g(x)|\overline{u}|^{q-2}\overline{u} + \frac{\alpha}{\alpha+\beta} f(x)|\overline{u}|^{\alpha-2}\overline{u}|\overline{v}|^{\beta} & \text{in } \mathbb{R}^N; \\ -\varepsilon^2 \Delta \overline{v} + \overline{v} = \mu h(x)|\overline{v}|^{q-2}\overline{v} + \frac{\beta}{\alpha+\beta} f(x)|\overline{u}|^{\alpha}|\overline{v}|^{\beta-2}\overline{v} & \text{in } \mathbb{R}^N; \\ \overline{u} > 0, \quad \overline{v} > 0, \end{cases} \qquad (\overline{E}_{\varepsilon,\lambda,\mu})$$

where $\varepsilon, \lambda, \mu > 0$.

Let f, g and h satisfy the following conditions:

(A1) f is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|x|\to\infty} f(x) = f_\infty > 0$.

(A2) there exist k points $a^1, a^2, \ldots, a^k$ in $\mathbb{R}^N$ such that

$$f(a^i) = \max_{x \in \mathbb{R}^N} f(x) = 1 \quad \text{for } 1 \leq i \leq k,$$

and $f_\infty < 1$.

(A3) $g, h \in L^m(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where $m = (\alpha+\beta)/(\alpha+\beta-q)$, and $g, h \gneqq 0$.

Springer

In [1], if Ω is a smooth and bounded domain in $\mathbb{R}^N$ ($N \leq 3$), they considered the following system:

$$\begin{cases} \varepsilon^2 \Delta \overline{u} - \lambda_1 \overline{u} = \mu_1 \overline{u}^3 + \beta \overline{u} \overline{v}^2 & \text{in } \Omega; \\ \varepsilon^2 \Delta \overline{v} - \lambda_2 \overline{v} = \mu_2 \overline{v}^3 + \beta \overline{u}^2 \overline{v} & \text{in } \Omega; \\ \overline{u} > 0, \qquad \overline{v} > 0, \end{cases}$$

and proved the existence of a least energy solution in Ω for sufficiently small $\varepsilon > 0$ and $\beta \in (-\infty, \beta_0)$. Lin and Wei also showed that this system has a least energy solution in $\mathbb{R}^N$ for $\varepsilon = 1$ and $\beta \in (0, \beta_0)$. In this paper, we study the effect of $f(z)$ of $(\overline{E}_{\varepsilon,\lambda,\mu})$. Recently, many authors [2–5] considered the elliptic systems with subcritical or critical exponents, and they proved the existence of a least energy positive solution or the existence of at least two positive solutions for these problems. In this paper, we construct the k compact Palais-Smale sequences which are suitably localized in correspondence of k maximum points of f. Then we could show that under some assumptions (A1)-(A3), for sufficiently small $\varepsilon, \lambda, \mu > 0$, there are at least k (≥ 1) positive solutions of the elliptic system $(E_{\varepsilon,\lambda,\mu})$. By the change of variables

$$x = \varepsilon z, \qquad u(z) = \overline{u}(\varepsilon z) \quad \text{and} \quad v(z) = \overline{v}(\varepsilon z),$$

System $(\overline{E}_{\varepsilon,\lambda,\mu})$ is transformed to

$$\begin{cases} -\Delta u + u = \lambda g(\varepsilon z)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta} f(\varepsilon z)|u|^{\alpha-2}u|v|^{\beta} & \text{in } \mathbb{R}^N; \\ -\Delta v + v = \mu h(\varepsilon z)|v|^{q-2}v + \frac{\beta}{\alpha+\beta} f(\varepsilon z)|u|^{\alpha}|v|^{\beta-2}v & \text{in } \mathbb{R}^N; \\ u > 0, \qquad v > 0. \end{cases} \tag{$E_{\varepsilon,\lambda,\mu}$}$$

Let $H = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ be the space with the standard norm

$$\|(u,v)\|_H = \left[\int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2 \right) dz + \int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2 \right) dz \right]^{1/2}.$$

Associated with the problem $(E_{\varepsilon,\lambda,\mu})$, we consider the C^1-functional $J_{\varepsilon,\lambda,\mu}$, for $(u,v) \in H$,

$$\begin{aligned} J_{\varepsilon,\lambda,\mu}(u,v) = {} & \frac{1}{2} \|(u,v)\|_H^2 - \frac{1}{\alpha+\beta} \int_{\mathbb{R}^N} f(\varepsilon z)|u|^{\alpha}|v|^{\beta}\, dz \\ & - \frac{1}{q} \int_{\mathbb{R}^N} \left(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q \right) dz. \end{aligned}$$

Actually, the weak solution $(u,v) \in H$ of $(E_{\varepsilon,\lambda,\mu})$ is the critical point of the functional $J_{\varepsilon,\lambda,\mu}$, that is, $(u,v) \in H$ satisfies

$$\begin{aligned} & \int_{\mathbb{R}^N} (\nabla u \nabla \varphi_1 + \nabla v \nabla \varphi_2 + u\varphi_1 + v\varphi_2)\, dz \\ & \quad - \lambda \int_{\mathbb{R}^N} g(\varepsilon z)|u|^{q-2}u\varphi_1\, dz - \mu \int_{\mathbb{R}^N} h(\varepsilon z)|v|^{q-2}v\varphi_2\, dz \\ & \quad - \frac{\alpha}{\alpha+\beta} \int_{\mathbb{R}^N} f(\varepsilon z)|u|^{\alpha-2}u|v|^{\beta}\varphi_1\, dz - \frac{\beta}{\alpha+\beta} \int_{\mathbb{R}^N} f(\varepsilon z)|u|^{\alpha}|v|^{\beta-2}v\varphi_2\, dz = 0 \end{aligned}$$

for any $(\varphi_1, \varphi_2) \in H$.

We consider the Nehari manifold

$$\mathbf{M}_{\varepsilon,\lambda,\mu} = \{(u,v) \in H \setminus \{(0,0)\} \mid \langle J'_{\varepsilon,\lambda,\mu}(u,v),(u,v)\rangle = 0\}, \tag{1.1}$$

where

$$\langle J'_{\varepsilon,\lambda,\mu}(u,v),(u,v)\rangle = \|(u,v)\|_H^2 - \int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz - \int_{\mathbb{R}^N} \left(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\right)dz.$$

The Nehari manifold $\mathbf{M}_{\varepsilon,\lambda,\mu}$ contains all nontrivial weak solutions of $(E_{\varepsilon,\lambda,\mu})$.

Let

$$S_{\alpha,\beta} = \inf_{u,v\in H^1(\mathbb{R}^N)\setminus\{(0)\}} \frac{\|(u,v)\|_H^2}{(\int_{\mathbb{R}^N} |u|^\alpha|v|^\beta\,dz)^{2/(\alpha+\beta)}}, \tag{1.2}$$

then by [2, Theorem 5], we have

$$S_{\alpha,\beta} = \left[\left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}\right] S_p,$$

where $p = \alpha + \beta$ and S_p is the best Sobolev constant defined by

$$S_p = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\,dz}{(\int_{\mathbb{R}^N} |u|^p\,dz)^{2/p}}.$$

For the semilinear elliptic systems ($\lambda = \mu = 0$)

$$\begin{cases} -\Delta u + u = \frac{\alpha}{\alpha+\beta} f(\varepsilon z)|u|^{\alpha-2}u|v|^\beta & \text{in } \mathbb{R}^N; \\ -\Delta v + v = \frac{\beta}{\alpha+\beta} f(\varepsilon z)|u|^\alpha|v|^{\beta-2}v & \text{in } \mathbb{R}^N; \\ (u,v) \in H, \end{cases} \tag{E_ε}$$

we define the energy functional $I_\varepsilon(u,v) = \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz$, and

$$\mathbf{N}_\varepsilon = \{(u,v) \in H \setminus \{(0,0)\} \mid \langle I'_\varepsilon(u,v),(u,v)\rangle = 0\}.$$

If $f \equiv \max_{z\in\mathbb{R}^N} f(z)$ $(= 1)$, then we define $I_{\max}(u,v) = \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{\alpha+\beta}\int_{\mathbb{R}^N} |u|^\alpha|v|^\beta\,dz$ and

$$\theta_{\max} = \inf_{(u,v)\in\mathbf{N}_{\max}} I_{\max}(u,v),$$

where $\mathbf{N}_{\max} = \{(u,v) \in H \setminus \{(0,0)\} \mid \langle I'_{\max}(u,v),(u,v)\rangle = 0\}$.

