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Hausdorff measure of noncompactness in some sequence spaces of a triple band matrix

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Abstract
The sequence spaces $c_0(B)$, $\ell_\infty(B)$ have recently been introduced by Sömez (Comput. Math. Appl. 62:641-650, 2011). In this paper, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(B)$, $\ell_\infty(B)$ and by using the Hausdorff measure of noncompactness, we characterize some classes of compact operators on these spaces.

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Keywords: Hausdorff measure of noncompactness; triple band matrix; sequence space

1 Introduction

By w, we shall denote the space of all real- or complex-valued sequences. Any vector subspace of w is called a sequence space. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also, by ℓ_1 and ℓ_p $(1 < p < \infty)$, we denote the spaces of all absolutely and p-absolutely convergent series, respectively. Further, we shall write bs, cs for the spaces of all sequences associated with bounded and convergent series.

The β-duals of a subset X of w are defined by

$$X^\beta = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}.$$

Let μ and γ be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from μ into γ, and we denote it by writing $A : \mu \to \gamma$ if for every sequence $x = (x_k) \in \mu$, the sequence $Ax = (Ax)_n$, the A-transform of x is in γ, where

$$(Ax)_n = \sum_k a_{nk} x_k \quad (n \in \mathbb{N}). \tag{1.1}$$

The notation $(\mu : \gamma)$ denotes the class of all matrices A such that $A : \mu \to \gamma$. Thus, $A \in (\mu : \gamma)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \gamma$ for all $x \in \mu$. The matrix domain μ_A of an infinite matrix A in a sequence space μ is defined by

$$\mu_A = \{x = (x_k) \in \omega : Ax \in \mu\}.$$

The theory of BK-spaces is the most powerful tool in the characterization of the matrix transformation between sequence spaces. A sequence space X is called a BK-space if it is a Banach space with the maps $p_i : \mu \longrightarrow \mathbb{C}$ defined by $p_i(x) = x_i$ being continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The sequence spaces c_0, c, ℓ_∞ and ℓ_1 are BK-spaces with the usual sup-norm defined by $\|x\|_\infty = \sup_k |x_k|$ and $\|x\|_{\ell_1} = \sum_k |x_k|$ [1].

2 The sequence spaces $c_0(B)$ and $\ell_\infty(B)$

Let r, s and t be non-zero real numbers, and define the triple band matrix $B(r,s,t) = \{b_{nk}(r,s,t)\}$

$$b_{nk} = \begin{cases} r, & k = n, \\ s, & k = n-1, \\ t, & k = n-2, \\ 0, & \text{otherwise.} \end{cases}$$

Recently, Sömez [2] introduced the sequence spaces $c_0(B)$ and $\ell_\infty(B)$ as the matrix domain of the triangle $B(r,s,t)$ in the spaces c_0 and ℓ_∞, respectively. It obvious that $c_0(B)$ and $\ell_\infty(B)$ are BK-spaces with the same norm by

$$\|x\|_{\ell_\infty(B(r,s,t))} = \left\|B(r,s,t)(x)\right\|_{\ell_\infty} = \sup_n \left|B_n(r,s,t)(x)\right|. \tag{2.1}$$

Throughout, for any sequence $x = (x_k) \in w$, we define the sequence $y = (y_k)$ which will be frequently used, as the $B(r,s,t)$-transform of a sequence $x = (x_k)$, *i.e.*,

$$y_k = rx_k + sx_{k-1} + tx_{k-2}. \tag{2.2}$$

Since the spaces $\lambda_{B(r,s,t)}$ and λ are norm isomorphic, one can easily observe that $x = (x_k) \in \lambda_{B(r,s,t)}$ if and only if $y = (y_k) \in \lambda$, where the sequences $x = (x_k)$ and $y = (y_k)$ are connected with relation (2.2); furthermore, $\|x\|_{\ell_\infty(B(r,s,t))} = \|y\|_{\ell_\infty}$, where λ is any of the sequences c_0 or ℓ_∞.

