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Generalized metrics and Caristi's theorem

William A Kirk[1] and Naseer Shahzad[2*]

*Correspondence: nshahzad@kau.edu.sa
[2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

A 'generalized metric space' is a semimetric space which does not satisfy the triangle inequality, but which satisfies a weaker assumption called the quadrilateral inequality. After reviewing various related axioms, it is shown that Caristi's theorem holds in complete generalized metric spaces without further assumptions. This is noteworthy because Banach's fixed point theorem seems to require more than the quadrilateral inequality, and because standard proofs of Caristi's theorem require the triangle inequality.

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1 Introduction

In an effort to generalize Banach's contraction mapping principle, which holds in all complete metric spaces, to a broader class of spaces, Branciari [1] conceived of the notion to replace the triangle inequality with a weaker assumption he called the quadrilateral inequality. He called these spaces 'generalized metric spaces'. These spaces retain the fundamental notion of distance. However, as we shall see, the quadrilateral inequality, while useful in some sense, ignores the importance of such things as the continuity of the distance function, uniqueness of limits, *etc.* In fact it has been asserted (see, *e.g.*, [2]) that for an accurate generalization of Banach's fixed point theorem along the lines envisioned by Branciari, one needs the quadrilateral inequality in conjunction with the assumption that the space is Hausdorff.

We begin by discussing the relationship of Branciari's concept to the classical axioms of semimetric spaces. Then we show that Caristi's fixed point theorem holds within Branciari's framework without any additional assumptions. This is possibly surprising. All proofs of Caristi's theorem that the writers are aware of rely in some way on use of the triangle inequality. (In contrast, it has been noted that the proof of the first author's fundamental fixed point theorem for nonexpansive mappings does not require the triangle inequality; see [3].)

2 Semimetric spaces

In the absence of relevant examples, it is not clear whether Branciari's concept of weakening the triangle inequality will prove useful in analysis. However, the notion of assigning a 'distance' between each two points of an abstract set is fundamental in geometry. According to Blumenthal [4, p.31], this notion has its origins in the late nineteenth century in axiomatic studies of de Tilly [5]. In his 1928 treatise [6], Karl Menger used the term *halb-metrischer Raume, or semimetric space*, to describe the same concept. We begin by summarizing the results of Wilson's seminal paper [7] on semimetric spaces.

Definition 1 Let X be a set and let $D : X \times X \to \mathbb{R}$ be a mapping satisfying for each $a, b \in X$:

I. $d(a,b) \geq 0$, and $d(a,b) = 0 \Leftrightarrow a = b$;

II. $d(a,b) = d(b,a)$. Then the pair (X,d) is called a *semimetric space.*

In such a space, convergence of sequences is defined in the usual way: A sequence $\{x_n\} \subseteq X$ is said to *converge* to $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. Also, a sequence is said to be *Cauchy* (or d-Cauchy) if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $m, n \geq N \Rightarrow d(x_m, x_n) < \varepsilon$. The space (X,d) is said to be *complete* if every Cauchy sequence has a limit.

With such a broad definition of distance, three problems are immediately obvious: (i) *There is nothing to assure that limits are unique* (*thus the space need not be Hausdorff*); (ii) *a convergent sequence need not be a Cauchy sequence*; (iii) *the mapping* $d(a,\cdot) : X \to \mathbb{R}$ *need not even be continuous*. Therefore it is unlikely there could be an effective topological theory in such a setting.

With the introduction of the triangle inequality, problems (i), (ii), and (iii) are simultaneously eliminated.

VI. (Triangle inequality) *With X and d as in Definition* 1, *assume also that for each* $a, b, c \in X$,

$$d(a,b) \leq d(a,c) + d(c,b).$$

Definition 2 A pair (X,d) satisfying Axioms I, II, and VI is called a *metric space*.[a]

In his study [7], Wilson introduces three axioms in addition to I and II which are weaker than VI. These are the following.