It is well known that this problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \mathbb{R}^N; \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$E0$}$$

has the unique, radially symmetric and positive ground state solution $w \in H^1(\mathbb{R}^N)$. Define $\overline{I}_{\max}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\,dz - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p\,dz$ and $\overline{\theta}_{\max} = \inf_{u\in\overline{\mathbf{N}}_{\max}} \overline{I}_{\max}(u)$, where

$$\overline{\mathbf{N}}_{\max} = \{u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle \overline{I}'_{\max}(u),u\rangle = 0\}.$$

Moreover, we have that

$$\overline{\theta}_{\max} = \frac{p-2}{2p} S_p^{\frac{p}{p-2}} > 0. \quad \text{(See Wang [6, Theorems 4.12 and 4.13].)}$$

This paper is organized as follows. First of all, we study the argument of the Nehari manifold $\mathbf{M}_{\varepsilon,\lambda,\mu}$. Next, we prove that the existence of a positive solution $(u_0, v_0) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ of $(E_{\varepsilon,\lambda,\mu})$. Finally, in Section 4, we show that the condition (A2) affects the number of positive solutions of $(E_{\varepsilon,\lambda,\mu})$; that is, there are at least k critical points $(u_i, v_i) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ of $J_{\varepsilon,\lambda,\mu}$ such that $J_{\varepsilon,\lambda,\mu}(u_i, v_i) = \beta^i_{\varepsilon,\lambda,\mu}$ ((PS)-value) for $1 \leq i \leq k$.

Theorem 1.1 *$(E_{\varepsilon,\lambda,\mu})$ has at least one positive solution (u_0, v_0), that is, $(\overline{E}_{\varepsilon,\lambda,\mu})$ admits at least one positive solution.*

Theorem 1.2 *There exist two positive numbers ε_0 and Λ^* such that $(E_{\varepsilon,\lambda,\mu})$ has at least k positive solutions for any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$, that is, $(\overline{E}_{\varepsilon,\lambda,\mu})$ admits at least k positive solutions.*

2 Preliminaries

By studying the argument of Han [7, Lemma 2.1], we obtain the following lemma.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ (possibly unbounded) be a smooth domain. If $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$, and $u_n \to u$, $v_n \to v$ almost everywhere in Ω, then*

$$\lim_{n\to\infty} \int_\Omega |u_n - u|^\alpha |v_n - v|^\beta \, dz = \lim_{n\to\infty} \int_\Omega |u_n|^\alpha |v_n|^\beta \, dz - \int_\Omega |u|^\alpha |v|^\beta \, dz.$$

Note that $J_{\varepsilon,\lambda,\mu}$ is not bounded from below in H. From the following lemma, we have that $J_{\varepsilon,\lambda,\mu}$ is bounded from below on $\mathbf{M}_{\varepsilon,\lambda,\mu}$.

Lemma 2.2 *The energy functional $J_{\varepsilon,\lambda,\mu}$ is bounded from below on $\mathbf{M}_{\varepsilon,\lambda,\mu}$.*

Proof For $(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$, by (1.1), we obtain that

$$J_{\varepsilon,\lambda,\mu}(u, v) = \left(\frac{1}{2} - \frac{1}{q}\right) \|(u, v)\|_H^2 + \left(\frac{1}{q} - \frac{1}{p}\right) \int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha |v|^\beta \, dz > 0,$$

where $p = \alpha + \beta$. Hence, we have that $J_{\varepsilon,\lambda,\mu}$ is bounded from below on $\mathbf{M}_{\varepsilon,\lambda,\mu}$. □

We define

$$\theta_{\varepsilon,\lambda,\mu} = \inf_{(u,v)\in\mathbf{M}_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u, v).$$

Lemma 2.3 (i) *There exist positive numbers σ and d_0 such that $J_{\varepsilon,\lambda,\mu}(u, v) \geq d_0$ for $\|(u, v)\|_H = \sigma$;*

(ii) *There exists $(\overline{u}, \overline{v}) \in H \backslash \{(0, 0)\}$ such that $\|(\overline{u}, \overline{v})\|_H > \sigma$ and $J_{\varepsilon,\lambda,\mu}(\overline{u}, \overline{v}) < 0$.*

Proof (i) By (1.2), the Hölder inequality ($p_1 = \frac{p}{p-q}$, $p_2 = \frac{p}{q}$) and the Sobolev embedding theorem, we have that

$$\begin{aligned} J_{\varepsilon,\lambda,\mu}(u,v) &= \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz \\ &\quad - \frac{1}{q}\int_{\mathbb{R}^N}\bigl(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\bigr)\,dz \\ &\geq \frac{1}{2}\|(u,v)\|_H^2 - \frac{1}{p}S_{\alpha,\beta}^{-p/2}\|(u,v)\|_H^p \\ &\quad - \frac{1}{q}\operatorname{Max} S_p^{-\frac{q}{2}}(\lambda+\mu)\|(u,v)\|_H^q, \end{aligned}$$

where $p = \alpha + \beta$ and $\operatorname{Max} = \max\{\|g\|_m, \|h\|_m\}$. Hence, there exist positive σ and d_0 such that $J_{\varepsilon,\lambda,\mu}(u,v) \geq d_0$ for $\|(u,v)\|_H = \sigma$.

(ii) For any $(u,v) \in H\backslash\{(0,0)\}$, since

$$\begin{aligned} J_{\varepsilon,\lambda,\mu}(tu,tv) &= \frac{t^2}{2}\|(u,v)\|_H^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz \\ &\quad - \frac{t^q}{q}\int_{\mathbb{R}^N}\bigl(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\bigr)\,dz, \end{aligned}$$

then $\lim_{t\to\infty} J_{\varepsilon,\lambda,\mu}(tu,tv) = -\infty$. Fix some $(u,v) \in H\backslash\{(0,0)\}$, there exists $\bar{t} > 0$ such that $\|(\bar{t}u,\bar{t}v)\|_H > \sigma$ and $J_{\varepsilon,\lambda,\mu}(\bar{t}u,\bar{t}v) < 0$. Let $(\bar{u},\bar{v}) = (\bar{t}u,\bar{t}v)$. □

Define

$$\psi(u,v) = \langle J'_{\varepsilon,\lambda,\mu}(u,v),(u,v)\rangle.$$

Then for $(u,v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$, we obtain that

$$\begin{aligned} \langle\psi'(u,v),(u,v)\rangle &= 2\|(u,v)\|_H^2 - p\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz \\ &\quad - q\int_{\mathbb{R}^N}\bigl(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\bigr)\,dz \\ &= (p-q)\int_{\mathbb{R}^N}\bigl(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\bigr)\,dz - (p-2)\|(u,v)\|_H^2 \qquad (2.1) \\ &= (2-q)\|(u,v)\|_H^2 + (q-p)\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz < 0. \qquad (2.2) \end{aligned}$$

Lemma 2.4 *For each $(u,v) \in H\backslash\{(0,0)\}$, there exists a unique positive number $t_{u,v}$ such that $(t_{u,v}u, t_{u,v}v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ and $J_{\varepsilon,\lambda,\mu}(t_{u,v}u,t_{u,v}v) = \sup_{t\geq 0} J_{\varepsilon,\lambda,\mu}(tu,tv)$.*

Proof Fixed $(u,v) \in H\backslash\{(0,0)\}$, we consider

$$\begin{aligned} R(t) &= J_{\varepsilon,\lambda,\mu}(tu,tv) \\ &= \frac{t^2}{2}\|(u,v)\|_H^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz - \frac{t^q}{q}\int_{\mathbb{R}^N}\bigl(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\bigr)\,dz. \end{aligned}$$

Since $R(0) = 0$, $\lim_{t\to\infty} R(t) = -\infty$, by Lemma 2.3(i), then $\sup_{t\geq 0} R(t)$ is achieved at some $t_{u,v} > 0$. Moreover, we have that $R'(t_{u,v}) = 0$, that is, $(t_{u,v}u, t_{u,v}v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$. Next, we claim that $t_{u,v}$ is a unique positive number such that $R'(t_{u,v}) = 0$. Consider

$$r(t) = \|(u,v)\|_H^2 - t^{p-2}\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz - t^{q-2}\int_{\mathbb{R}^N}\bigl(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\bigr)\,dz,$$

then $R'(t) = tr(t)$. Since $r(0) = \|(u,v)\|_H^2 > 0$,

$$\begin{aligned} r'(t) = {}&-(p-2)t^{p-3}\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz \\ &-(q-2)t^{q-3}\int_{\mathbb{R}^N}\bigl(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\bigr)\,dz < 0, \end{aligned}$$

there exists a unique positive number $\bar{t}_{u,v}$ such that $r(\bar{t}_{u,v}) = 0$. It follows that $R'(\bar{t}_{u,v}) = 0$. Hence, $\bar{t}_{u,v} = t_{u,v}$. □

Remark 2.5 By Lemma 2.3(i) and Lemma 2.4, then $\theta_{\varepsilon,\lambda,\mu} \geq d_0 > 0$ for some constant d_0.