3 Compactness by the Hausdorff measure of noncompactness

In the present paper, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(B)$ and $\ell_\infty(B)$. Further, by using the Hausdorff measure of noncompactness, we characterize some classes of compact operators on these spaces. It is quite natural to find condition for a matrix map between BK-spaces to define a compact operator since a matrix transformation between BK-spaces is continuous. This can be achieved by applying the Hausdorff measure of noncompactness. Recently, several authors characterized classes of compact operators given by infinite matrices on some sequence spaces by using this method. For example, in [3, 4], Mursaleen and Noman, Malkowsky and Rakočević [5], Djolović and Malkowsky [6] and Kara and Başarır [7, 8] established some identities or estimates for the operator norms and the Hausdorff measure of noncompactness of the linear operator given by infinite matrices that map an arbitrary BK-space or the matrix domain of triangles in an arbitrary BK-space. Further, they characterized some classes of compact operators on these spaces by using the Hausdorff measure of noncompactness. Now, we give some related definitions, notation and preliminary result.

Let X and Y be Banach spaces. Then we write $\mathcal{B}(X,Y)$ for the set of all bounded (continuous) linear operators $L : X \longrightarrow Y$, which is a Banach space with the operator norm given by $\|L\| = \sup_{x\in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X,Y)$, where S_X denotes the unit sphere in X, the sequence $(L(x_n))$ has a subsequence which converges in Y. By $\mathcal{C}(X,Y)$, we denote the class of all compact operators in $\mathcal{B}(X,Y)$. An operator $L \in \mathcal{B}(X,Y)$ is said to be of finite rank if $\dim R(L) < \infty$, where $R(L)$ denotes the range of L. An operator of finite rank is clearly compact.

If $(\|\cdot\|, X)$ is a normed sequence space, then we write

$$\|a\|_X^* \sup_{x\in S_X} \sum_{k=n}^{\infty} |a_k x_k| \tag{3.1}$$

for $a \in w$ provided the expression on the right-hand side exists and is finite, which is the case whenever X is a BK-space and $a \in X^\beta$ [9]. Let S and M be subsets of a metric space (X,d) and $\varepsilon > 0$. Then S is called an ε-net of M in X if for every $x \in M$ there exists $s \in S$ such that $d(x,s) < \varepsilon$. Further the set S is finite, then the ε-net S of M is called a finite ε-net of M, and we say that M has a finite ε-net in X. A subset of a metric space is said to be totally bounded if it has a finite ε-net for every $\varepsilon > 0$. By $\mathcal{M}_X$ we denote the collection of all bounded subsets of a metric space (X,d). If $Q \in \mathcal{M}_X$, then the Hausdorff measure of noncompactness of the set Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\}.$$

The function $\chi : \mathcal{M}_X \longrightarrow [0,\infty)$ is called the Hausdorff measure of noncompactness [9, p.387].

The basic properties of the Hausdorff measure of noncompactness can be found in [10, Lemma 2]; for example, if Q, Q_2 and Q are bounded subsets of a metric space (X,d), then

$$\chi(Q) = 0 \quad \text{if and only if} \quad Q \text{ is totally bounded,}$$
$$Q_1 \subset Q_2 \quad \text{implies} \quad \chi(Q_1) \le \chi(Q_2).$$

Further, if X is a normed space, then the function χ has some additional properties connected with the linear structure, that is,

$$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2),$$
$$\chi(\alpha Q) = |\alpha|\chi(Q) \quad \text{for all } \alpha \in \mathbb{C}.$$

We shall need the following known result for our investigation.

Lemma 3.1 [10, Lemma 15(a)] *Let $\varphi \supset X$ and Y be a BK-space. Then we also have $(X,Y) \subset \mathcal{B}(X,Y)$, that is, every matrix $A \in (X,Y)$ defines an operator $L_A \in \mathcal{B}(X,Y)$ by $L_A(x) = Ax$ for all $x \in X$.*

Lemma 3.2 [11, Theorem 3.8] *Let T be a triangle. Then we have*

(a) *For arbitrary subsets X and Y of w, $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*
(b) *Further, if X and Y are BK-spaces and $A \in (X, Y_T)$, then $\|L_A\| = \|L_B\|$.*

Lemma 3.3 [12, Lemma 5.2] *Let $\varphi \supset X$ be a BK-space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X, Y)$, then we have*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n |A_n|_X^* < \infty.$$