III. *For each pair of* (*distinct*) *points* $a, b \in X$, *there is a number* $r_{a,b} > 0$ *such that for every* $c \in X$,

$$r_{a,b} \leq d(a,c) + d(c,b).$$

IV. *For each point* $a \in X$ *and each* $k > 0$, *there is a number* $r_{a,k} > 0$ *such that if* $b \in X$ satisfies $d(a,b) \geq k$, then *for every* $c \in X$,

$$r_{a,k} \leq d(a,c) + d(c,b).$$

V. *For each* $k > 0$, there is a number $r_k > 0$ such that if $a, b \in X$ satisfy $d(a,b) \geq k$, then for every $c \in X$,

$$r_k \leq d(a,c) + d(c,b).$$

Obviously, if Axiom V is strengthened to $r_k = k$, then the space becomes metric. Chittenden [8] has shown (using an equivalent definition) that a semimetric space satisfying Axiom V is always *homeomorphic* to a metric space.

Axiom III is equivalent to the assertion that there do not exist distinct points $a, b \in X$ and a sequence $\{c_n\} \subseteq X$ such that $d(a, c_n) + d(b, c_n) \to 0$ as $n \to \infty$. Thus, as Wilson observes, the following is self-evident.

Proposition 1 *In a semimetric space, Axiom* III *is equivalent to the assertion that limits are unique.*

For $r > 0$, let $U(p; r) = \{x \in X : d(x, p) < r\}$. Then Axiom III is also equivalent to the assertion that X is Hausdorff in the sense that given any two distinct points $a, b \in X$, there exist positive numbers r_a and r_b such that $U(a; r_a) \cap U(b; r_b) = \emptyset$. This suggests the presence of a topology.

Definition 3 Let (X, d) be a semimetric space. Then the distance function d is said to be *continuous* if for any sequences $\{p_n\}, \{q_n\} \subseteq X$, $\lim_n d(p_n, p) = 0$ and $\lim_n d(q_n, q) = 0 \Rightarrow \lim_n d(p_n, q_n) = d(p, q)$.

Remark Some writers call a space satisfying Axioms I and II a 'symmetric space' and reserve the term semimetric space for a symmetric space with a continuous distance function (see, *e.g.*, [9]; *cf.* also [10, 11]). Here we use Menger's original terminology.

A point p in a semimetric space X is said to be an *accumulation point* of a subset E of X if, given any $\varepsilon > 0$, $U(p; \varepsilon) \cap E \neq \emptyset$. A subset of a semimetric space is said to be *closed* if it contains each of its accumulation points. A subset of a semimetric space is said to be *open* if its complement is closed. With these definitions, if X is a semimetric space with a continuous distance function, then $U(p; r)$ is an open set for each $p \in X$ and $r > 0$ and, moreover, X is a Hausdorff topological space [4].

We now turn to the concept introduced by Branciari.

Definition 4 ([1]) Let X be a nonempty set, and let $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and all distinct points $u, v \in X$, each distinct from x and y:

(i) $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then X is called a *generalized metric space* (g.m.s.).

Proposition 2 *If* (X, d) *is a generalized metric space which satisfies Axiom* III, *then the distance function is continuous.*

Proof Suppose that $\{p_n\}, \{q_n\} \subseteq X$ satisfy $\lim_n d(p_n, p) = 0$ and $\lim_n d(q_n, q) = 0$, where $p \neq q$. Also assume that for n arbitrarily large, $p_n \neq p$ and $q_n \neq q$. In view of Axiom III, we may also assume that for n sufficiently large, $p_n \neq q_n$. Then

$$d(p, q) \leq d(p, p_n) + d(p_n, q_n) + d(q_n, q)$$

and

$$d(p_n, q_n) \leq d(p_n, p) + d(p, q) + d(q, q_n).$$

Together these inequalities imply

$$\liminf_n d(p_n, q_n) \geq d(p,q) \geq \limsup_n d(p_n, q_n).$$

Thus $\lim_n d(p_n, q_n) = d(p,q)$. □

Therefore if a generalized metric space satisfies Axiom III, it is a Hausdorff topological space. However, the following observation shows that the quadrilateral inequality implies a weaker but useful form of distance continuity. (This is a special case of Proposition 1 of [12].)