Lemma 2.6 *Let* $(u_0, v_0) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ *satisfy*

$$J_{\varepsilon,\lambda,\mu}(u_0,v_0) = \min_{(u,v)\in\mathbf{M}_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u,v) = \theta_{\varepsilon,\lambda,\mu},$$

then (u_0, v_0) *is a solution of* $(E_{\varepsilon,\lambda,\mu})$.

Proof By (2.2), $\langle\psi'(u,v),(u,v)\rangle < 0$ for $(u,v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$. Since $J_{\varepsilon,\lambda,\mu}(u_0,v_0) = \min_{(u,v)\in\mathbf{M}_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u,v)$, by the Lagrange multiplier theorem, there is $\tau \in \mathbb{R}$ such that $J'_{\varepsilon,\lambda,\mu}(u_0,v_0) = \tau\psi'(u_0,v_0)$ in H^{-1}. Then we have

$$0 = \bigl\langle J'_{\varepsilon,\lambda,\mu}(u_0,v_0),(u_0,v_0)\bigr\rangle = \tau\bigl\langle\psi'(u_0,v_0),(u_0,v_0)\bigr\rangle.$$

It follows that $\tau = 0$ and $J'_{\varepsilon,\lambda,\mu}(u_0,v_0) = 0$ in H^{-1}. Therefore, (u_0,v_0) is a nontrivial solution of $(E_{\varepsilon,\lambda,\mu})$ and $J_{\varepsilon,\lambda,\mu}(u_0,v_0) = \theta_{\varepsilon,\lambda,\mu}$. □

3 $(\mathrm{PS})_\gamma$-condition in H for $J_{\varepsilon,\lambda,\mu}$

First of all, we define the Palais-Smale (denoted by (PS)) sequence and (PS)-condition in H for some functional J.

Definition 3.1 (i) For $\gamma \in \mathbb{R}$, a sequence $\{(u_n,v_n)\}$ is a $(\mathrm{PS})_\gamma$-sequence in H for J if $J(u_n,v_n) = \gamma + o_n(1)$ and $J'(u_n,v_n) = o_n(1)$ strongly in H^{-1} as $n\to\infty$, where H^{-1} is the dual space of H;

(ii) J satisfies the $(\mathrm{PS})_\gamma$-condition in H if every $(\mathrm{PS})_\gamma$-sequence in H for J contains a convergent subsequence.

Applying Ekeland's variational principle and using the same argument as in Cao-Zhou [8] or Tarantello [9], we have the following lemma.

Lemma 3.2 (i) *There exists a* $(PS)_{\theta_{\varepsilon,\lambda,\mu}}$*-sequence* $\{(u_n,v_n)\}$ *in* $\mathbf{M}_{\varepsilon,\lambda,\mu}$ *for* $J_{\varepsilon,\lambda,\mu}$.

In order to prove the existence of positive solutions, we want to prove that $J_{\varepsilon,\lambda,\mu}$ satisfies the $(\mathrm{PS})_\gamma$-condition in H for $\gamma \in (0, \frac{p-2}{2p}\frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}})$.

Lemma 3.3 *$J_{\varepsilon,\lambda,\mu}$ satisfies the $(PS)_\gamma$-condition in H for $\gamma \in (0, \frac{p-2}{2p}\frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}})$.*

Proof Let $\{(u_n, v_n)\}$ be a $(\mathrm{PS})_\gamma$-sequence in H for $J_{\varepsilon,\lambda,\mu}$ such that $J_{\varepsilon,\lambda,\mu}(u_n, v_n) = \gamma + o_n(1)$ and $J'_{\varepsilon,\lambda,\mu}(u_n, v_n) = o_n(1)$ in H^{-1}. Then

$$\begin{aligned}\gamma + c_n + \frac{d_n\|(u_n,v_n)\|_H}{q} &\geq J_{\varepsilon,\lambda,\mu}(u_n,v_n) - \frac{1}{q}\langle J'_{\varepsilon,\lambda,\mu}(u_n,v_n),(u_n,v_n)\rangle \\ &= \left(\frac{1}{2}-\frac{1}{q}\right)\|(u_n,v_n)\|_H^2 + \left(\frac{1}{q}-\frac{1}{p}\right)\int_{\mathbb{R}^N} f(\varepsilon z)|u_n|^\alpha|v_n|^\beta\,dz \\ &\geq \frac{q-2}{2q}\|(u_n,v_n)\|_H^2,\end{aligned}$$

where $c_n = o_n(1)$, $d_n = o_n(1)$ as $n \to \infty$. It follows that $\{(u_n, v_n)\}$ is bounded in H. Hence, there exist a subsequence $\{(u_n, v_n)\}$ and $(u, v) \in H$ such that

$$\begin{aligned}&u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{weakly in } H^1(\mathbb{R}^N);\\ &u_n \to u, \quad v_n \to v \quad \text{strongly in } L^s_{\mathrm{loc}}(\mathbb{R}^N) \text{ for any } 1 \leq s < 2^*;\\ &u_n \to u, \quad v_n \to v \quad \text{a.e. in } \mathbb{R}^N.\end{aligned}$$

Moreover, we have that $J'_{\varepsilon,\lambda,\mu}(u, v) = 0$ in H^{-1}. We use the Brézis-Lieb lemma to obtain (3.1) and (3.2) as follows:

$$\int_{\mathbb{R}^N} g(\varepsilon z)|u_n - u|^q\,dz = \int_{\mathbb{R}^N} g(\varepsilon z)|u_n|^q\,dz - \int_{\mathbb{R}^N} g(\varepsilon z)|u|^q\,dz + o_n(1); \tag{3.1}$$

$$\int_{\mathbb{R}^N} h(\varepsilon z)|v_n - v|^q\,dz = \int_{\mathbb{R}^N} h(\varepsilon z)|v_n|^q\,dz - \int_{\mathbb{R}^N} h(\varepsilon z)|v|^q\,dz + o_n(1). \tag{3.2}$$

Next, we claim that

$$\int_{\mathbb{R}^N} g(\varepsilon z)|u_n - u|^q\,dz \to 0 \quad \text{as } n \to \infty \tag{3.3}$$

and

$$\int_{\mathbb{R}^N} h(\varepsilon z)|v_n - v|^q\,dz \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Since $g \in L^m(\mathbb{R}^N)$, where $m = p/(p-q)$, then for any $\sigma > 0$, there exists $r > 0$ such that $\int_{[B_r^N(0)]^c} g(\varepsilon z)^{\frac{p}{p-q}}\,dz < \sigma$. By the Hölder inequality and the Sobolev embedding theorem, we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} g(\varepsilon z)|u_n - u|^q\,dz\right| \\ &\quad \leq \int_{B_r^N(0)} g(\varepsilon z)|u_n - u|^q\,dz\end{aligned}$$

$$
\begin{aligned}
&+\int_{[B_r^N(0)]^c} g(\varepsilon z)|u_n-u|^q\,dz \\
&\le \|g\|_m\left(\int_{B_r^N(0)}|u_n-u|^p\,dz\right)^{q/p} \\
&\quad + S_p^{-\frac{q}{2}}\left(\int_{[B_r^N(0)]^c} g(\varepsilon z)^{\frac{p}{p-q}}\,dz\right)^{\frac{p-q}{p}}\left(\int_{\mathbb{R}^N}\left|\nabla(u_n-u)\right|^2+|u_n-u|^2\,dz\right)^{q/2} \\
&\le C'\sigma+o_n(1)\quad \left(\because \{u_n\} \text{ is bounded in } H^1(\mathbb{R}^N) \text{ and } u_n\to u \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N)\right).
\end{aligned}
$$

