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Schur quadratic concavity of the elliptic Neuman mean and its application

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Abstract

For $x, y > 0$ and $k \in [0, 1]$, we prove that the elliptic Neuman mean $N_k(x, y)$ is strictly Schur quadratically concave on $(0, \infty) \times (0, \infty)$ if and only if $k \in [\sqrt{2}/2, 1]$. As an application, the bounds for elliptic Neuman mean $N_k(x, y)$ in terms of the quadratic mean $Q(x, y) = \sqrt{(x^2 + y^2)/2}$ are presented.

MSC: 26B25; 26E60

Keywords: elliptic Neuman mean; Schur quadratically concave; quadratic mean

1 Introduction

Let $(x, y) \in (0, \infty) \times (0, \infty)$ and $k \in [0, 1]$. Then the elliptic Neuman mean $N_k(x, y)$, see [1], is defined by

$$N_k(x,y) = \begin{cases} \frac{\sqrt{y^2-x^2}}{cn^{-1}(x/y,k)}, & x < y, \\ x, & x = y, \\ \frac{\sqrt{x^2-y^2}}{nc^{-1}(x/y,k)}, & y < x, \end{cases} \tag{1.1}$$

where $cn^{-1}(x,k) = \int_x^1 \frac{du}{\sqrt{(1-u^2)(k'^2+k^2u^2)}}$ and $nc^{-1}(x,k) = \int_1^x \frac{du}{\sqrt{(u^2-1)(k^2+k'^2u^2)}}$ are the inverse functions of Jacobian elliptic functions cn and nc, see [2, 3], respectively, and $k' = \sqrt{1-k^2}$. In particular, $cn^{-1}(0,k) = \mathcal{K}(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2\sin^2 t}}$ is the well-known complete elliptic integral of the first kind.

In [1] Neuman proved that $N_k(x, y)$ is symmetric and homogeneous on $(0, \infty) \times (0, \infty)$, and strictly decreasing with respect to $k \in [0, 1]$ for fixed $(x, y) \in (0, \infty) \times (0, \infty)$ with $x \neq y$. In this context let us note that if a mean is homogeneous, then the order of its homogeneity must be 1; see [4].

Let us recall the notion of Schur quadratic convexity (concavity) [5–7] for a real-valued function on $(0, \infty) \times (0, \infty)$.

A real-valued function $f : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is said to be strictly Schur quadratically convex on $(0, \infty) \times (0, \infty)$ if $f(x_1, x_2) < f(y_1, y_2)$ for each pair of 2-tuples $(x_1, x_2), (y_1, y_2) \in (0, \infty) \times (0, \infty)$ with $\max\{x_1, x_2\} < \max\{y_1, y_2\}$ and $x_1^2 + x_2^2 = y_1^2 + y_2^2$. f is said to be strictly Schur quadratically concave if $-f$ is strictly Schur quadratically convex.

The main purpose of this paper is to present the range of k such that the elliptic Neuman mean $N_k(x, y)$ is strictly Schur quadratically concave on $(0, \infty) \times (0, \infty)$. As an application, an inequality between the elliptic Neuman mean $N_k(x,y)$ and the quadratic mean $Q(x,y) = \sqrt{(x^2+y^2)/2}$ is also given.

2 Two lemmas

In order to prove our main results we need two lemmas, which we present in this section.

Lemma 2.1 (See [5, Corollary 2.1], [6, Corollary 1], [7, Corollary 1]) *Suppose that* $f : (0,\infty)\times(0,\infty)\to(0,\infty)$ *is a continuous symmetric function. If* f *is differentiable in* $(0,\infty)\times(0,\infty)$, *then* f *is strictly Schur quadratically convex on* $(0,\infty)\times(0,\infty)$ *if and only if*

$$(x-y)\left(y\frac{\partial f(x,y)}{\partial x}-x\frac{\partial f(x,y)}{\partial y}\right)>0 \tag{2.1}$$

for all $x,y\in(0,\infty)$ *with* $x\neq y$, *and* f *is strictly Schur quadratically concave on* $(0,\infty)\times(0,\infty)$ *if and only if inequality* (2.1) *is reversed.*

Lemma 2.2 *Let* $t\in(0,1)$, $k\in[0,1]$, *and*

$$f_k(t)=cn^{-1}(t,k)-\frac{(1+t^2)\sqrt{1-t^2}}{2t\sqrt{1-k^2+k^2t^2}}. \tag{2.2}$$

Then $f_k(t)<0$ *for all* $t\in(0,1)$ *if and only if* $\sqrt{2}/2\le k\le 1$, *and there exists* $\lambda=\lambda(k)\in(0,1)$ *such that* $f_k(t)<0$ *for* $t\in(0,\lambda)$ *and* $f_k(t)>0$ *for* $t\in(\lambda,1)$ *if* $k\in[0,\sqrt{2}/2)$.

