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Research Article

New Existence Results for Fractional Integrodifferential Equations with Nonlocal Integral Boundary Conditions

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We consider a boundary value problem of fractional integrodifferential equations with new nonlocal integral boundary conditions of the form: $x(0) = \beta x(\theta)$, $x(\xi) = \alpha \int_\eta^1 x(s)ds$, and $0 < \theta < \xi < \eta < 1$. According to these conditions, the value of the unknown function at the left end point $t = 0$ is proportional to its value at a nonlocal point θ while the value at an arbitrary (local) point ξ is proportional to the contribution due to a substrip of arbitrary length $(1 - \eta)$. These conditions appear in the mathematical modelling of physical problems when different parts (nonlocal points and substrips of arbitrary length) of the domain are involved in the input data for the process under consideration. We discuss the existence of solutions for the given problem by means of the Sadovski fixed point theorem for condensing maps and a fixed point theorem due to O'Regan. Some illustrative examples are also presented.

1. Introduction

We consider a boundary value problem of fractional differential equations with nonlocal integral boundary conditions given by

$$
\begin{aligned}
&{}^{c}D^{q}x(t) = Af(t, x(t)) + BI^{r}g(t, x(t)), \quad t \in [0, 1],\\
&x(0) = \beta x(\theta), \qquad x(\xi) = \alpha \int_{\eta}^{1} x(s)\, ds, \qquad (1)\\
&0 < \theta < \xi < \eta < 1,
\end{aligned}
$$

where ${}^{c}D^{q}$ denotes the Caputo fractional derivative of order q, $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $1 < q \leq 2$, $0 < r < 1$, and α, β, A, B are real constants.

Here we remark that the boundary conditions introduced in the problem (1) are of nonlocal strip type and describe the situation when the receptors at the end points of the boundary are influenced by the nonlocal contributions due to interior points and strips of the domain for the problem. For practical examples, see [1, 2]. The problem (1) can also be termed as a five-point nonlocal fractional boundary value problem.

In recent years, several aspects of fractional boundary value problems, ranging from theoretical analysis to numerical simulation, have been investigated. The nonlocal nature of fractional order differential operators has significantly contributed to the popularity and development of the subject. As a matter of fact, this characteristic of such operators help to understand the memory and hereditary properties of many useful materials and processes. For details and applications of fractional differential equations in physical and technical sciences such as biology, physics, biophysics, chemistry, statistics, economics, blood flow phenomena, control theory, and signal and image processing, see [3–6]. For some recent works on nonlocal fractional boundary value problems, we refer the reader to the papers [7–11], while the results based on monotone method for such problems can be found in [12, 13]. In [14], the limit properties of positive solutions of fractional boundary value problems have been discussed.

Fractional differential inclusions supplemented with different kinds of boundary conditions have also been studied by several researchers, for instance, see [15–20].

The paper is organized as follows. In Section 2, we recall some basic definitions from fractional calculus and establish an auxiliary lemma which plays a pivotal role in the sequel. Section 3.1 contains an existence result for the problem (1) which is established by applying Sadovskii's fixed point theorem for condensing maps. In Section 3.2, we show the existence of solutions for the problem (1) by means of a fixed point theorem due to O'Regan.

2. Preliminaries

In this section, some basic definitions on fractional calculus and an auxiliary lemma are presented [3, 4].

Definition 1. The Riemann-Liouville fractional integral of order q for a continuous function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\, ds, \quad q > 0, \tag{2}$$

provided the integral exists.

Definition 2. For at least n-times continuously differentiable function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$\begin{aligned} {}^cD^q g(t) &= \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\, ds, \\ & n-1 < q < n, \quad n = [q] + 1, \end{aligned} \tag{3}$$

where $[q]$ denotes the integer part of the real number q.

Lemma 3. *For any $y \in C([0,1], \mathbb{R})$ the unique solution of the linear fractional boundary value problem*

$$\begin{aligned} &{}^cD^q x(t) = y(t), \quad t \in [0,1], \ 1 < q \le 2, \\ &x(0) = \beta x(\theta), \qquad x(\xi) = \alpha \int_\eta^1 x(s)\, ds, \\ &0 < \theta < \xi < \eta < 1, \end{aligned} \tag{4}$$

is

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds \\ & + \frac{\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)}{Q} \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} y(s)\, ds \\ & + \frac{\alpha\beta\theta}{Q}\left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} y(s)\, ds - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} y(s)\, ds\right) \\ & - \frac{\beta\theta}{Q} \int_0^\xi \frac{(\xi-s)^q}{\Gamma(q)} y(s)\, ds \\ & + \frac{t}{Q}\left[\alpha(1-\beta)\left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} y(s)\, ds \right.\right. \\ & \qquad \left. - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} y(s)\, ds\right) \\ & - (1-\beta) \int_0^\xi \frac{(\xi-s)^q}{\Gamma(q)} y(s)\, ds \\ & \left. - \beta(1-\alpha(1-\eta)) \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} y(s)\, ds\right], \end{aligned} \tag{5}$$

where

$$Q = (1-\beta)\left(\xi - \frac{\alpha}{2}\left(1-\eta^2\right)\right) + \beta\theta(1-\alpha(1-\eta)) \neq 0. \tag{6}$$