Proposition 3 *Suppose that* $\{q_n\}$ *is a Cauchy sequence in a generalized metric space* X *and suppose* $\lim_n d(q_n, q) = 0$. *Then* $\lim_n d(p, q_n) = d(p,q)$ *for all* $p \in X$. *In particular,* $\{q_n\}$ *does not converge to* p *if* $p \neq q$.

Proof We may assume that $p \neq q$. If $q_n = p$ for arbitrarily large n, it must be the case that $p = q$. So, we may also assume that $p \neq q_n$ for all n. Also, $q_n \neq q$ for infinitely many n; otherwise, the result is trivial. So, we may assume that $q_n \neq q_m \neq q$ and $q_n \neq q_m \neq p$ for all $m, n \in \mathbb{N}$ with $m \neq n$. Then, by the quadrilateral inequality,

$$d(p,q) \leq d(p, q_n) + d(q_n, q_{n+1}) + d(q_{n+1}, q)$$

and

$$d(p, q_n) \leq d(p,q) + d(q, q_{n+1}) + d(q_{n+1}, q_n).$$

Since $\{q_n\}$ is a Cauchy sequence, $\lim_n d(q_n, q_{n+1}) = 0$. Therefore, letting $n \to \infty$ in the above inequalities,

$$\limsup_n d(p, q_n) \leq d(p,q) \leq \liminf_n d(p, q_n).$$ □

We now come to Branciari's extension of Banach's contraction mapping theorem. Although in his proof Branciari makes the erroneous assertion that a g.m.s. is a Hausdorff topological space with a neighborhood basis given by

$$\mathbb{B} = \{B(x;r) : x \in S, r \in \mathbb{R}^+ \backslash 0\},$$

with the aid of Proposition 3, Branciari's proof carries over with only a minor change. The assertion in [2] that the space needs to be Hausdorff is superfluous, a fact first noted in [12]. See also the example in [13].

Theorem 1 ([1]) *Let* (X,d) *be a complete generalized metric space, and suppose that the mapping* $f : X \to X$ *satisfies* $d(f(x), f(y)) \leq \lambda d(x,y)$ *for all* $x, y \in X$ *and fixed* $\lambda \in (0,1)$. *Then* f *has a unique fixed point* x_0, *and* $\lim_n f^n(x) = x_0$ *for each* $x \in X$.

It is possible to prove this theorem by following the proof given by Branciari up to the point of showing that $\{f^n(x)\}$ is a Cauchy sequence for each $x \in X$. Then, by completeness of X, there exists $x_0 \in X$ such that $\lim_n f^n(x) = x_0$. But $\lim_n d(f^{n+1}(x), f(x_0)) \leq \lambda \lim_n d(f^n(x), x_0) = 0$, so $\lim_n f^{n+1}x = f(x_0)$. In view of Proposition 3, $f(x_0) = x_0$.

3 Caristi's theorem

We now turn to a proof of Caristi's theorem in a complete g.m.s.

Theorem 2 (*cf.* Caristi [14]) *Let (X,d) be a complete g.m.s. Let $f : X \to X$ be a mapping, and let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. Suppose that*

$$d\bigl(x, f(x)\bigr) \leq \varphi(x) - \varphi\bigl(f(x)\bigr), \quad x \in X.$$

Then f has a fixed point.

Typically, proofs of Caristi's theorem (and there have been many) involve assigning a partial order $\preceq$ to X by setting $x \preceq y \Leftrightarrow d(x,y) \leq \varphi(x) - \varphi(y)$, and then either using Zorn's lemma or the Brézis-Browder order principle (see Section 4). However, the triangle inequality is needed for these approaches in order to show that $(X, \preceq)$ is transitive. The proof we give below is based on Wong's modification [15] of Caristi's original transfinite induction argument [14]. (Recall that if M is a metric space, a mapping $\varphi : M \to \mathbb{R}$ is said to be *lower semicontinuous* (l.s.c.) if given $x \in X$ and a net $\{x_\alpha\}$ in M, the conditions $x_\alpha \to x$ and $\varphi(x_\alpha) \to r$ imply $\varphi(x) \leq r$.)