Similarly, $\int_{\mathbb{R}^N} h(\varepsilon z)|v_n-v|^q\,dz\to 0$ as $n\to\infty$. By (A1) and $u_n\to u$, $v_n\to v$ strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, we have that

$$
\int_{\mathbb{R}^N} f(\varepsilon z)|u_n-u|^\alpha|v_n-v|^\beta\,dz=\int_{\mathbb{R}^N} f_\infty|u_n-u|^\alpha|v_n-v|^\beta\,dz=o_n(1). \tag{3.5}
$$

Let $p_n=(u_n-u,v_n-v)$. By (3.1)-(3.4) and Lemma 2.1, we deduce that

$$
\begin{aligned}
\|p_n\|_H^2 &= \left(\|u_n\|_H^2+\|v_n\|_H^2\right)-\left(\|u\|_H^2+\|v\|_H^2\right)+o_n(1) \\
&= \int_{\mathbb{R}^N} f(\varepsilon z)|u_n|^\alpha|v_n|^\beta\,dz+\int_{\mathbb{R}^N}\left(\lambda g(\varepsilon z)|u_n|^q+\mu h(\varepsilon z)|v_n|^q\right)dz \\
&\quad -\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz-\int_{\mathbb{R}^N}\left(\lambda g(\varepsilon z)|u|^q+\mu h(\varepsilon z)|v|^q\right)dz+o_n(1) \\
&= \int_{\mathbb{R}^N} f(\varepsilon z)|u_n-u|^\alpha|v_n-v|^\beta\,dz+o_n(1),
\end{aligned}
$$

and

$$
\frac{1}{2}\|p_n\|_H^2-\frac{1}{\alpha+\beta}\int_{\mathbb{R}^N} f(\varepsilon z)|u_n-u|^\alpha|v_n-v|^\beta\,dz=\gamma-J_{\varepsilon,\lambda,\mu}(u,v)+o_n(1). \tag{3.6}
$$

We may assume that

$$
\|p_n\|_H^2\to l\quad\text{and}\quad\int_{\mathbb{R}^N} f(\varepsilon z)|u_n-u|^\alpha|v_n-v|^\beta\,dz\to l\quad\text{as } n\to\infty. \tag{3.7}
$$

Recall that

$$
S_{\alpha,\beta}=\inf_{u,v\in H^1(\mathbb{R}^N)\setminus\{(0)\}}\frac{\|(u,v)\|_H^2}{(\int_{\mathbb{R}^N}|u|^\alpha|v|^\beta\,dz)^{2/p}},\quad\text{where } p=\alpha+\beta.
$$

If $l>0$, by (3.5), then

$$
\begin{aligned}
S_{\alpha,\beta}l^{\frac{2}{p}} &= S_{\alpha,\beta}\left(\int_{\mathbb{R}^N} f(\varepsilon z)|u_n-u|^\alpha|v_n-v|^\beta\,dz\right)^{2/p}+o_n(1) \\
&= S_{\alpha,\beta}\left(\int_{\mathbb{R}^N} f_\infty|u_n-u|^\alpha|v_n-v|^\beta\,dz\right)^{2/p}+o_n(1) \\
&\le (f_\infty)^{2/p}\|p_n\|_H^2+o_n(1)=(f_\infty)^{2/p}l.
\end{aligned}
$$

This implies $l \geq (S_{\alpha,\beta})^{p/(p-2)}/(f_\infty)^{2/(p-2)}$. By (3.6) and (3.7), we obtain that

$$\gamma = \left(\frac{1}{2} - \frac{1}{p}\right) l + J_{\varepsilon,\lambda,\mu}(u,v) \geq \frac{p-2}{2p} \frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}},$$

which is a contradiction. Hence, $l = 0$, that is, $(u_n, v_n) \to (u, v)$ strongly in H. □

4 Existence of *k* solutions

Let $w \in H^1(\mathbb{R}^N)$ be the unique, radially symmetric and positive ground state solution of equation $(E0)$ in $\mathbb{R}^N$. Recall the facts (or see Bahri-Li [10], Bahri-Lions [11], Gidas-Ni-Nirenberg [12] and Kwong [13]):

(i) $w \in L^\infty(\mathbb{R}^N) \cap C^{2,\theta}_{\text{loc}}(\mathbb{R}^N)$ for some $0 < \theta < 1$ and $\lim_{|z|\to\infty} w(z) = 0$;

(ii) for any $\varepsilon > 0$, there exist positive numbers C_1, C_2^ε and C_3^ε such that for all $z \in \mathbb{R}^N$

$$C_2^\varepsilon \exp\bigl(-(1+\varepsilon)|z|\bigr) \leq w(z) \leq C_1 \exp\bigl(-|z|\bigr)$$

and

$$\bigl|\nabla w(z)\bigr| \leq C_3^\varepsilon \exp\bigl(-(1-\varepsilon)|z|\bigr).$$

By Lien-Tzeng-Wang [14], then

$$S_p = \frac{\int_{\mathbb{R}^N} (|\nabla w|^2 + w^2)\,dz}{(\int_{\mathbb{R}^N} w^p\,dz)^{2/p}}. \tag{4.1}$$

For $1 \leq i \leq k$, we define

$$w_\varepsilon^i(z) = w\left(z - \frac{a^i}{\varepsilon}\right), \quad \text{where } f\bigl(a^i\bigr) = \max_{z\in\mathbb{R}^N} f(z) = 1.$$

Clearly, $w_\varepsilon^i(z) \in H^1(\mathbb{R}^N)$.

First of all, we want to prove that

$$\lim_{\varepsilon\to 0^+} \sup_{t\geq 0} J_{\varepsilon,\lambda,\mu}\bigl(t\sqrt{\alpha} w_\varepsilon^i, t\sqrt{\beta} w_\varepsilon^i\bigr) \leq \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

Lemma 4.1 *For $\lambda > 0$ and $\mu > 0$, then*

$$\lim_{\varepsilon\to 0^+} \sup_{t\geq 0} J_{\varepsilon,\lambda,\mu}\bigl(t\sqrt{\alpha} w_\varepsilon^i, t\sqrt{\beta} w_\varepsilon^i\bigr) \leq \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)} \quad \textit{uniformly in } i.$$

Moreover, we have that

$$0 < \theta_{\varepsilon,\lambda,\mu} \leq \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)}.$$

Proof Part I: Since $J_{\varepsilon,\lambda,\mu}$ is continuous in H, $J_{\varepsilon,\lambda,\mu}(0,0) = 0$, and $\{(\sqrt{\alpha} w_\varepsilon^i, \sqrt{\beta} w_\varepsilon^i)\}$ is uniformly bounded in H for any $\varepsilon > 0$ and $1 \leq i \leq k$, then there exists $t_0 > 0$ such that for $0 \leq t < t_0$ and any $\varepsilon > 0$,

$$J_{\varepsilon,\lambda,\mu}\bigl(t\sqrt{\alpha} w_\varepsilon^i, t\sqrt{\beta} w_\varepsilon^i\bigr) < \frac{p-2}{2p} (S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

From (A1), we have that $\inf_{z\in\mathbb{R}^N} f(z) > 0$. Then

$$\begin{aligned} J_{\varepsilon,\lambda,\mu}\big(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i\big) &\le \frac{t^2}{2}\big\|(\sqrt{\alpha}w, \sqrt{\beta}w)\big\|_H^2 \\ &\quad - \frac{t^{\alpha+\beta}}{\alpha+\beta}\Big(\inf_{z\in\mathbb{R}^N} f(z)\Big)\int_{\mathbb{R}^N} |\sqrt{\alpha}w|^\alpha |\sqrt{\beta}w|^\beta \, dz \\ &\to -\infty \quad \text{as } t \to \infty. \end{aligned}$$

It follows that there exists $t_1 > 0$ such that for $t > t_1$ and any $\varepsilon > 0$,

$$J_{\varepsilon,\lambda,\mu}\big(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i\big) < \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

From now on, we only need to show that

$$\lim_{\varepsilon\to 0^+} \sup_{t_0\le t\le t_1} J_{\varepsilon,\lambda,\mu}\big(tw_\varepsilon^i\big) \le \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