Lemma 3.4 [11, Theorem 1.29] *Let X denote any of the spaces c, c_0 or ℓ_∞. If $X^\beta = \ell_1$ and $\|a\|_X^* = \|a\|_{\ell_1}$ for all $a \in \ell_1$.*

Lemma 3.5 *Let X denote any of the spaces $c_0(B)$ and $\ell_\infty(B)$. If $a = (a_k) \in X^\beta$, then we have $\widehat{a} = (\widehat{a}_k) \in \ell_1$ and the equality*

$$\sum_{k=0}^{\infty} a_k x_k = \sum_{k=0}^{\infty} \widehat{a}_k y_k \tag{3.2}$$

holds for every $x = (x_k) \in X$, where $y = B(r,s,t)(x)$ is the associated sequence defined by (2.2) *and*

$$\widehat{a}_k = \sum_{j=k}^{n} \sum_{i=0}^{k-j} \left(\frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{j-k-i} \left(\frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^{i} \frac{a_j}{r}.$$

Theorem 3.6 *Let X denote any of the spaces $c_0(B)$ or $\ell_\infty(B)$. Then we have*

$$\|a\|_X = \|\widehat{a}\|_{\ell_1} = \sum_{k=0}^{\infty} |\widehat{a}_k| < \infty$$

for all $a = (a_k) \in X^\beta$, where $\widehat{a} = (\widehat{a}_k)$ is as in Lemma 3.5.

Proof Let Y be the respective one of the spaces c_0 or ℓ_∞, and take any $a = (a_k) \in X^\beta$. Then we have by Lemma 3.5 that $\widehat{a} = (\widehat{a}_k) \in \ell_1$ and equality (3.2) holds for all sequences $x = (x_k) \in X$ and $y = (y_k) \in Y$ which are connected by relation (2.2). Further, it follows by (2.1) that $x \in S_X$ if and only if $y \in S_Y$. Therefore, we derive from (3.1) and (3.2) that

$$\|a\|_X = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right| = \sup_{y \in S_Y} \left| \sum_{k=0}^{\infty} \widehat{a}_k y_k \right| = \|\widehat{a}\|_Y,$$

and since $\widehat{a} \in \ell_1$, we obtain from Lemma 3.4 that

$$\|a\|_X^* = \|\widehat{a}\|_Y^* = \|\widehat{a}\|_{\ell_1}^* < \infty. \qquad \square$$

Lemma 3.7 *Let X be any of the spaces $c_0(B)$ or $\ell_\infty(B)$, let Y be the respective one of the spaces c_0 or ℓ_∞, Z be a sequence space and $A = (a_{nk})$ be an infinite matrix. If $A \in (X, Z)$,* *then $\widehat{A} \in (Y, Z)$ such that $Ax = \widehat{A}y$ for all sequences $x \in X$ and $y \in Y$ which are connected by relation* (2.2), *where $\widehat{A} = (\widehat{a}_{nk})$ is the associated matrix defined by*

$$\widehat{a}_{nk} = \sum_{j=k}^{\infty} \sum_{i=0}^{k-j} \left(\frac{-s + \sqrt{s^2 - 4tr}}{2r} \right)^{j-k-i} \left(\frac{-s - \sqrt{s^2 - 4tr}}{2r} \right)^{i} \frac{a_{nj}}{r}. \tag{3.3}$$

Proof It can be similarly proved by the same technique as in [4, Lemma 2.3]. □

Theorem 3.8 *Let X be any of the spaces $c_0(B)$ or $\ell_\infty(B)$, let $A = (a_{nk})$ be an infinite matrix and $\widehat{A} = (\widehat{a}_{nk})$ be the associated matrix. If A is any of the classes (X, c_0), (X, c) or (X, ℓ_∞), then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \left(\sum_n^{\infty} |\widehat{a}_{nk}| \right) < \infty.$$

Proof This is immediate by combining Lemmas 3.2 and 3.6. □

The following result shows how to compute the Hausdorff measure of noncompactness in the *BK*-space c_0.