Proof. It is well known that the general solution of the fractional differential equation in (4) can be written as

$$x(t) = c_0 + c_1 t + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)\, ds, \tag{7}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants.

Applying the given boundary conditions, we obtain the following system

$$\begin{aligned} &(1-\beta) c_0 - \beta\theta c_1 = \beta \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} y(s)\, ds \\ &(1-\alpha(1-\eta)) c_0 + \left(\xi - \frac{\alpha}{2}\left(1-\eta^2\right)\right) c_1 \\ &\quad = \alpha\left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} y(s)\, ds - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} y(s)\, ds\right) \\ &\qquad - \int_0^\xi \frac{(\xi-s)^q}{\Gamma(q)} y(s)\, ds, \end{aligned} \tag{8}$$

from which we get

$$\begin{aligned} c_0 = {} & \frac{\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)}{Q} \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} y(s)\, ds \\ & + \frac{\alpha\beta\theta}{Q}\left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} y(s)\, ds - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} y(s)\, ds\right) \\ & - \frac{\beta\theta}{Q} \int_0^\xi \frac{(\xi-s)^q}{\Gamma(q)} y(s)\, ds, \\ c_1 = {} & \frac{1}{Q}\left[\alpha(1-\beta)\left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} y(s)\, ds \right.\right. \\ & \qquad \left. - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} y(s)\, ds\right) \end{aligned}$$

$$-(1-\beta)\int_0^{\xi}\frac{(\xi-s)^q}{\Gamma(q)}y(s)\,ds$$

$$-\beta(1-\alpha(1-\eta))\int_0^{\theta}\frac{(\theta-s)^{q-1}}{\Gamma(q)}y(s)\,ds\Bigg]. \tag{9}$$

Substituting the values of c_0, c_1 in (7), we get (5). This completes the proof. □

3. Existence Results

We denote by $\mathscr{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Also by $L^1([0,1],\mathbb{R})$ we denote the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)|dt$.

In the following we will give two existence results for the problem (1), one with the help of Sadovskii's fixed point theorem and the other based on a fixed point theorem due to O'Regan in [21].

3.1. Existence Results via Sadovskii's Fixed Point Theorem

Definition 4. Let M be a bounded set in metric space (X,d); then Kuratowskii measure of noncompactness, $\alpha(M)$, is defined as $\inf\{\epsilon : M$ covered by finitely many sets such that the diameter of each set $\le \epsilon\}$.

Definition 5 (see [22]). Let $\Phi : D(\Phi) \subseteq X \to X$ be a bounded and continuous operator on Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset D(\Phi)$, where α denotes the Kuratowski measure of noncompactness.

Lemma 6 (see [23, Example 11.7]). *The map $K + C$ is a k-set contraction with $0 \le k < 1$ and is thus condensing, if*

(i) *$K, C : D \subseteq X \to X$ are operators on the Banach space X;*

(ii) *K is k-contractive; that is,*

$$\|Kx - Ky\| \le k\|x - y\| \tag{10}$$

for all $x, y \in D$ and fixed $k \in [0,1)$;

(iii) *C is compact.*

Theorem 7 (see [24]). *Let B be a convex, bounded, and closed subset of a Banach space X and let $\Phi : B \to B$ be a condensing map. Then Φ has a fixed point.*

In view of Lemma 3, we define an operator $\mathscr{P} : \mathscr{C} \to \mathscr{C}$ by

$$(\mathscr{P}x)(t) = (\mathscr{P}_1x)(t) + (\mathscr{P}_2x)(t), \quad t \in [0,1], \tag{11}$$