Since

$$\sup_{t\ge 0}\left(\frac{t^2}{2}a - \frac{t^{\alpha+\beta}}{\alpha+\beta}b\right) = \frac{\alpha+\beta-2}{2(\alpha+\beta)}\left(\frac{a}{b^{\frac{2}{\alpha+\beta}}}\right)^{\frac{\alpha+\beta}{\alpha+\beta-2}}, \quad \text{where } a, b > 0,$$

and by (4.1), then

$$\begin{aligned} &\sup_{t\ge 0}\left\{\frac{t^2}{2}\big\|\big(\sqrt{\alpha}w_\varepsilon^i, \sqrt{\beta}w_\varepsilon^i\big)\big\|_H^2 - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} \big|\sqrt{\alpha}w_\varepsilon^i\big|^\alpha \big|\sqrt{\alpha}w_\varepsilon^i\big|^\beta \, dz\right\} \\ &\quad = \frac{p-2}{2p}\left[\frac{(\alpha+\beta)\int_{\mathbb{R}^N}(|\nabla w|^2 + w^2)\,dz}{(\alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}\int_{\mathbb{R}^N} w^p\,dz)^{2/p}}\right]^{\frac{p}{p-2}} = \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)}. \end{aligned} \tag{4.2}$$

For $t_0 \le t \le t_1$, by (4.2), we have that

$$\begin{aligned} J_{\varepsilon,\lambda,\mu}\big(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i\big) &= \frac{t^2}{2}\big\|\big(\sqrt{\alpha}w_\varepsilon^i, \sqrt{\beta}w_\varepsilon^i\big)\big\|_H^2 - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_{\mathbb{R}^N} f(\varepsilon z)\big|\sqrt{\alpha}w_\varepsilon^i\big|^\alpha \big|\sqrt{\beta}w_\varepsilon^i\big|^\beta \, dz \\ &\quad - \frac{t^q}{q}\int_{\mathbb{R}^N}\big(\lambda g(\varepsilon z)\big|\sqrt{\alpha}w_\varepsilon^i\big|^q + \mu h(\varepsilon z)\big|\sqrt{\beta}w_\varepsilon^i\big|^q\big)\,dz \\ &\le \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)} \\ &\quad + \frac{t_1^p}{p}\int_{\mathbb{R}^N}\big(1 - f(\varepsilon z)\big)\big|\sqrt{\alpha}w_\varepsilon^i\big|^\alpha \big|\sqrt{\alpha}w_\varepsilon^i\big|^\beta \, dz. \end{aligned}$$

Since

$$\begin{aligned} &\int_{\mathbb{R}^N}\big(1 - f(\varepsilon z)\big)\big|\sqrt{\alpha}w_\varepsilon^i\big|^\alpha \big|\sqrt{\beta}w_\varepsilon^i\big|^\beta \, dz \\ &\quad = \alpha^{\frac{\alpha}{2}}\beta^{\frac{\beta}{2}}\int_{\mathbb{R}^N}\big(1 - f\big(\varepsilon z + a^i\big)\big)w^p \, dz = o(1) \quad \text{as } \varepsilon \to 0^+ \text{ uniformly in } i, \end{aligned}$$

then

$$\lim_{\varepsilon\to 0^+}\sup_{t_0\le t\le t_1} J_{\varepsilon,\lambda,\mu}\left(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i\right) \le \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)},$$

that is, for $\lambda > 0$ and $\mu > 0$,

$$\lim_{\varepsilon\to 0^+}\sup_{t\ge 0} J_{\varepsilon,\lambda,\mu}\left(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i\right) \le \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)} \quad \text{uniformly in } i.$$

Part II: By Lemma 2.4, there is a number $t_\varepsilon^i > 0$ such that $(t_\varepsilon^i\sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i\sqrt{\beta}w_\varepsilon^i) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$, where $1 \le i \le k$. Hence, from the result of Part I, we have that for $\lambda > 0$ and $\mu > 0$,

$$0 < \theta_{\varepsilon,\lambda,\mu} \le \lim_{\varepsilon\to 0^+}\sup_{t\ge 0} J_{\varepsilon,\lambda,\mu}\left(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i\right) \le \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)}. \qquad \square$$

Proof of Theorem 1.1 By Lemma 3.2, there exists a $(\mathrm{PS})_{\theta_{\varepsilon,\lambda,\mu}}$-sequence $\{(u_n, v_n)\}$ in $\mathbf{M}_{\varepsilon,\lambda,\mu}$ for $J_{\varepsilon,\lambda,\mu}$. Since $0 < \theta_{\varepsilon,\lambda,\mu} \le \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)} < \frac{p-2}{2p}\frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}$ for $\lambda > 0$ and $\mu > 0$, by Lemma 3.3, there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0, v_0) \in H$ such that $(u_n, v_n) \to (u_0, v_0)$ strongly in H. It is easy to check that (u_0, v_0) is a nontrivial solution of $(E_{\varepsilon,\lambda,\mu})$ and $J_{\varepsilon,\lambda,\mu}(u_0, v_0) = \theta_{\varepsilon,\lambda,\mu}$. Since $J_{\varepsilon,\lambda,\mu}(u_0, v_0) = J_{\lambda,\mu}(|u_0|, |v_0|)$ and $(|u_0|, |v_0|) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$, by Lemma 2.6, we may assume that $u_0 \ge 0$, $v_0 \ge 0$. Applying the maximum principle, $u_0 > 0$ and $v_0 > 0$ in Ω. $\square$

Choosing $0 < \rho_0 < 1$ such that

$$\overline{B^N_{\rho_0}(a^i)} \cap \overline{B^N_{\rho_0}(a^j)} = \varnothing \quad \text{for } i \ne j \text{ and } 1 \le i, j \le k,$$

where $\overline{B^N_{\rho_0}(a^i)} = \{z \in \mathbb{R}^N \mid |z - a^i| \le \rho_0\}$ and $f(a^i) = \max_{z\in\mathbb{R}^N} f(z) = 1$, define $\mathbf{K} = \{a^i \mid 1 \le i \le k\}$ and $\mathbf{K}_{\rho_0/2} = \bigcup_{i=1}^k \overline{B^N_{\rho_0/2}(a^i)}$. Suppose $\bigcup_{i=1}^k \overline{B^N_{\rho_0}(a^i)} \subset B^N_{r_0}(0)$ for some $r_0 > 0$. Let Q_ε be given by

$$Q_\varepsilon(u, v) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z)|u|^\alpha |v|^\beta\, dz}{\int_{\mathbb{R}^N} |u|^\alpha |v|^\beta\, dz},$$

where $\chi : \mathbb{R}^N \to \mathbb{R}^N$, $\chi(z) = z$ for $|z| \le r_0$ and $\chi(z) = r_0 z/|z|$ for $|z| > r_0$.

For each $1 \le i \le k$, we define

$$O^i_{\varepsilon,\lambda,\mu} = \left\{(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu} \mid \left|Q_\varepsilon(u, v) - a^i\right| < \rho_0\right\},$$
$$\partial O^i_{\varepsilon,\lambda,\mu} = \left\{(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu} \mid \left|Q_\varepsilon(u, v) - a^i\right| = \rho_0\right\},$$
$$\beta^i_{\varepsilon,\lambda,\mu} = \inf_{(u,v)\in O^i_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u, v) \quad \text{and} \quad \widetilde{\beta}^i_{\varepsilon,\lambda,\mu} = \inf_{(u,v)\in \partial O^i_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u, v).$$

By Lemma 2.4, there exists $t_\varepsilon^i > 0$ such that $(t_\varepsilon^i\sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i\sqrt{\beta}w_\varepsilon^i) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ for each $1 \le i \le k$. Then we have the following result.

Lemma 4.2 *There exists $\varepsilon_1 > 0$ such that if $\varepsilon \in (0, \varepsilon_1)$, then $Q_\varepsilon(t_\varepsilon^i\sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i\sqrt{\beta}w_\varepsilon^i) \in \mathbf{K}_{\rho_0/2}$ for each $1 \le i \le k$.*

Proof Since

$$\begin{aligned} Q_\varepsilon\bigl(t_\varepsilon^i\sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i\sqrt{\beta}w_\varepsilon^i\bigr) &= \frac{\int_{\mathbb{R}^N}\chi(\varepsilon z)|w(z-\frac{a^i}{\varepsilon})|^p\,dz}{\int_{\mathbb{R}^N}|w(z-\frac{a^i}{\varepsilon})|^p\,dz} \\ &= \frac{\int_{\mathbb{R}^N}\chi(\varepsilon z+a^i)|w(z)|^p\,dz}{\int_{\mathbb{R}^N}|w(z)|^p\,dz} \\ &\to a^i \quad \text{as } \varepsilon\to 0^+, \end{aligned}$$

there exists $\varepsilon_1 > 0$ such that

$$Q_\varepsilon\bigl(t_\varepsilon^i\sqrt{\alpha}w_\varepsilon^i, t_\varepsilon^i\sqrt{\beta}w_\varepsilon^i\bigr)\in \mathbf{K}_{\rho_0/2} \quad \text{for any } \varepsilon\in(0,\varepsilon_1) \text{ and each } 1\le i\le k. \qquad \square$$

We need the following lemmas to prove that $\beta^i_{\lambda,\mu} < \widetilde{\beta}^i_{\lambda,\mu}$ for sufficiently small ε, λ, μ.