Proof of Theorem 2 Let $n \in \mathbb{N}$. Then

$$\begin{aligned} &\varphi(x) - \varphi\bigl(f^n(x)\bigr) \\ &\quad = \varphi(x) - \varphi\bigl(f(x)\bigr) + \varphi\bigl(f(x)\bigr) - \varphi\bigl(f^2(x)\bigr) + \cdots + \varphi\bigl(f^{n-1}(x)\bigr) - \varphi\bigl(f^n(x)\bigr) \\ &\quad \geq d\bigl(x, f(x)\bigr) + d\bigl(f(x), f^2(x)\bigr) + \cdots + d\bigl(f^{n-1}(x), f^n(x)\bigr). \end{aligned}$$

Hence

$$\sum_{i=0}^{n-1} d\bigl(f^i(x), f^{i+1}(x)\bigr) \leq \varphi(x) - \varphi\bigl(f^n(x)\bigr) \leq \varphi(x),$$

so

$$\sum_{i=0}^{\infty} d\bigl(f^i(x), f^{i+1}(x)\bigr) < \infty.$$

This proves that $\{f^n(x)\}$ is a Cauchy sequence. If f were continuous, one could immediately conclude that there exists $x_0 \in X$ such that $\lim_n f^n(x) = x_0 = f(x_0)$. (The quadrilateral inequality is not needed in this case, but it is necessary for Cauchy sequences to have unique limits.)

Let Γ denote the set of countable ordinals. For $\alpha, \beta \in \Gamma$, $\alpha < \beta$, we use $|[\alpha, \beta]|$ to denote the cardinality of the set

$$\{\mu : \alpha \leq \mu \leq \beta\}.$$

Now let $x_0 \in X$, let $\beta \in \Gamma$, and suppose that the net $\{x_\alpha\}_{\alpha<\beta}$ has been defined so that

(i) $x_{\alpha+1} = f(x_\alpha)$ for all $\alpha < \beta$;
(ii) if $\gamma < \beta$ is a limit ordinal, then the net $\{x_\alpha\}_{\alpha<\gamma}$ converges to x_γ;
(iii) if $0 \le \alpha \le \mu < \beta$ and $|[\alpha,\mu]| \ge 4$, then $d(x_\alpha, x_\mu) \le \varphi(x_\alpha) - \varphi(x_\mu)$.

If $\beta = \gamma + 1$, define $x_\beta = f(x_\gamma)$. If $\alpha < \beta$ and $|[\alpha,\beta]| \ge 4$, then $|[\alpha+1,\gamma]| \ge 4$ and by the quadrilateral inequality,

$$\begin{aligned} d(x_\alpha, x_\beta) &\le d(x_\alpha, x_{\alpha+1}) + d(x_{\alpha+1}, x_\gamma) + d(x_\gamma, x_\beta) \\ &= d(x_\alpha, x_{\alpha+1}) + d(x_{\alpha+1}, x_\gamma) + d(x_\gamma, x_{\gamma+1}). \end{aligned}$$

Thus if $|[\alpha+1,\gamma]| \ge 4$, by the inductive assumption,

$$d(x_\alpha, x_\beta) \le \varphi(x_\alpha) - \varphi(x_\beta).$$

Otherwise, $|[\alpha+1,\gamma]| \le 3$. If $\gamma = \alpha+1$, $|[\alpha,\beta]| = |\{\alpha,\alpha+1,\alpha+2\}| = 3 < 4$. If $\gamma = \alpha+2$, then $\beta = \alpha+3$ and we have

$$\begin{aligned} d(x_\alpha, x_\beta) &= d(x_\alpha, x_{\alpha+3}) \\ &\le d(x_\alpha, x_{\alpha+1}) + d(x_{\alpha+1}, x_{\alpha+2}) + d(x_{\alpha+2}, x_{\alpha+3}) \\ &\le \varphi(x_\alpha) - \varphi(x_\beta). \end{aligned}$$

Finally, if $\gamma = \alpha+3$, we can write (here order 3 is needed!)