where

$$\begin{aligned}
&(\mathscr{P}_1x)(t)\\
&= A\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds\\
&\quad + A\frac{\beta\left(\xi-(\alpha/2)\left(1-\eta^2\right)\right)}{Q}\int_0^{\theta}\frac{(\theta-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds\\
&\quad + A\frac{\alpha\beta\theta}{Q}\left(\int_0^1\frac{(1-s)^q}{\Gamma(q+1)}f(s,x(s))\,ds\right.\\
&\qquad\qquad \left.-\int_0^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}f(s,x(s))\,ds\right)\\
&\quad - A\frac{\beta\theta}{Q}\int_0^{\xi}\frac{(\xi-s)^q}{\Gamma(q)}f(s,x(s))\,ds\\
&\quad + \frac{At}{Q}\left[\alpha(1-\beta)\left(\int_0^1\frac{(1-s)^q}{\Gamma(q+1)}f(s,x(s))\,ds\right.\right.\\
&\qquad\qquad \left.-\int_0^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}f(s,x(s))\,ds\right)\\
&\qquad - (1-\beta)\int_0^{\xi}\frac{(\xi-s)^q}{\Gamma(q)}f(s,x(s))\,ds\\
&\qquad \left.-\beta(1-\alpha(1-\eta))\int_0^{\theta}\frac{(\theta-s)^{q-1}}{\Gamma(q)}f(s,x(s))\,ds\right],
\end{aligned} \tag{12}$$

$$\begin{aligned}
&(\mathscr{P}_2x)(t)\\
&= B\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\int_0^s\frac{(s-u)^{r-1}}{\Gamma(r)}g(u,x(u))\,du\,ds\\
&\quad + B\frac{\beta\left(\xi-(\alpha/2)\left(1-\eta^2\right)\right)}{Q}\\
&\quad \times\int_0^{\theta}\frac{(\theta-s)^{q-1}}{\Gamma(q)}\int_0^s\frac{(s-u)^{r-1}}{\Gamma(r)}g(u,x(u))\,du\,ds\\
&\quad + B\frac{\alpha\beta\theta}{Q}\left(\int_0^1\frac{(1-s)^q}{\Gamma(q+1)}\int_0^s\frac{(s-u)^{r-1}}{\Gamma(r)}g(u,x(u))\,du\,ds\right.\\
&\qquad\qquad \left.-\int_0^{\eta}\frac{(\eta-s)^q}{\Gamma(q+1)}\int_0^s\frac{(s-u)^{r-1}}{\Gamma(r)}g(u,x(u))\,du\,ds\right)\\
&\quad - B\frac{\beta\theta}{Q}\int_0^{\xi}\frac{(\xi-s)^q}{\Gamma(q)}\int_0^s\frac{(s-u)^{r-1}}{\Gamma(r)}g(u,x(u))\,du\,ds\\
&\quad + \frac{Bt}{Q}\left[\alpha(1-\beta)\right.\\
&\qquad \times\left(\int_0^1\frac{(1-s)^q}{\Gamma(q+1)}\int_0^s\frac{(s-u)^{r-1}}{\Gamma(r)}g(u,x(u))\,du\,ds\right.
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^{\eta} \frac{(\eta - s)^q}{\Gamma(q+1)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \\
&\qquad \times g(u, x(u))\, du\, ds \Bigg) \\
&- (1-\beta) \int_0^{\xi} \frac{(\xi - s)^q}{\Gamma(q)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \\
&\qquad \times g(u, x(u))\, du\, ds \\
&- \beta(1 - \alpha(1-\eta)) \\
&\times \int_0^{\theta} \frac{(\theta - s)^{q-1}}{\Gamma(q)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} g(u, x(u))\, du\, ds \Bigg].
\end{aligned} \tag{13}
$$

Theorem 8. *Let $f, g : [0,1]\times\mathbb{R} \to \mathbb{R}$ be continuous functions satisfying the following conditions.*

(H_1) f satisfies the Lipschitz condition:

$$
\begin{aligned}
&\left|f(t,x) - f(t,y)\right| \le L\left|x - y\right|, \\
&L > 0, \quad \forall (t,x), (t,y) \in [0,1]\times\mathbb{R},
\end{aligned} \tag{14}
$$

(H_2) there exist a function $m \in C([0,1], \mathbb{R}^+)$ and a nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$
\left|g(t,x)\right| \le m(t)\,\psi(\|x\|), \quad \forall (t,x) \in [0,1]\times\mathbb{R}. \tag{15}
$$

Then the boundary value problem (1) has at least one solution provided that

$$
\begin{aligned}
\gamma :=\ & |A|\, L \left\{ \frac{1}{\Gamma(q+1)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^q}{|Q|\,\Gamma(q+1)} \right. \\
&+ \frac{|\alpha\beta|\,\theta}{|Q|}\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \\
&+ \frac{|\beta|\,\theta\xi^q}{|Q|\,\Gamma(q+1)} + \frac{1}{|Q|} \\
&\times \left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \right. \\
&\left. \left. + \frac{|1-\beta|\,\xi^q}{\Gamma(q+1)} + \frac{\left|\beta(1-\alpha(1-\eta))\right|\theta^q}{\Gamma(q+1)} \right] \right\} \\
<\ & 1.
\end{aligned} \tag{16}
$$