Lemma 4.3 $\theta_{\max} = \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)}$.

Proof From Part I of Lemma 4.1, we obtain $\sup_{t\ge0} I_{\max}(t\sqrt{\alpha}w_\varepsilon^i, t\sqrt{\beta}w_\varepsilon^i) = \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)}$ uniformly in i. Similarly to Lemma 2.4, there is a sequence $\{s^i_{\max}\}\subset\mathbb{R}^+$ such that $(s^i_{\max}\sqrt{\alpha}w_\varepsilon^i, s^i_{\max}\sqrt{\beta}w_\varepsilon^i)\in\mathbf{N}_{\max}$ and

$$\theta_{\max}\le I_{\max}\bigl(s^i_{\max}\sqrt{\alpha}u_\varepsilon^i, s^i_{\max}\sqrt{\beta}u_\varepsilon^i\bigr) = \sup_{t\ge0} J_{\max}\bigl(t\sqrt{\alpha}u_\varepsilon^i, t\sqrt{\beta}u_\varepsilon^i\bigr) = \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)}.$$

Let $\{(u_n,v_n)\}\subset\mathbf{N}_{\max}$ be a minimizing sequence of $\theta_{\max}$ for $I_{\max}$. It follows that $\|(u_n,v_n)\|_H^2 = \int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,dz$ and

$$\begin{aligned} \theta_{\max} &= \frac12\bigl\|(u_n,v_n)\bigr\|_H^2 - \frac1p\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,dz + o_n(1) \\ &= \frac{p-2}{2p}\bigl\|(u_n,v_n)\bigr\|_H^2 + o_n(1). \end{aligned}$$

We may assume that $\|(u_n,v_n)\|_H^2\to l$ and $\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,dz\to l$ as $n\to\infty$, where $l = \frac{2p}{p-2}\theta_{\max} > 0$. By the definition of $S_{\alpha,\beta}$, then $S_{\alpha,\beta}l^{\frac2p}\le l$. We can deduce that $S_{\alpha,\beta}\le l^{\frac{p-2}{p}} = (\frac{2p}{p-2}\theta_{\max})^{\frac{p-2}{p}}$, that is, $\frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)}\le\theta_{\max}$. $\square$

Lemma 4.4 *There exists a number* $\delta_0 > 0$ *such that if* $(u,v)\in\mathbf{N}_\varepsilon$ *and* $I_\varepsilon(u,v)\le\theta_{\max}+\delta_0$, *then* $Q_\varepsilon(u,v)\in\mathbf{K}_{\rho_0/2}$ *for any* $0<\varepsilon<\varepsilon_1$.

Proof On the contrary, there exist the sequences $\{\varepsilon_n\}\subset\mathbb{R}^+$ and $\{(u_n,v_n)\}\subset\mathbf{N}_{\varepsilon_n}$ such that $\varepsilon_n\to0$, $I_{\varepsilon_n}(u_n,v_n) = \theta_{\max}\ (>0) + o_n(1)$ as $n\to\infty$ and $Q_{\varepsilon_n}(u_n,v_n)\notin\mathbf{K}_{\rho_0/2}$ for all $n\in\mathbb{N}$. It is easy to check that $\{(u_n,v_n)\}$ is bounded in H. Suppose that $\int_{\mathbb{R}^N}|u_n|^\alpha|v_n|^\beta\,dz\to0$ as $n\to\infty$. Since

$$\bigl\|(u_n,v_n)\bigr\|_H^2 = \int_{\mathbb{R}^N} f(\varepsilon_n z)|u_n|^\alpha|v_n|^\beta\,dz \quad \text{for each } n\in\mathbb{N},$$

then

$$\theta_{\max}+o_n(1) = I_{\varepsilon_n}(u_n,v_n) = \left(\frac12-\frac1p\right)\int_{\mathbb{R}^N} f(\varepsilon_n z)|u_n|^\alpha|v_n|^\beta\,dz \le o_n(1),$$

which is a contradiction. Thus, $\int_{\mathbb{R}^N} |u_n|^\alpha |v_n|^\beta \, dz \nrightarrow 0$ as $n \to \infty$. Similarly to the concentration-compactness principle (see Lions [15, 16] or Wang [6, Lemma 2.16]), then there exist a constant $c_0 > 0$ and a sequence $\{\widetilde{z}_n\} \subset \mathbb{R}^N$ such that

$$\int_{B^N(\widetilde{z}_n;1)} |u_n|^{\frac{\alpha l}{p}} |v_n|^{\frac{\beta l}{p}} \, dz \geq c_0 > 0, \tag{4.3}$$

where $2 < l < p = \alpha + \beta < 2^*$ and $p = l(1-t) + 2^* t$ for some $t \in ((N-2)/N, 1)$. Let $(\widetilde{u}_n(z), \widetilde{v}_n(z)) = (u_n(z + \widetilde{z}_n), v_n(z + \widetilde{z}_n))$. Then there are a subsequence $\{(\widetilde{u}_n, \widetilde{v}_n)\}$ and $(\widetilde{u}, \widetilde{v}) \in H$ such that $\widetilde{u}_n \rightharpoonup \widetilde{u}$ and $\widetilde{v}_n \rightharpoonup \widetilde{v}$ weakly in $H^1(\mathbb{R}^N)$. Using the similar computation of Lemma 2.4, there is a sequence $\{s^n_{\max}\} \subset \mathbb{R}^+$ such that $(s^n_{\max}\widetilde{u}_n, s^n_{\max}\widetilde{v}_n) \in \mathbf{N}_{\max}$ and

$$\begin{aligned} 0 < \theta_{\max} &\leq I_{\max}\left(s^n_{\max}\widetilde{u}_n, s^n_{\max}\widetilde{v}_n\right) = I_{\max}\left(s^n_{\max}u_n, s^n_{\max}v_n\right) \\ &\leq I_{\varepsilon_n}\left(s^n_{\max}u_n, s^n_{\max}v_n\right) \leq I_{\varepsilon_n}(u_n, v_n) = \theta_{\max} + o_n(1) \quad \text{as } n \to \infty. \end{aligned}$$

We deduce that a subsequence $\{s^n_{\max}\}$ satisfies $s^n_{\max} \to s_0 > 0$. Then there are a subsequence $\{(s^n_{\max}\widetilde{u}_n, s^n_{\max}\widetilde{v}_n)\}$ and $(s_0\widetilde{u}, s_0\widetilde{v}) \in H$ such that $s^n_{\max}\widetilde{u}_n \rightharpoonup s_0\widetilde{u}$ and $s^n_{\max}\widetilde{v}_n \rightharpoonup s_0\widetilde{v}$ weakly in $H^1(\mathbb{R}^N)$. By (4.3), then $\widetilde{u} \neq 0$ and $\widetilde{v} \neq 0$. Applying Ekeland's variational principle, there exists a $(PS)_{\theta_{\max}}$-sequence $\{(U_n, V_n)\}$ for $I_{\max}$ and $\|(U_n - s^n_{\max}\widetilde{u}_n, V_n - s^n_{\max}\widetilde{v}_n)\|_H = o_n(1)$. Similarly to the proof of Lemma 3.3, there exist a subsequence $\{(U_n, V_n)\}$ and $(U_0, V_0) \in H$ such that $U_n \to U_0$, $V_n \to V_0$ strongly in $H^1(\mathbb{R}^N)$ and $I_{\max}(U_0, V_0) = \theta_{\max}$. Now, we want to show that there exists a subsequence $\{z_n\} = \{\varepsilon_n \widetilde{z}_n\}$ such that $z_n \to z_0 \in \mathbf{K}$.