Lemma 3.9 [13, Theorem 3.3] *Let Q be a bounded subset of the normed space X, where X is ℓ_p for $1 \le p < \infty$ or c_0. If $p_r : X \longrightarrow X$ ($r \in \mathbb{N}$) is an operator defined by $p_r(x) = (x_0, x_1, \ldots, x_r, 0, 0, \ldots)$ for all $x = (x_k) \in X$, then we have*

$$\chi(Q) = \lim_{r\to\infty} \left(\sup_{x\in Q} \bigl\| (I - p_r)(x) \bigr\|_{\ell_\infty} \right),$$

where I is the identity operator on X.

Further, we know by [11, Theorem 1.10] *that every $z = (z_k) \in c$ has a unique representation $z = \overline{z}e + \sum_n^{\infty} (z_n - \overline{z}) e^{(n)}$, where $\overline{z} = \lim_{n\to\infty} z_n$. Thus, we define the projectors $p_r : c \longrightarrow c$ ($r \in \mathbb{N}$) by*

$$p_r = \overline{z}e + \sum_{n=0}^{r} (z_n - \overline{z}) e^{(n)} \quad (r \in \mathbb{N})$$

for all $z = (z_k) \in c$ with $\overline{z} = \lim_{n\to\infty} z_n$, where $e = (1, 1, 1, \ldots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. In this situation, the following result gives an estimate for the Hausdorff measure of noncompactness in the BK-space c.

Lemma 3.10 [10, Theorem 5(b)] *Let $Q \in \mathcal{M}_c$ and $p_r : c \longrightarrow c$ ($r \in \mathbb{N}$) be the projector onto the linear span of $(e^{(0)}, e^{(1)}, \ldots, e^{(r)})$. Then we have*

$$\frac{1}{2} \lim_{r\to\infty} \left(\sup_{x\in Q} \bigl\| (I - p_r)(x) \bigr\|_{\ell_\infty} \right) \le \chi(Q) \le \lim_{r\to\infty} \left(\sup_{x\in Q} \bigl\| (I - p_r)(x) \bigr\|_{\ell_\infty} \right), \tag{3.4}$$

where I is the identity operator on c.

The next lemma is related to the Hausdorff measure of noncompactness of a bounded linear operator.

Lemma 3.11 [11, Theorem 2.25] *Let X and Y be Banach spaces and $L \in B(X,Y)$. Then we have*

$$\|L_A\|_\chi = \chi\big(L(S_X)\big) \tag{3.5}$$

and

$$L \in \mathcal{C}(X,Y) \quad \text{if and only if} \quad \|L_A\|_\chi = 0. \tag{3.6}$$

4 Compact operators on the spaces $c_0(B)$ and $\ell_\infty(B)$

In this subsection, we establish some identities or estimates for the Hausdorff measures of noncompactness of certain matrix operators on the spaces $c_0(B)$ and $\ell_\infty(B)$. Further, we apply our results to characterize some classes of compact operators on those spaces. We begin with the following lemmas which will be used in proving our results.

Lemma 4.1 [3, Lemma 3.1] *Let X denote any of the spaces c_0 or ℓ_∞. If $A \in (X,c)$, then*

$$\alpha_k = \lim_{n\to\infty} a_{nk} \quad \text{exists for every } k \in \mathbb{N},$$

$$\alpha = (\alpha_k) \in \ell_1,$$

$$\sup_n \left(\sum_k^\infty |a_{nk} - \alpha_k|\right) < \infty,$$

$$\lim_{n\to\infty} A_n = \sum_k^\infty \alpha_k x_k \quad \text{for all } x = (x_k) \in X.$$

Lemma 4.2 [14, Theorem 3.7] *Let $X \supset \phi$ be a BK-space. Then we have*

(a) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty} \|A_n\|_X^*.$$

(b) *If $A \in (X, \ell_\infty)$, then*

$$0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty} \|A_n\|_X^*.$$

Theorem 4.3 *Let X denote any of the spaces $c_0(B)$ and $\ell_\infty(B)$. Then we have*

(a) *If $A \in (X, c_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^\infty |\widehat{a}_{nk}|\right) \tag{4.1}$$

and

$$L_A \text{ is compact} \quad \text{if and only if} \quad \lim_{n\to\infty}\left(\sum_{k=0}^\infty |\widehat{a}_{nk}|\right) = 0. \tag{4.2}$$