Proof. Let $B_\nu = \{x \in \mathscr{C} : \|x\| \le \nu\}$ be a closed bounded and convex subset of $\mathscr{C} := C([0,1], \mathbb{R})$, where ν will be fixed later. We define a map $\mathscr{P} : B_\nu \to \mathscr{C}$ as

$$
(\mathscr{P}x)(t) = (\mathscr{P}_1 x)(t) + (\mathscr{P}_2 x)(t), \quad t \in [0,1], \tag{17}
$$

where $\mathscr{P}_1$ and $\mathscr{P}_2$ are defined by (12) and (13), respectively. Notice that the problem (1) is equivalent to a fixed point problem $\mathscr{P}(x) = x$.

Step 1 ($(\mathscr{P}x)(B_\nu) \subset B_\nu$). For that, we set $M = \sup_{t\in[0,1]}|f(t,0)|$ and select $\nu \ge \omega/(1-\gamma)$, where

$$
\begin{aligned}
\omega =\ & |A|\, M \left[\frac{1}{\Gamma(q+1)} \right. \\
&+ \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^q}{|Q|\,\Gamma(q+1)} \\
&+ \frac{|\alpha\beta|\,\theta}{|Q|}\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \\
&+ \frac{|\beta|\,\theta\xi^q}{|Q|\,\Gamma(q+1)} + \frac{1}{|Q|} \\
&\times \left\{ |\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \right. \\
&\left. \left. + \frac{|1-\beta|\,\xi^q}{\Gamma(q+1)} + \frac{\left|\beta(1-\alpha(1-\eta))\right|\theta^q}{\Gamma(q+1)} \right\} \right] \\
&+ |B|\,\|m\|\,\psi(r) \\
&\times \left[\frac{1}{\Gamma(q+r+1)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^{q+r}}{|Q|\,\Gamma(q+r+1)} \right. \\
&+ \frac{|\alpha\beta|\,\theta}{|Q|}\left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \\
&+ \frac{|\beta|\,\theta\xi^{q+r}}{|Q|\,\Gamma(q+r+1)} + \frac{1}{|Q|} \\
&\times \left\{ |\alpha(1-\beta)| \right. \\
&\times \left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \\
&+ \frac{|1-\beta|\,\xi^{q+r}}{\Gamma(q+r+1)} \\
&\left. \left. + \frac{\left|\beta(1-\alpha(1-\eta))\right|\theta^{q+r}}{\Gamma(q+r+1)} \right\} \right].
\end{aligned} \tag{18}
$$

Using $|f(t,x(t))| \le |f(t,x(t)) - f(t,0)| + |f(t,0)| \le L\nu + M$, for $x \in B_\nu$, $t \in [0,1]$, we get

$$
\begin{aligned}
&\left|(\mathscr{P}_1 x)(t)\right| \\
&\le |A|\,(L\nu + M) \\
&\times \left\{ \frac{1}{\Gamma(q+1)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^q}{|Q|\,\Gamma(q+1)} \right. \\
&\left. + \frac{|\alpha\beta|\,\theta}{|Q|}\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \right.
\end{aligned}
$$

$$+\frac{|\beta|\theta\xi^q}{|Q|\Gamma(q+1)}+\frac{1}{|Q|}\times\left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+2)}+\frac{\eta^{q+1}}{\Gamma(q+2)}\right)+\frac{|1-\beta|\xi^q}{\Gamma(q+1)}+\frac{|\beta(1-\alpha(1-\eta))|\theta^q}{\Gamma(q+1)}\right]\Bigg\},$$

$$\begin{aligned}&|(\mathscr{P}_2x)(t)|\\&\le|B|\,\|m\|\,\psi(\nu)\times\left\{\frac{1}{\Gamma(q+r+1)}+\frac{\left|\beta\left(\xi-(\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^{q+r}}{|Q|\Gamma(q+r+1)}+\frac{|\alpha\beta|\theta}{|Q|}\left(\frac{1}{\Gamma(q+r+2)}+\frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right)+\frac{|\beta|\theta\xi^{q+r}}{|Q|\Gamma(q+r+1)}+\frac{1}{|Q|}\right.\\&\left.\times\left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+r+2)}+\frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right)+\frac{|1-\beta|\xi^{q+r}}{\Gamma(q+r+1)}+\frac{|\beta(1-\alpha(1-\eta))|\theta^{q+r}}{\Gamma(q+r+1)}\right]\right\},\end{aligned}\tag{19}$$

where we have used the following relations:

$$\begin{aligned}&\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\int_0^s\frac{(s-u)^{r-1}}{\Gamma(r)}du\,ds\\&=\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\frac{(s)^r}{\Gamma(r+1)}ds\\&=\frac{t^{q+r}}{\Gamma(q)\Gamma(r+1)}\int_0^1(1-v)^{q-1}v^r\,dv\\&=\frac{t^{q+r}}{\Gamma(q+r+1)},\end{aligned}\tag{20}$$

$$\int_0^1(1-u)^{\alpha-1}u^\beta du=\frac{\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}.$$