(i) Claim that the sequence $\{z_n\}$ is bounded in $\mathbb{R}^N$. On the contrary, assume that $|z_n| \to \infty$, then

$$\begin{aligned} \theta_{\max} = I_{\max}(U_0, V_0) &< \frac{1}{2}\left\|(U_0, V_0)\right\|^2_H - \frac{1}{p}\int_{\mathbb{R}^N} f_\infty |U_0|^\alpha |V_0|^\beta \, dz \\ &\leq \liminf_{n\to\infty}\left[\frac{(s^n_{\max})^2}{2}\left\|(\widetilde{u}_n, \widetilde{v}_n)\right\|^2_H - \frac{(s^n_{\max})^p}{p}\int_{\mathbb{R}^N} f(\varepsilon_n z + z_n)|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta \, dz\right] \\ &= \liminf_{n\to\infty}\left[\frac{(s^n_{\max})^2}{2}\left\|(u_n, v_n)\right\|^2_H - \frac{(s^n_{\max})^p}{p}\int_{\mathbb{R}^N} f(\varepsilon_n z)|u_n|^\alpha |v_n|^\beta \, dz\right] \\ &= \liminf_{n\to\infty} I_{\varepsilon_n}\left(s^n_{\max}u_n, s^n_{\max}v_n\right) \leq \liminf_{n\to\infty} I_{\varepsilon_n}(u_n, v_n) = \theta_{\max}, \end{aligned}$$

which is a contradiction.

(ii) Claim that $z_0 \in \mathbf{K}$. On the contrary, assume that $z_0 \notin \mathbf{K}$, that is, $f(z_0) < 1 = \max_{z\in\mathbb{R}^N} f(z)$. Then use the argument of (i) to obtain that

$$\begin{aligned} \theta_{\max} &= I_{\max}(U_0, V_0) \leq I_{\max}(s_0 U_0, s_0 V_0) \\ &< \frac{(s_0)^2}{2}\left\|(U_0, V_0)\right\|^2_H - \frac{(s_0)^p}{p}\int_{\mathbb{R}^N} f(z_0)|U_0|^\alpha |V_0|^\beta \, dz \\ &\leq \liminf_{n\to\infty}\left[\frac{(s^n_{\max})^2}{2}\left\|(\widetilde{u}_n, \widetilde{v}_n)\right\|^2_H - \frac{(s^n_{\max})^p}{p}\int_{\mathbb{R}^N} f(\varepsilon_n z + z_n)|\widetilde{u}_n|^\alpha |\widetilde{v}_n|^\beta \, dz\right] \\ &\leq \theta_{\max}, \end{aligned}$$

which is a contradiction.

Since $\|(U_n - s^n_{\max}\widetilde{u_n}, V_n - s^n_{\max}\widetilde{v_n})\|_H = o_n(1)$ and $U_n \to U_0$, $V_n \to V_0$ strongly in $H^1(\mathbb{R}^N)$, we have that

$$\begin{aligned} Q_{\varepsilon_n}(u_n, v_n) &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n z)|\widetilde{u_n}(z-\widetilde{z_n})|^\alpha |\widetilde{v_n}(z-\widetilde{z_n})|^\beta\,dz}{\int_{\mathbb{R}^N} |\widetilde{u_n}(z-\widetilde{z_n})|^\alpha |\widetilde{v_n}(z-\widetilde{z_n})|^\beta\,dz} \\ &= \frac{\int_{\mathbb{R}^N} \chi(\varepsilon_n z + \varepsilon_n \widetilde{z_n})|U_0|^\alpha |V_0|^\beta\,dz}{\int_{\mathbb{R}^N} |U_0|^\alpha |V_0|^\beta\,dz} \to z_0 \in \mathbf{K}_{\rho_0/2} \quad \text{as } n \to \infty, \end{aligned}$$

which is a contradiction.

Hence, there exists $\delta_0 > 0$ such that if $(u, v) \in \mathbf{N}_\varepsilon$ and $I_\varepsilon(u, v) \le \theta_{\max} + \delta_0$, then $Q_\varepsilon(u, v) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \varepsilon_1$. □

Lemma 4.5 *If* $(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$ *and* $J_{\varepsilon,\lambda,\mu}(u, v) \le \theta_{\max} + \delta_0/2$, *then there exists a number* $\Lambda^* > 0$ *such that* $Q_\varepsilon(u, v) \in \mathbf{K}_{\rho_0/2}$ *for any* $0 < \varepsilon < \varepsilon_1$ *and* $0 < \lambda + \mu < \Lambda^*$.

Proof Using the similar computation in Lemma 2.4, we obtain that there is the unique positive number

$$s_\varepsilon = \left(\frac{\|(u, v)\|_H^2}{\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha |v|^\beta\,dz} \right)^{1/(p-2)}$$

such that $(s_\varepsilon u, s_\varepsilon v) \in \mathbf{N}_\varepsilon$. We want to show that there exists $\Lambda_0 > 0$ such that if $0 < \lambda + \mu < \Lambda_0$, then $s_\varepsilon < c$ for some constant $c > 0$ (independent of u and v). First, for $(u, v) \in \mathbf{M}_{\varepsilon,\lambda,\mu}$,

$$0 < d_0 \le \theta_{\varepsilon,\lambda,\mu} \le J_{\varepsilon,\lambda,\mu}(u, v) \le \theta_{\max} + \delta_0/2.$$

Since $\langle J'_{\varepsilon,\lambda,\mu}(u, v), (u, v)\rangle = 0$, then

$$\begin{aligned} \theta_{\max} + \delta_0/2 &\ge J_{\varepsilon,\lambda,\mu}(u, v) \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|(u, v)\|_H^2 + \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha |v|^\beta\,dz \\ &\ge \frac{q-2}{2q}\|(u, v)\|_H^2, \quad \text{that is,} \quad \|(u, v)\|_H^2 \le c_1 = \frac{2q}{q-2}(\theta_{\max} + \delta_0/2), \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} d_0 &\le J_{\varepsilon,\lambda,\mu}(u, v) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\|(u, v)\|_H^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_\Omega \left(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\right)dz \\ &\le \frac{p-2}{2p}\|(u, v)\|_H^2, \quad \text{that is,} \quad \|(u, v)\|_H^2 \ge c_2 = \frac{2p}{p-2}d_0. \end{aligned} \tag{4.5}$$

Moreover, we have that

$$\begin{aligned} \int_\Omega f(\varepsilon z)|u|^\alpha |v|^\beta\,dz &= \|(u, v)\|_H^2 - \int_{\mathbb{R}^N} \left(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q\right)dz \\ &\ge c_2 - \operatorname{Max} S_p^{-\frac{q}{2}}(\lambda + \mu)c_1^{q/2}, \end{aligned}$$

where Max = $\max\{\|g\|_m, \|h\|_m\}$. It follows that there exists $\Lambda_0 > 0$ such that for $0 < \lambda + \mu < \Lambda_0$

$$\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz \geq c_2 - \operatorname{Max} S_p^{-\frac{q}{2}}(\lambda+\mu)(c_1)^{q/2} > 0. \tag{4.6}$$

Hence, by (4.4), (4.5) and (4.6), $s_\varepsilon < c$ for some constant $c > 0$ (independent of u and v) for $0 < \lambda + \mu < \Lambda_0$. Now, we obtain that

$$\begin{aligned}
\theta_{\max} + \delta_0/2 \geq J_{\varepsilon,\lambda,\mu}(u,v) &= \sup_{t\geq 0} J_{\varepsilon,\lambda,\mu}(tu,tv) \geq J_{\varepsilon,\lambda,\mu}(s_\varepsilon u, s_\varepsilon v) \\
&= \frac{1}{2}\left\|(s_\varepsilon u, s_\varepsilon v)\right\|_H^2 - \frac{1}{p}\int_{\mathbb{R}^N} f(\varepsilon z)|s_\varepsilon u|^\alpha|s_\varepsilon v|^\beta\,dz \\
&\quad - \frac{1}{q}\int_{\mathbb{R}^N}\left(\lambda g(\varepsilon z)|s_\varepsilon u|^q + \mu h(\varepsilon z)|s_\varepsilon v|^q\right)dz \\
&\geq I_\varepsilon(s_\varepsilon u, s_\varepsilon v) - \frac{1}{q}\int_{\mathbb{R}^N}\left(\lambda g(\varepsilon z)|s_\varepsilon u|^q + \mu h(\varepsilon z)|s_\varepsilon v|^q\right)dz.
\end{aligned}$$

From the above inequality, we deduce that for any $0 < \varepsilon < \varepsilon_1$ and $0 < \lambda + \mu < \Lambda_0$,

$$\begin{aligned}
I_\varepsilon(s_\varepsilon u, s_\varepsilon v) &\leq \theta_{\max} + \delta_0/2 + \frac{1}{q}\int_{\mathbb{R}^N}\left(\lambda g(\varepsilon z)|s_\varepsilon u|^q + \mu h(\varepsilon z)|s_\varepsilon v|^q\right)dz \\
&\leq \theta_{\max} + \delta_0/2 + \operatorname{Max}(\lambda+\mu)S_p^{-\frac{q}{2}}\left\|(s_\varepsilon u, s_\varepsilon v)\right\|_H^q \\
&< \theta_{\max} + \delta_0/2 + \operatorname{Max} S_p^{-\frac{q}{2}}(\lambda+\mu)c^q(c_1)^{q/2}.
\end{aligned}$$