$$d(x_\alpha, x_\beta) \le d(x_\alpha, x_{\alpha+1}) + d(x_{\alpha+1}, x_{\alpha+2}) + d(x_{\alpha+2}, x_{\alpha+3}) + d(x_{\alpha+3}, x_{\alpha+4}).$$

Now suppose β is a limit ordinal. We claim that $\{x_\alpha\}_{\alpha<\beta}$ is a Cauchy net. If not, there exists $\varepsilon > 0$ and a strictly increasing sequence $\{\alpha_n\}$ in $(0,\beta)$ such that $|[\alpha_n,\alpha_{n+1}]| \ge 4$ and $d(x_{\alpha_n}, x_{\alpha_{n+1}}) \ge \varepsilon$. This leads to the contradiction

$$\begin{aligned} \infty &= \sum_{n=1}^{\infty} d(x_{\alpha_n}, x_{\alpha_{n+1}}) \\ &\le \sum_{n=1}^{\infty} \bigl(\varphi(x_{\alpha_n}) - \varphi(x_{\alpha_{n+1}})\bigr) \\ &\le \varphi(x_{\alpha_1}). \end{aligned}$$

Therefore $\{x_\alpha\}_{\alpha<\beta}$ is a Cauchy net and, since X is complete, it is possible to take $x_\beta = \lim_{\alpha<\beta} x_\alpha$.

Since β is a limit ordinal, the cardinality of $[\alpha,\beta]$ is infinite for all $\alpha < \beta$. Consequently, since φ is lower semicontinuous,

$$\begin{aligned} d(x_\alpha, x_\beta) &= \lim_{\gamma<\beta} d(x_\alpha, x_\gamma) \\ &\le \liminf_{\gamma<\beta} \bigl(\varphi(x_\alpha) - \varphi(x_\gamma)\bigr) \\ &= \varphi(x_\alpha) - \limsup_{\gamma<\beta} \varphi(x_\gamma) \\ &\le \varphi(x_\alpha) - \varphi(x_\beta). \end{aligned}$$

Therefore a net $\{x_\alpha\}$ has been defined satisfying (i), (ii), and (iii) for all $\alpha \in \Gamma$. Let Γ' denote the set of limit ordinals in Γ. If f has no fixed point, the net $\{\varphi(x_\alpha)\}_{\alpha\in\Gamma'}$ is strictly decreasing. This is a contradiction because Γ' is uncountable and any strictly decreasing net of real numbers must be countable. □

4 Another approach

We now examine an easy proof of Caristi's original theorem based on Zorn's lemma. (A more constructive proof which uses the Brézis-Browder order principle is given in [16].)

Theorem 3 *Let (X,d) be a complete metric space. Let $f : X \to X$ be a mapping, and let $\varphi : X \to \mathbb{R}^+$ be a lower semicontinuous function. Suppose that*

$$d\bigl(x,f(x)\bigr) \le \varphi(x) - \varphi\bigl(f(x)\bigr) \quad x \in X. \tag{C}$$

Then f has a fixed point.

Proof Introduce the Brøndsted partial order on X by setting $x \preceq y \Leftrightarrow d(x,y) \le \varphi(x) - \varphi(y)$. Let I be a totally ordered set, and let $\{x_\gamma\}_{\gamma\in I}$ be a chain in $(X,\preceq)$. Then $\alpha \le \beta \Rightarrow x_\alpha \preceq x_\beta \Leftrightarrow d(x_\alpha,x_\beta) \le \varphi(x_\alpha) - \varphi(x_\beta)$. Therefore $\{\varphi(x_\gamma)\}_{\gamma\in I}$ is decreasing. Since φ is bounded below, $\lim_\gamma \varphi(x_\gamma) = r$. This implies $\lim_{\alpha,\beta} d(x_\alpha,x_\beta) = 0$; hence $\{x_\gamma\}_{\gamma\in I}$ is a Cauchy net. Since X is complete, there exists $x \in X$ such that $\lim_\gamma x_\gamma = x$. Thus for $\alpha \in I$,

$$\begin{aligned} d(x_\alpha,x) &= \lim_\gamma d(x_\alpha,x_\gamma) \\ &\le \lim_\gamma \bigl(\varphi(x_\alpha) - \varphi(x_\gamma)\bigr) \\ &= \varphi(x_\alpha) - r \\ &\le \varphi(x_\alpha) - \varphi(x). \end{aligned}$$