(b) *If* $A \in (X, \ell_\infty)$, *then*

$$0 \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{a}_{nk}|\right)$$

and

$$L_A \text{ is compact} \quad \text{if and only if} \quad \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{a}_{nk}|\right) = 0.$$

Proof Let $A \in (X, c_0)$. Since $A_n \in X^\beta$ for all $k \in \mathbb{N}$, we have from Lemma 3.6 that

$$\|A_n\|_X = \|\widehat{A}_n\|_{\ell_1} = \sum_{k=0}^{\infty}|\widehat{a}_{nk}| \tag{4.3}$$

for all $k \in \mathbb{N}$. Thus, we get (4.1) and (4.2) from (4.3) and Lemma 4.2(a). Part (b) can be proved similarly by using Lemma 4.2(b). □

Theorem 4.4 *Let X denote any of the spaces* $c_0(B)$ *or* $\ell_\infty(B)$. *If* $A \in (X, c)$, *then we have*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{a}_{nk} - \widehat{\alpha}_k|\right) \leq \|L_A\|_\chi \leq \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{a}_{nk} - \widehat{\alpha}_k|\right) \tag{4.4}$$

and

$$L_A \text{ is compact} \quad \text{if and only if} \quad \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{a}_{nk} - \widehat{\alpha}_k|\right) = 0, \tag{4.5}$$

where $\lim_{n\to\infty}\widehat{a}_{nk} = \widehat{\alpha}_k$.

Proof By combining Lemma 3.7 and Lemma 4.1, we deduce that the expression in (4.4) exists. We write $S = S_X$ for short. Then we obtain by (3.5) and Lemma 3.1 that

$$\|L_A\|_\chi = \chi(AS) \tag{4.6}$$

and $AS \in \mathcal{M}_c$, where is the class of all bounded subsets of c. Then we are going to apply Lemma 3.10 to get an estimate for the value of $\chi(AS)$ in (4.3). For this, let $p_r : c \longrightarrow c$ be the projectors defined by (3.4). Then we have for every $r \in \mathbb{N}$ that $(I - p_r)(z) = \sum_{n=r+1}^{\infty}(z_n - z)e^n$ and hence

$$\big\|(I - p_r)(z)\big\|_{\ell_\infty} = \sup_{n>r}|z - \overline{z}| \tag{4.7}$$

for all $z \in c$ and every $r \in \mathbb{N}$. Thus, from (4.6) and applying Lemma 3.10, we get that

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{x\in S}\big\|(I - p_r)(Ax)\big\|_{\ell_\infty}\right) \leq \|L_A\|_\chi \leq \lim_{r\to\infty}\left(\sup_{x\in S}\big\|(I - p_r)(Ax)\big\|_{\ell_\infty}\right). \tag{4.8}$$

Now, for every given $x \in X$, let $y \in Y$ be an associated sequence space defined by (2.2), where Y is the respective one of the spaces c_0 or ℓ_∞. Since $A \in (X,c)$, we have by Lemma 3.7 that $\widehat{A} \in (Y,c)$ and $Ax = \widehat{A}y$. Further, it follows from Lemma 4.1 that the limits $\widehat{\alpha}_k = \lim_{n\to\infty} \widehat{a}_{nk}$ exist for all $k, \widehat{\alpha} = (\widehat{\alpha}_k) \in \ell_1 = Y^\beta$ and $\lim_{n\to\infty} \widehat{A}_n(y) = \sum_{k=0}^\infty \widehat{\alpha}_k y_k$. Thus we derive from (4.7) that

$$\left\|(I-p_r)(Ax)\right\|_{\ell_\infty} = \left\|(I-p_r)(\widehat{A}y)\right\|_{\ell_\infty} = \sup_{n>r}\left|\widehat{A}_n(y) - \sum_{k=0}^\infty \widehat{\alpha}_k y_k\right| = \sup_{n>r}\left|\sum_{k=0}^\infty (\widehat{a}_{nk} - \widehat{\alpha}_k) y_k\right|$$

for $r \in \mathbb{N}$. Furthermore, since $x \in S = S_X$ if and only if $y \in S_Y$, we obtain by (3.1) and Lemma 3.1

$$\sup_{X\in S}\left\|(I-p_r)(Ax)\right\|_{\ell_\infty} = \sup_{n>r}\left(\sup_{Y\in S_Y}\left|\sum(\widehat{a}_{nk} - \widehat{\alpha}_k)y_k\right|\right) = \sup_{n>r}\|\widehat{A}_n - \widehat{\alpha}\|_Y^* = \sup_{n>r}\|\widehat{A}_n - \widehat{\alpha}\|_{\ell_1}$$

for all $r \in \mathbb{N}$. Thus, we get (4.4) and (4.5) from (4.8) and (3.6), respectively and this concludes the proof. □