Consequently

$$\begin{aligned}&|(\mathscr{P}x)(t)|\\&\le\left|(\mathscr{P}_1x)(t)\right|+\left|(\mathscr{P}_2x)(t)\right|\\&\le|A|(L\nu+M)\times\left\{\frac{1}{\Gamma(q+1)}+\frac{\left|\beta\left(\xi-(\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^q}{|Q|\Gamma(q+1)}+\frac{|\alpha\beta|\theta}{|Q|}\left(\frac{1}{\Gamma(q+2)}+\frac{\eta^{q+1}}{\Gamma(q+2)}\right)+\frac{|\beta|\theta\xi^q}{|Q|\Gamma(q+1)}+\frac{1}{|Q|}\right.\\&\left.\times\left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+2)}+\frac{\eta^{q+1}}{\Gamma(q+2)}\right)+\frac{|1-\beta|\xi^q}{\Gamma(q+1)}+\frac{|\beta(1-\alpha(1-\eta))|\theta^q}{\Gamma(q+1)}\right]\right\}\\&+|B|\,\|m\|\,\psi(\nu)\times\left\{\frac{1}{\Gamma(q+r+1)}+\frac{\left|\beta\left(\xi-(\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^{q+r}}{|Q|\Gamma(q+r+1)}+\frac{|\alpha\beta|\theta}{|Q|}\left(\frac{1}{\Gamma(q+r+2)}+\frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right)+\frac{|\beta|\theta\xi^{q+r+r}}{|Q|\Gamma(q+r+1)}+\frac{1}{|Q|}\right.\\&\left.\times\left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+r+2)}+\frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right)+\frac{|1-\beta|\xi^{q+r}}{\Gamma(q+r+1)}+\frac{|\beta(1-\alpha(1-\eta))|\theta^{q+r}}{\Gamma(q+r+1)}\right]\right\}\\&\le\gamma\nu+\omega\le\nu,\end{aligned}\tag{21}$$

which implies that $(\mathscr{P}x)(B_\nu)\subset B_\nu$.

Step 2 ($\mathscr{P}_1$ is continuous and γ-contractive). To show the continuity of $\mathscr{P}_1$ for $t\in[0,1]$, let us consider a sequence x_n converging to x. Then, by the assumption (H_1), we have

$$\begin{aligned}&\left|(\mathscr{P}_1x_n)(t)-(\mathscr{P}_1x)(t)\right|\\&\le|A|L\left\{\frac{1}{\Gamma(q+1)}+\frac{\left|\beta\left(\xi-(\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^q}{|Q|\Gamma(q+1)}+\frac{|\alpha\beta|\theta}{|Q|}\left(\frac{1}{\Gamma(q+2)}+\frac{\eta^{q+1}}{\Gamma(q+2)}\right)+\frac{|\beta|\theta\xi^q}{|Q|\Gamma(q+1)}+\frac{1}{|Q|}\right.\end{aligned}$$

$$\times \left[\left|\alpha\left(1-\beta\right)\right| \left(\frac{1}{\Gamma\left(q+2\right)} + \frac{\eta^{q+1}}{\Gamma\left(q+2\right)} \right) + \frac{\left|1-\beta\right|\xi^{q}}{\Gamma\left(q+1\right)} + \frac{\left|\beta\left(1-\alpha\left(1-\eta\right)\right)\right|\theta^{q}}{\Gamma\left(q+1\right)} \right] \Bigg\} \times \left\|x_n - x\right\|. \tag{22}$$