Therefore $x_\alpha \preceq x$ for each $\alpha \in I$, so x is an upper bound for the chain $\{\varphi(x_\gamma)\}_{\gamma\in I}$. By Zorn's lemma, $(X,\preceq)$ has a maximal element $\bar{x}$. But condition (C) implies $\bar{x} \preceq f(\bar{x})$, so it must be the case that $\bar{x} = f(\bar{x})$. □

The above argument fails in the setting of Theorem 2 because it is not possible to show that $(X,\preceq)$ is transitive in a g.m.s. In a metric space, transitivity follows directly from the triangle inequality. A way to circumvent this difficulty is to only consider points of X that are limits of nontrivial Cauchy sequences. The proof of Theorem 2 implies that nontrivial Cauchy sequences exist. So, let

$$X_C = \{x \in X : x \text{ is the limit of an infinite Cauchy sequence in } X\}$$

and define

$$x \preceq y \quad \Leftrightarrow \quad x,y \in X_C \quad \text{and} \quad \varphi(x) \le \varphi(y).$$

Now let x, y, and z be three distinct points in (X_C,d), and let $\{z_n\}$ be a Cauchy sequence converging to z. Then, by the quadrilateral inequality,

$$d(x,y) \le d(x,z_n) + d(z_n,z_{n+1}) + d(z_{n+1},y).$$

Letting $n \to \infty$ and applying Proposition 3, we see that $d(x,y) \leq d(x,z) + d(z,y)$. Therefore (X_C, d) is a metric space. In the proof of Theorem 3 $\bar{x} \in X_C$. To show that $\bar{x} \preceq f(\bar{x})$, it is necessary to show that $f(\bar{x}) \in X_C$. Assume that $\bar{x} \neq f(\bar{x})$. Then $\{f^n(\bar{x})\}$ is a Cauchy sequence. So, let $x_\infty = \lim_n f^n(\bar{x})$.

By induction,

$$d\bigl(\bar{x}, f^{2n+1}(\bar{x})\bigr) \leq \varphi(\bar{x}) - \varphi\bigl(f^{2n+1}(\bar{x})\bigr).$$

Then

$$\begin{aligned} d(\bar{x}, x_\infty) &= \lim_n d\bigl(\bar{x}, f^{2n+1}(\bar{x})\bigr) \\ &\leq \lim_n \bigl(\varphi(\bar{x}) - \varphi\bigl(f^{2n+1}(\bar{x})\bigr)\bigr) \\ &= \varphi(\bar{x}) - \lim_n \varphi\bigl(f^{2n+1}(\bar{x})\bigr) \\ &\leq \varphi(\bar{x}) - \varphi(x_\infty). \end{aligned}$$

This leads to the contradiction $\bar{x} \preceq x_\infty$. The other alternative is that there exists a periodic point. This is impossible because

$$f^n(x) \neq f^{n+1}(x) \quad \Rightarrow \quad \varphi\bigl(f^{n+1}(x)\bigr) < \varphi\bigl(f^n(x)\bigr).$$

Remark In view of Proposition 3, it seems reasonable to introduce the following definition.

Definition 5 A point p in a generalized metric space X is said to be an *accumulation point* of a subset E of X if some infinite Cauchy sequence in E converges to p. A set E in X is said to be *closed* if it contains all of its accumulation points.

Observe that with convergence defined as above, $\lim_n x_n = x \Leftrightarrow \{x_n\}$ is a Cauchy sequence and $\lim_n d(x_n, x) = 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1] Department of Mathematics, University of Iowa, Iowa City, IA 52242, USA. [2] Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia.

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Endnote
[a] The term 'metric space' for spaces satisfying Axioms I, II, and VI is apparently due to Hausdorff [17].

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