Now, let $\mathcal{F}$ denote the collection of all finite subsets of $\mathbb{N}$, and let $\mathcal{F}_r$ $(r \in \mathbb{N})$ be the subcollection of $\mathcal{F}$ consisting of all nonempty subsets of $\mathbb{N}$ with elements that are grater than r.

Lemma 4.5 [9, Proposition 4.3] *Let $X \supset \phi$ be a BK-space. If $A \in (X,\ell_1)$, then*

$$\|A\|_{(X,\ell_1)} = \|L_A\| \leq 4\|A\|_{(X,\ell_1)},$$

where

$$\|A\|_{(X,\ell_1)} = \sup_{N\in\mathcal{F}}\left\|\sum_{n\in N}^\infty A_n\right\|_X^* < \infty.$$

Lemma 4.6 [3, Lemma 3.5] *Let $x = (x_k) \in \ell_1$. Then the inequalities*

$$\sup_{N\in\mathcal{F}_r}\left|\sum_{n\in N}^\infty x_n\right| \leq \sum_{n=r+1}^\infty |x_n| \leq 4\cdot\sup_{N\in\mathcal{F}_r}\left|\sum_{n\in N}^\infty x_n\right|.$$

Theorem 4.7 *Let X denote any of the spaces $c_0(B)$ or $\ell_\infty(B)$. If $A \in (X,\ell_1)$, then we have*

$$\lim_{r\to\infty}\|A\|_{(X,\ell_1)}^{(r)} = \|L_A\| \leq 4\lim_{r\to\infty}\|A\|_{(X,\ell_1)}^{(r)}, \tag{4.9}$$

and

$$L_A \text{ is compact} \quad \text{if and only if} \quad \lim_{r\to\infty}\|A\|_{(X,\ell_1)}^{(r)} = 0,$$

where

$$\|A\|_{(X,\ell_1)}^{(r)} = \sup_{N\in\mathcal{F}}\sum_{k=0}^\infty\left|\sum_{n\in N}\widehat{a}_{nk}\right| \quad (r \in \mathbb{N}).$$

Proof Since $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_2 \cdots$, the sequence $(\|A\|_{(X,\ell_1)}^{(r)})_{r=0}^{\infty}$ of nonnegative reals is nonincreasing and bounded by Lemma 4.5. Thus, the limit in (4.9) exists.

Now, let $S = S_X$. Then we have by Lemma 3.2(a) that $L_A(S) = AS \in \mathcal{M}_{\ell_1}$. Hence, it follows from (3.5) and Lemma 3.9 that

$$\|L_A\|_\chi = \chi(AS) = \lim_{r\to\infty}\left(\sup_{x\in S}\left(\sum_{n=r+1}^{\infty}\left|A_n(x)\right|\right)\right). \tag{4.10}$$

Since, $A \in (X, \ell_1)$, we obtain by Lemma 4.6 that

$$\sup_{N\in\mathcal{F}_r}\left|\sum_{n\in N}^{\infty} A_n(x)\right| \le \sum_{n=r+1}^{\infty}\left|A_n(x)\right| \le 4\cdot \sup_{N\in\mathcal{F}_r}\left|\sum_{n\in N}^{\infty} A_n(x)\right| \tag{4.11}$$

for all $x \in X$ and every $r \in \mathbb{N}$. On the other hand, since $A_n \in X^\beta$ for all $n \in \mathbb{N}$, we derive from (3.1) and Lemma 3.4