Proof We distinguish for the proof two cases.

Case 1. $k=1$. Then from (2.2) one has

$$\begin{aligned} f_1(t)&=cn^{-1}(t,1)-\frac{(1+t^2)\sqrt{1-t^2}}{2t^2}\\ &=\cosh^{-1}\left(\frac{1}{t}\right)-\frac{(1+t^2)\sqrt{1-t^2}}{2t^2}\\ &=\log\bigl(1+\sqrt{1-t^2}\bigr)-\log t-\frac{(1+t^2)\sqrt{1-t^2}}{2t^2},\end{aligned}$$

$$f_1\bigl(1^-\bigr)=0, \tag{2.3}$$

$$f'(t)=\frac{(1-t^2)(2-t^2)}{2t^3\sqrt{1-t^2}}>0 \tag{2.4}$$

for all $t\in(0,1)$. (Here and in the sequel, $f(t^-)$ and $f(t^+)$ denote, respectively, the left and right limit of f at t.)

From (2.3) and (2.4) we clearly see that $f_1(t)<0$ for all $t\in(0,1)$.

Case 2. $0\le k<1$. Then (2.2) leads to

$$f_k\bigl(0^+\bigr)=-\infty, \tag{2.5}$$

$$f_k\bigl(1^-\bigr)=0, \tag{2.6}$$

$$\begin{aligned} f_k'(t) &= -\frac{1}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}} - \frac{-k^2t^6+(3k^2-2)t^4+(1-3k^2)t^2+k^2-1}{2t^2\sqrt{1-t^2}(1-k^2+k^2t^2)^{3/2}} \\ &= -\frac{-k^2t^6+(5k^2-2)t^4+(3-5k^2)t^2+k^2-1}{2t^2\sqrt{1-t^2}(1-k^2+k^2t^2)^{3/2}} \\ &= -\frac{\sqrt{1-t^2}[k^2t^4+(2-4k^2)t^2+k^2-1]}{2t^2(1-k^2+k^2t^2)^{3/2}}. \end{aligned} \tag{2.7}$$

Let

$$g_k(t) = k^2t^4 + \left(2-4k^2\right)t^2 + k^2 - 1. \tag{2.8}$$

Then simple computations lead to

$$g_k(0) = k^2 - 1 < 0, \tag{2.9}$$

$$g_k(1) = 2\left(\frac{\sqrt{2}}{2} - k\right)\left(\frac{\sqrt{2}}{2} + k\right), \tag{2.10}$$

$$g_k'(t) = 4k^2t^3 + 4\left(1-2k^2\right)t, \tag{2.11}$$

$$g_k'(0) = 0, \tag{2.12}$$

$$g_k'(1) = 4(1-k)(1+k) > 0, \tag{2.13}$$

$$g_k''(t) = 12k^2t^2 + 4\left(1-2k^2\right), \tag{2.14}$$

$$g_k''(0) = 8\left(\frac{\sqrt{2}}{2} - k\right)\left(\frac{\sqrt{2}}{2} + k\right), \tag{2.15}$$

$$g_k''(1) = 4\left(1+k^2\right) > 0. \tag{2.16}$$

We distinguish for the proof three subcases.

Subcase 2.1. $k = \sqrt{2}/2$. Then (2.7) leads to the conclusion that

$$f_{\sqrt{2}/2}'(t) = \frac{\sqrt{2}\sqrt{1-t^2}(1-t^4)}{2t^2(1+t^2)^{3/2}} > 0 \tag{2.17}$$

for all $t \in (0,1)$.

Therefore, $f_{\sqrt{2}/2}(t) < 0$ for all $t \in (0,1)$ follows from (2.6) and (2.17).