Next, we show that $\mathscr{P}_1$ is γ-contractive. For $x, y \in B_\nu$, we get

$$\begin{aligned} &\left|\left(\mathscr{P}_1 x\right)(t) - \left(\mathscr{P}_1 y\right)(t)\right| \\ &\le |A|\, L \Bigg\{ \frac{1}{\Gamma\left(q+1\right)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^{q}}{|Q|\,\Gamma\left(q+1\right)} \\ &\quad + \frac{\left|\alpha\beta\right|\theta}{|Q|}\left(\frac{1}{\Gamma\left(q+2\right)} + \frac{\eta^{q+1}}{\Gamma\left(q+2\right)} \right) + \frac{\left|\beta\right|\theta\xi^{q}}{|Q|\,\Gamma\left(q+1\right)} + \frac{1}{|Q|} \\ &\quad \times \left[\left|\alpha\left(1-\beta\right)\right| \left(\frac{1}{\Gamma\left(q+2\right)} + \frac{\eta^{q+1}}{\Gamma\left(q+2\right)} \right) + \frac{\left|1-\beta\right|\xi^{q}}{\Gamma\left(q+1\right)} + \frac{\left|\beta\left(1-\alpha\left(1-\eta\right)\right)\right|\theta^{q}}{\Gamma\left(q+1\right)} \right] \Bigg\} \\ &\quad \times \left\|x - y\right\|. \end{aligned} \tag{23}$$

By the given assumption

$$\begin{aligned} \gamma := |A|\, L \Bigg\{ &\frac{1}{\Gamma\left(q+1\right)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^{q}}{|Q|\,\Gamma\left(q+1\right)} \\ &+ \frac{\left|\alpha\beta\right|\theta}{|Q|}\left(\frac{1}{\Gamma\left(q+2\right)} + \frac{\eta^{q+1}}{\Gamma\left(q+2\right)} \right) + \frac{\left|\beta\right|\theta\xi^{q}}{|Q|\,\Gamma\left(q+1\right)} + \frac{1}{|Q|} \\ &\times \left[\left|\alpha\left(1-\beta\right)\right| \left(\frac{1}{\Gamma\left(q+2\right)} + \frac{\eta^{q+1}}{\Gamma\left(q+2\right)} \right) + \frac{\left|1-\beta\right|\xi^{q}}{\Gamma\left(q+1\right)} + \frac{\left|\beta\left(1-\alpha\left(1-\eta\right)\right)\right|\theta^{q}}{\Gamma\left(q+1\right)} \right] \Bigg\} < 1, \end{aligned} \tag{24}$$

it follows that $\mathscr{P}_1$ is γ-contractive.

Step 3 ($\mathscr{P}_2$ is compact). In Step 1, it has been shown that $\mathscr{P}_2$ is uniformly bounded. Now we show that $\mathscr{P}_2$ maps bounded sets into equicontinuous sets of $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $x \in B_\nu$. Then we obtain

$$\begin{aligned} &\left|\left(\mathscr{P}_2 x\right)(t_2) - \left(\mathscr{P}_2 x\right)(t_1)\right| \\ &\le \frac{|B|\,\psi(\nu)}{\Gamma\left(q+r+1\right)} \int_0^{t_1} \left[\left(t_2 - s\right)^{q-1} - \left(t_1 - s\right)^{q-1} \right] m(s)\, ds \\ &\quad + \frac{|B|\,\psi(\nu)}{\Gamma\left(q+r+1\right)} \int_{t_1}^{t_2} \left(t_2 - s\right)^{q-1} m(s)\, ds \\ &\quad + \frac{|B|\,\psi(\nu)\,\|m\|\left|t_2 - t_1\right|}{|Q|} \\ &\quad \times \left[\left|\alpha\left(1-\beta\right)\right| \left(\frac{1}{\Gamma\left(q+r+2\right)} + \frac{\eta^{q+r+1}}{\Gamma\left(q+r+2\right)} \right) + \frac{\left|1-\beta\right|\xi^{q+r}}{\Gamma\left(q+r+1\right)} + \frac{\left|\beta\left(1-\alpha\left(1-\eta\right)\right)\right|\theta^{q+r}}{\Gamma\left(q+r+1\right)} \right]. \end{aligned} \tag{25}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_\nu$ as $t_2 - t_1 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathscr{P}_2 : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous. Thus $\mathscr{P}_2$ is compact on $[0,1]$.

Step 4 ($\mathscr{P}$ is condensing). Since $\mathscr{P}_1$ is continuous, γ-contractive and $\mathscr{P}_2$ is compact, by Lemma 6, $\mathscr{P} : B_\nu \to B_\nu$ with $\mathscr{P} = \mathscr{P}_1 + \mathscr{P}_2$ is a condensing map on B_ν.