$$\sup_{x\in S}\left|\sum_{n\in N}^{\infty} A_n(x)\right| = \sup_{x\in S}\left|\sum_{k=0}^{\infty}\left(\sum_{n\in N} A_n\right)x_k\right| = \left\|\sum_{n\in N}^{\infty} A_n\right\|_X^* = \left\|\sum_{n\in N}\widehat{A}_n\right\|_{\ell_1}$$

for all $N \in \mathcal{F}_r$ $(r \in \mathbb{N})$. This, together with (4.11), implies that

$$\sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N}\widehat{A}_n\right\|_{\ell_1} \le \sup_{x\in S}\left(\sum_{n=r+1}^{\infty}|A_n|\right) \le 4\cdot \sup_{N\in\mathcal{F}_r}\left\|\sum_{n\in N}\widehat{A}_n\right\|_{\ell_1} \tag{4.12}$$

for every $(r \in \mathbb{N})$. Thus, we get (4.9) by passing to the limits in (4.12) as $r \longrightarrow \infty$ and using (4.10). This completes the proof. □

Theorem 4.8 *Let X denote any of the spaces $c_0(B)$ or $\ell_\infty(B)$. Then we have*

(a) *If $A \in (X, cs_0)$, then*

$$\|L_A\|_\chi = \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{b}_{nk}|\right) \tag{4.13}$$

and

$$L_A \text{ is compact} \quad \text{if and only if} \quad \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{b}_{nk}|\right) = 0. \tag{4.14}$$

(b) *If $A \in (X, bs)$, then*

$$0 \le \|L_A\|_\chi \le \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{b}_{nk}|\right) \tag{4.15}$$

and

$$L_A \text{ is compact} \quad \text{if and only if} \quad \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{b}_{nk}|\right) = 0. \tag{4.16}$$

(c) *If* $A \in (X, cs)$, *then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{b}_{nk}-\widehat{b}_k|\right) \le \|L_A\|_{\chi} \le \limsup_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{b}_{nk}-\widehat{b}_k|\right) \tag{4.17}$$

and

$$L_A \text{ is compact} \quad \text{if and only if} \quad \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}|\widehat{b}_{nk}-\widehat{b}_k|\right) = 0, \tag{4.18}$$

where $\widehat{b}_{nk} = \sum_{m=0}^{n}\widehat{a}_{mk}$ *and* $\lim_{n\to\infty}\widehat{b}_{nk} = \widehat{b}_k$.

Proof Let $A = (a_{nk})$ be an infinite matrix and $S = (s_{nk})$ be the summation matrix, and we define $B = (b_{nk})$ by

$$b_{nk} = \sum_{m=0}^{n} a_{mk} \quad \text{for all } (n, k \in \mathbb{N}),$$

that is, $B = SA$, and hence

$$B_n = \sum_{m=0}^{n} s_{nm}A_{mk} = \left(\sum_{m=0}^{n} a_{mk}\right)_{k=0}^{\infty} \quad (n, k \in \mathbb{N}).$$

Further, let $\widehat{A} = (\widehat{a}_{nk})$ and $\widehat{B} = (\widehat{b}_{nk})$ be the associated matrices, respectively. Then it can be easily seen that

$$\widehat{b}_{nk} = \sum_{m=0}^{n}\widehat{a}_{mk} \quad \text{for all } (n, k \in \mathbb{N}),$$

and hence

$$\widehat{B}_n = \sum_{m=0}^{n} s_{nm}\widehat{A}_{mk} = \left(\sum_{m=0}^{n}\widehat{a}_{mk}\right)_{k=0}^{\infty} \quad (n, k \in \mathbb{N}).$$

Furthermore, we define the sequence $\widehat{b} = (\widehat{b}_k)$ by

$$\widehat{b}_k = \lim_{n\to\infty}\sum_{m=0}^{n}\widehat{a}_{mk} \quad \text{for all } (n, k \in \mathbb{N}), \tag{4.19}$$

provided the limits in (4.19) exist for all $k \in \mathbb{N}$ which is the case whenever $A \in (X, cs)$.

Since $bs = (\ell_\infty)_S$, $cs_0 = (c_0)_S$ and $cs = (c)_S$, (4.13)-(4.18) are obtained from Theorems 4.3 and 4.4 by using Lemma 3.2. This completes the proof. □

Competing interests

The author declares that they have no competing interests.

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