Subcase 2.2. $\sqrt{2}/2 < k < 1$. Then (2.10) and (2.15) lead to

$$g_k(1) < 0, \tag{2.18}$$

$$g_k''(0) < 0. \tag{2.19}$$

It follows from (2.14) that g_k'' is strictly increasing on $(0,1)$, then (2.16) and (2.19) lead to the conclusion that there exists $\lambda_0 \in (0,1)$ such that g_k' is strictly decreasing on $(0,\lambda_0]$ and strictly increasing on $[\lambda_0,1)$.

From (2.12) and (2.13) together with the piecewise monotonicity of g_k' we clearly see that there exists $\lambda_1 \in (\lambda_0,1)$ such that g_k is strictly decreasing on $(0,\lambda_1]$ and strictly increasing on $[\lambda_1, 1)$. Then (2.9) and (2.18) lead to the conclusion that

$$g_k(t) < 0 \tag{2.20}$$

for all $t \in (0, 1)$.

It follows from (2.7) and (2.8) together with (2.20) that f_k is strictly increasing on $(0, 1)$. Therefore, $f_k(t) < 0$ for all $t \in (0, 1)$ follows easily from (2.6) and the monotonicity of f_k.

Subcase 2.3. $0 \leq k < \sqrt{2}/2$. Then from (2.10) and (2.11) we know that g_k is strictly increasing on $(0, 1)$ and

$$g_k(1) > 0. \tag{2.21}$$

It follows from (2.9) and (2.21) together with the monotonicity of g_k that there exists $\mu_0 \in (0, 1)$ such that $g_k(t) > 0$ for $t \in (0, \mu_0)$ and $g_k(t) < 0$ for $t \in (\mu_0, 1)$. Then (2.7) and (2.8) lead to the conclusion that f_k is strictly increasing on $(0, \mu_0]$ and strictly decreasing on $[\mu_0, 1)$. Therefore, there exists $\lambda = \lambda(k) \in (0, \mu_0) \subset (0, 1)$ such that $f_k(t) < 0$ for $t \in (0, \lambda)$ and $f_k(t) > 0$ for $t \in (\lambda, 1)$ follows from (2.5) and (2.6) together with the piecewise monotonicity of f_k. □

3 Main results

Theorem 3.1 *The elliptic Neuman mean $N_k(x, y)$ is strictly Schur quadratically concave on $(0, \infty) \times (0, \infty)$ if and only if $k \in [\sqrt{2}/2, 1]$, and $N_k(x, y)$ is not Schur quadratically convex on $(0, \infty) \times (0, \infty)$ if $k \in [0, \sqrt{2}/2)$.*

Proof Since $N_k(x, y)$ is symmetric and homogeneous of degree 1, without loss of generality, we assume that $x < y$. Let $t = x/y \in (0, 1)$, then

$$N_k(x, y) = yN_k(t, 1), \qquad \frac{\partial t}{\partial y} = -\frac{x}{y^2}, \qquad \frac{\partial t}{\partial x} = \frac{1}{y}, \tag{3.1}$$

$$\frac{\partial N_k(x, y)}{\partial y} = N_k(t, 1) - t\frac{dN_k(t, 1)}{dt}, \qquad \frac{\partial N_k(x, y)}{\partial x} = \frac{dN_k(t, 1)}{dt}. \tag{3.2}$$

Note that

$$\frac{dN_k(t, 1)}{dt} = -\frac{t}{\sqrt{1 - t^2}cn^{-1}(t, k)} + \frac{1}{(cn^{-1}(t, k))^2\sqrt{1 - k^2 + k^2t^2}}. \tag{3.3}$$

It follows from (1.1) and (3.2) together with (3.3) that

$$\begin{aligned}
&(y - x)\left(x\frac{\partial N_k(x, y)}{\partial y} - y\frac{\partial N_k(x, y)}{\partial x}\right) \\
&\quad = x(y - x)\left[N_k(t, 1) - \left(t + \frac{1}{t}\right)\frac{dN_k(t, 1)}{dt}\right] \\
&\quad = \frac{2x(y - x)}{\sqrt{1 - t^2}(cn^{-1}(t, k))^2}\left(cn^{-1}(t, k) - \frac{(1 + t^2)\sqrt{1 - t^2}}{2t\sqrt{1 - k^2 + k^2t^2}}\right) \\
&\quad = \frac{2x(y - x)}{\sqrt{1 - t^2}(cn^{-1}(t, k))^2}f_k(t),
\end{aligned} \tag{3.4}$$

where $f_k(t)$ is defined as in Lemma 2.2.