Consequently, by Theorem 7, the map $\mathscr{P}$ has a fixed point which, in turn, implies that the problem (1) has a solution. □

Example 9. Consider a nonlocal integral boundary value problem of fractional integrodifferential equations given by

$$\begin{aligned} &D^{3/2}x(t) = f\left(t, x(t)\right) + I^{3/4} g\left(t, x(t)\right), \quad t \in [0,1], \\ &x(0) = \frac{1}{2}x\left(\frac{1}{4}\right), \qquad x\left(\frac{1}{3}\right) = \int_{2/3}^{1} x(s)\, ds, \end{aligned} \tag{26}$$

where $q = 3/2, A = B = 1, r = 3/4, \theta = 1/4, \xi = 1/3, \eta = 2/3, \alpha = 1, \beta = 1/2, f(t,x) = (1/(2+t)^3)\tan^{-1}x + t + 1$, and $g(t,x) = (t^2/(1+t^3))(1 + (|x|/(1+|x|)))$.

Clearly $L = 1/8$ as $|f(t,x) - f(t,y)| \le (1/8)|x-y|$, and $|g(t,x)| \le m(t)\psi(\|x\|)$ with $m(t) = t^2/(1+t^3)$ and $\psi(\|x\|) = 2$. Furthermore $|Q| = 1/9$, and the condition (16) yields $\gamma \simeq 0.522371 < 1$. Thus all the conditions of Theorem 8 are satisfied and consequently the problem (26) has a solution.

3.2. Existence Results via O'Regan's Fixed Point Theorem. Our next existence result relies on a fixed point theorem due to O'Regan in [21].

Lemma 10. *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\overline{U})$ is bounded and that $F : \overline{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \overline{U} \to E$ is continuous and completely continuous and $F_2 : \overline{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \overline{U}$). Then, either*

(C1) F has a fixed point $u \in \overline{U}$, or

(C2) there exist a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, where $\overline{U}$ and ∂U, respectively, represent the closure and boundary of U.

For convenience we set

$$\begin{aligned} p_0 = |A| &\left\{ \frac{1}{\Gamma(q+1)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^q}{|Q|\,\Gamma(q+1)} \right. \\ &+ \frac{|\alpha\beta|\,\theta}{|Q|}\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \\ &+ \frac{|\beta|\,\theta\xi^q}{|Q|\,\Gamma(q+1)} + \frac{1}{|Q|} \\ &\times \left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \right. \\ &\left.\left. + \frac{|1-\beta|\,\xi^q}{\Gamma(q+1)} + \frac{\left|\beta\left(1-\alpha\left(1-\eta\right)\right)\right|\theta^q}{\Gamma(q+1)} \right] \right\}, \\ k_0 = |B| &\left\{ \frac{1}{\Gamma(q+r+1)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^{q+r}}{|Q|\,\Gamma(q+r+1)} \right. \\ &+ \frac{|\alpha\beta|\,\theta}{|Q|}\left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \\ &+ \frac{|\beta|\,\theta\xi^{q+r}}{|Q|\,\Gamma(q+r+1)} + \frac{1}{|Q|} \\ &\times \left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \right. \\ &\left.\left. + \frac{|1-\beta|\,\xi^{q+r}}{\Gamma(q+r+1)} + \frac{\left|\beta\left(1-\alpha\left(1-\eta\right)\right)\right|\theta^{q+r}}{\Gamma(q+r+1)} \right] \right\}. \end{aligned} \tag{27}$$

Let

$$\Omega_\sigma = \{x \in C([0,1],\mathbb{R}) : \|x\| < \sigma\} \tag{28}$$

and denote the maximum number by

$$M_\sigma = \max\left\{\left|f(t,x)\right| : (t,x) \in [0,1] \times [-\sigma,\sigma]\right\}. \tag{29}$$

Theorem 11. *Let $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that*

(A_1) there exist a nonnegative function $p \in C([0,1],\mathbb{R})$ and a nondecreasing function $\zeta : [0,\infty) \to (0,\infty)$ such that

$$\left|f(t,u)\right| \le p(t)\,\zeta(\|u\|) \quad \text{for any } (t,u) \in [0,1] \times \mathbb{R}; \tag{30}$$

(A_2) there exist a positive constant $\ell < k_0^{-1}$ and a continuous function $\phi : [0,\infty) \to (0,\infty)$ such that $\phi(z) \le \ell z$ and $|g(t,u) - g(t,v)| \le \phi(\|u - v\|)$ for all $t \in [0,1]$ and $u, v \in \mathbb{R}$;

(A_3) $\sup_{r\in(0,\infty)}(r/(k_0K + p_0\zeta(r)\|p\|)) > (1/(1-k_0\ell))$, where $K = \sup_{t\in[0,1]}|g(t,0)|$.

Then the boundary value problem (1) has at least one solution on $[0,1]$.

Proof. By the assumption (A_3), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{k_0K + p_0\zeta(r_0)\|p\|} > \frac{1}{1-k_0\ell}. \tag{31}$$

We will show that the operators $\mathscr{P}_1$ and $\mathscr{P}_2$ defined by (12) and (13), respectively, satisfy all the conditions of Lemma 10. The proof consists of a series of steps.