Hence, there exists $\Lambda^* \in (0, \Lambda_0)$ such that for $0 < \lambda + \mu < \Lambda^*$,

$$I_\varepsilon(s_\varepsilon u, s_\varepsilon v) \leq \theta_{\max} + \delta_0, \quad \text{where } (s_\varepsilon u, s_\varepsilon v) \in \mathbf{N}_\varepsilon.$$

By Lemma 4.4, we obtain

$$Q_\varepsilon(s_\varepsilon u, s_\varepsilon v) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon z)|s_\varepsilon u|^\alpha|s_\varepsilon v|^\beta\,dz}{\int_{\mathbb{R}^N} |s_\varepsilon u|^\alpha|s_\varepsilon v|^\beta\,dz} \in \mathbf{K}_{\rho_0/2},$$

or $Q_\varepsilon(u,v) \in \mathbf{K}_{\rho_0/2}$ for any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$. □

Since $f_\infty < 1$, then by Lemma 4.3,

$$\theta_{\max} = \frac{p-2}{2p}(S_{\alpha,\beta})^{p/(p-2)} < \frac{p-2}{2p}\frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}. \tag{4.7}$$

By Lemmas 4.1, 4.2 and (4.7), for any $0 < \varepsilon < \varepsilon_0$ ($< \varepsilon_1$) and $0 < \lambda + \mu < \Lambda^*$,

$$\beta^i_{\varepsilon,\lambda,\mu} \leq J_{\varepsilon,\lambda,\mu}\left(t^i_\varepsilon\sqrt{\alpha}w^i_\varepsilon, t^i_\varepsilon\sqrt{\beta}w^i_\varepsilon\right) < \frac{p-2}{2p}\frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}. \tag{4.8}$$

Applying above Lemma 4.5, we get that

$$\widetilde{\beta}^i_{\varepsilon,\lambda,\mu} \geq \theta_{\max} + \delta_0/2 \quad \text{for any } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \lambda + \mu < \Lambda^*. \tag{4.9}$$

For each $1 \le i \le k$, by (4.8) and (4.9), we obtain that

$$\beta^i_{\varepsilon,\lambda,\mu} < \widetilde{\beta}^i_{\varepsilon,\lambda,\mu} \quad \text{for any } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \lambda + \mu < \Lambda^*.$$

It follows that

$$\beta^i_{\varepsilon,\lambda,\mu} = \inf_{(u,v)\in O^i_{\varepsilon,\lambda,\mu}\cup\partial O^i_{\varepsilon,\lambda,\mu}} J_{\varepsilon,\lambda,\mu}(u,v) \quad \text{for any } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \lambda + \mu < \Lambda^*.$$

Then applying Ekeland's variational principle and using the standard computation, we have the following lemma.

Lemma 4.6 *For each $1 \le i \le k$, there is a $(PS)_{\beta^i_{\varepsilon,\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset O^i_{\varepsilon,\lambda,\mu}$ in H for $J_{\varepsilon,\lambda,\mu}$.*

Proof See Cao-Zhou [8]. □

Proof of Theorem 1.2 For any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$, by Lemma 4.6, there is a $(\mathrm{PS})_{\beta^i_{\varepsilon,\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset O^i_{\varepsilon,\lambda,\mu}$ for $J_{\varepsilon,\lambda,\mu}$ where $1 \le i \le k$. By (4.8), we obtain that

$$\beta^i_{\varepsilon,\lambda,\mu} < \frac{p-2}{2p}\frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}}.$$

Since $J_{\varepsilon,\lambda,\mu}$ satisfies the $(\mathrm{PS})_\gamma$-condition for $\gamma \in (-\infty, \frac{p-2}{2p}\frac{(S_{\alpha,\beta})^{p/(p-2)}}{(f_\infty)^{2/(p-2)}})$, then $J_{\varepsilon,\lambda,\mu}$ has at least k critical points in $\mathbf{M}_{\varepsilon,\lambda,\mu}$ for any $0 < \varepsilon < \varepsilon_0$ and $0 < \lambda + \mu < \Lambda^*$. Set $u_+ = \max\{u, 0\}$ and $v_+ = \max\{v, 0\}$. Replace the terms $\int_{\mathbb{R}^N} f(\varepsilon z)|u|^\alpha|v|^\beta\,dz$ and $\int_{\mathbb{R}^N}(\lambda g(\varepsilon z)|u|^q + \mu h(\varepsilon z)|v|^q)\,dz$ of the functional $J_{\varepsilon,\lambda,\mu}$ by $\int_{\mathbb{R}^N} f(\varepsilon z)u_+^\alpha v_+^\beta\,dz$ and $\int_{\mathbb{R}^N}(\lambda g(\varepsilon z)u_+^q + \mu h(\varepsilon z)v_+^q)\,dz$, respectively. It follows that $(E_{\varepsilon,\lambda,\mu})$ has k nonnegative solutions. Applying the maximum principle, $(E_{\varepsilon,\lambda,\mu})$ admits at least k positive solutions. □

Competing interests
The author declares that he has no competing interests.

Acknowledgements
The author was grateful for the referee's helpful suggestions and comments.

Received: 29 March 2012 Accepted: 4 October 2012 Published: 24 October 2012

References

1. Lin, TC, Wei, J: Spikes in two coupled nonlinear Schrödinger equations. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **22**, 403-439 (2005)
2. Alves, CO, de Morais Filho, DC, Souto, MAS: On systems of elliptic equations involving subcritical or critical Sobolev exponents. Nonlinear Anal. **42**, 771-787 (2000)
3. Han, P: Multiple positive solutions of nonhomogeneous elliptic system involving critical Sobolev exponents. Nonlinear Anal. **64**, 869-886 (2006)
4. Hsu, TS: Multiple positive solutions for a critical quasilinear elliptic system with concave-convex nonlinearities. Nonlinear Anal. **71**, 2688-2698 (2009)
5. Wu, TF: The Nehari manifold for a semilinear elliptic system involving sign-changing weight function. Nonlinear Anal. **68**, 1733-1745 (2008)
6. Wang, HC: Palais-Smale approaches to semilinear elliptic equations in unbounded domains. Electron. J. Differ. Equ. **Monograph 06** (2004)
7. Han, P: The effect of the domain topology on the number of positive solutions of elliptic systems involving critical Sobolev exponents. Houst. J. Math. **32**, 1241-1257 (2006)
8. Cao, DM, Zhou, HS: Multiple positive solutions of nonhomogeneous semilinear elliptic equations in $\mathbb{R}^N$. Proc. R. Soc. Edinb., Sect. A, Math. **126**, 443-463 (1996)

9. Tarantello, G: On nonhomogeneous elliptic involving critical Sobolev exponent. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **9**, 281-304 (1992)
10. Bahri, A, Li, YY: On a min-max procedure for the existence of a positive solution for certain scalar field equations in $\mathbb{R}^N$. Rev. Mat. Iberoam. **6**, 1-15 (1990)
11. Bahri, A, Lions, PL: On the existence of a positive solution of semilinear elliptic equations in unbounded domains. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **14**, 365-413 (1997)
12. Gidas, B, Ni, WM, Nirenberg, L: Symmetry and related properties via the maximum principle. Commun. Math. Phys. **68**, 209-243 (1979)
13. Kwong, MK: Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$. Arch. Ration. Mech. Anal. **105**, 234-266 (1989)
14. Lien, WC, Tzeng, SY, Wang, HC: Existence of solutions of semilinear elliptic problems on unbounded domains. Differ. Integral Equ. **6**, 1281-1298 (1993)
15. Lions, PL: The concentration-compactness principle in the calculus of variations. The locally compact case. I. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1**, 109-145 (1984)
16. Lions, PL: The concentration-compactness principle in the calculus of variations. The locally compact case. II. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **1**, 223-283 (1984)

doi:10.1186/1687-2770-2012-118
Cite this article as: Lin: Multiple positive solutions for semilinear elliptic systems involving subcritical nonlinearities in $\mathbb{R}^N$. *Boundary Value Problems* 2012 **2012**:118.