Therefore, Theorem 3.1 follows easily from Lemmas 2.1 and 2.2 together with (3.4). □

Theorem 3.2 *The elliptic Neuman mean $N_k(x,y)$ is strictly Schur quadratically concave (or convex, respectively) on $(0,\infty)\times(0,\infty)$ if and only if the function $N_k(t,1)/Q(t,1)$ is strictly increasing (or decreasing, respectively) in $(0,1)$, where $Q(x,y)=\sqrt{(x^2+y^2)/2}$ is the quadratic mean of x and y.*

Proof Without loss of generality, we assume that $x<y$. Let $t=x/y\in(0,1)$, then from (3.1) and (3.2) together with (3.4) we get

$$\begin{aligned}\frac{d(N_k(t,1)/Q(t,1))}{dt} &= -\frac{\sqrt{2}t}{(t^2+1)^{3/2}}\left(N_k(t,1)-\left(t+\frac{1}{t}\right)\frac{dN_k(t,1)}{dt}\right)\\ &= -\frac{\sqrt{2}}{y(y-x)(t^2+1)^{3/2}}(y-x)\left(x\frac{\partial N_k(x,y)}{\partial y}-y\frac{\partial N_k(x,y)}{\partial x}\right).\end{aligned}\tag{3.5}$$

Therefore, Theorem 3.2 follows easily from Lemma 2.1 and (3.5). □

Theorem 3.3 *The inequalities*

$$Q(x,y)>N_{\sqrt{2}/2}(x,y)\tag{3.6}$$

and

$$Q(x,y)<\frac{\sqrt{2}\mathcal{K}(k)}{2}N_k(x,y)\tag{3.7}$$

hold for all $x,y>0$ with $x\neq y$, and $k\in[0,1]$, and $N_{\sqrt{2}/2}(x,y)$ is the best possible lower elliptic Neuman mean bound for the quadratic mean $Q(x,y)$.

Proof Without loss of generality, we assume that $y>x>0$. Let $t=x/y\in(0,1)$ and $L_k(t)=N_k(t,1)/Q(t,1)$. Then

$$L_k(t)=\frac{N_k(x,y)}{Q(x,y)},\tag{3.8}$$

$$L_k(0)=\frac{\sqrt{2}}{\mathcal{K}(k)},\tag{3.9}$$

$$L_k(1)=1.\tag{3.10}$$

We distinguish for the proof two cases.

Case 1. $k\in[\sqrt{2}/2,1]$. Then from Theorems 3.1 and 3.2 we clearly see that L_k is strictly increasing on $(0,1)$. Then (3.8)-(3.10) lead to the conclusion that

$$\frac{\sqrt{2}}{\mathcal{K}(k)}<\frac{N_k(x,y)}{Q(x,y)}<1.\tag{3.11}$$

In particular, for $k=\sqrt{2}/2$ we have

$$\frac{N_{\sqrt{2}/2}(x,y)}{Q(x,y)}<1.\tag{3.12}$$

Therefore, inequalities (3.6) and (3.7) follow from (3.11) and (3.12).

Case 2. $k \in [0, \sqrt{2}/2)$. Then (3.4) and (3.5) together with the Subcase 2.3 in Lemma 2.2 lead to the conclusion that there exists $\lambda = \lambda(k) \in (0,1)$ such that $L_k'(t) > 0$ for $t \in (0,\lambda)$ and $L_k'(t) < 0$ for $t \in (\lambda, 1)$, hence L_k is strictly increasing on $(0,\lambda]$ and strictly decreasing on $[\lambda, 1)$. Therefore, $N_k(x,y) > Q(x,y)$ for all $x, y > 0$ with $x/y \in (\lambda, 1)$ follows from (3.8) and (3.10) together with the monotonicity of L_k on $[\lambda, 1)$, and the optimality of inequality (3.6) follows.

Note that

$$L_k(0) = \frac{\sqrt{2}}{\mathcal{K}(k)} < \frac{\sqrt{2}}{\mathcal{K}(0)} = \frac{2\sqrt{2}}{\pi} = 0.90031\cdots < 1. \tag{3.13}$$

From (3.8), (3.9), (3.13), and the piecewise monotonicity of L_k we clearly see that

$$\frac{N_k(x,y)}{Q(x,y)} = L_k(t) > L_k(0) = \frac{\sqrt{2}}{\mathcal{K}(k)}. \tag{3.14}$$

Therefore, inequality (3.7) follows from (3.14). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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