Step 1 (the operator $\mathscr{P}_1$ is continuous and completely continuous). We first show that $\mathscr{P}_1(\overline{\Omega}_{r_0})$ is bounded. For any $x \in \overline{\Omega}_{r_0}$, we have

$$\begin{aligned} &\|(\mathscr{P}_1 x)\| \\ &\le |A|\,M_\sigma \\ &\quad\times \left\{ \frac{1}{\Gamma(q+1)} + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right|\theta^q}{|Q|\,\Gamma(q+1)} \right. \\ &\qquad + \frac{|\alpha\beta|\,\theta}{|Q|}\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \\ &\qquad + \frac{|\beta|\,\theta\xi^q}{|Q|\,\Gamma(q+1)} + \frac{1}{|Q|} \\ &\qquad \times \left[|\alpha(1-\beta)|\left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \right. \\ &\qquad\left.\left. + \frac{|1-\beta|\,\xi^q}{\Gamma(q+1)} + \frac{\left|\beta\left(1-\alpha\left(1-\eta\right)\right)\right|\theta^q}{\Gamma(q+1)} \right] \right\}. \end{aligned} \tag{32}$$

Thus the operator $\mathscr{P}_1(\overline{\Omega}_{r_0})$ is uniformly bounded. For any $t_1, t_2 \in [0,1]$, $t_1 < t_2$, we have

$$\begin{aligned}
&\left|(\mathscr{P}_1 x)(t_2) - (\mathscr{P}_1 x)(t_1)\right| \\
&\le |A| M_r \left[\int_0^{t_1} \left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1}\right] ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} (t_2 - s)^{q-1} ds\right] \\
&\quad + \frac{|A| M_r |t_2 - t_1|}{|Q|} \\
&\quad \times \left[\left|\alpha(1-\beta)\right| \left(\frac{1}{\Gamma(q+2)} + \frac{\eta^{q+1}}{\Gamma(q+2)}\right) \right. \\
&\qquad \left. + \frac{\left|1-\beta\right| \xi^q}{\Gamma(q+1)} + \frac{\left|\beta(1-\alpha(1-\eta))\right| \theta^q}{\Gamma(q+1)}\right],
\end{aligned} \tag{33}$$

which is independent of x and tends to zero as $t_2 - t_1 \to 0$. Thus, $\mathscr{P}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli theorem, $\mathscr{P}_1(\overline{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \in \overline{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ is uniformly valid on $[0,1]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,1] \times [-r_0, r_0]$, it follows that $\|f(t, x_n(t)) - f(t, x(t))\| \to 0$ is uniformly valid on $[0,1]$. Hence $\|\mathscr{P}_1 x_n - \mathscr{P}_1 x\| \to 0$ as $n \to \infty$. This shows the continuity of $\mathscr{P}_1$.

Step 2 (the operator $\mathscr{P}_2 : \overline{\Omega}_{r_0} \to C([0,1], \mathbb{R})$ is contractive). Consider

$$\begin{aligned}
&\left|(\mathscr{P}_2 x)(t) - (\mathscr{P}_2 y)(t)\right| \\
&\le |B| \left\{\frac{1}{\Gamma(q+r+1)} \right. \\
&\qquad + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right| \theta^{q+r}}{|Q| \Gamma(q+r+1)} \\
&\qquad + \frac{|\alpha\beta| \theta}{|Q|} \left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \\
&\qquad + \frac{|\beta| \theta \xi^{q+r}}{|Q| \Gamma(q+r+1)} + \frac{1}{|Q|} \\
&\qquad \times \left[\left|\alpha(1-\beta)\right| \left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \right. \\
&\qquad\quad \left. \left. + \frac{\left|1-\beta\right| \xi^{q+r}}{\Gamma(q+r+1)} + \frac{\left|\beta(1-\alpha(1-\eta))\right| \theta^{q+r}}{\Gamma(q+r+1)}\right]\right\} \\
&\quad \times \phi(\|x - y\|).
\end{aligned} \tag{34}$$

This, together with (A_2), implies that

$$\|(\mathscr{P}_2 x) - (\mathscr{P}_2 y)\| \le \phi(\|x - y\|), \tag{35}$$

so $\mathscr{P}_2 : \overline{\Omega}_{r_0} \to C([0,1], \mathbb{R})$ is a nonlinear contraction.

Step 3 (the set $\mathscr{P}(\overline{\Omega}_{r_0})$ is bounded). Using the inequality

$$\begin{aligned}
|g(t,x)| &\le |g(t,x) - g(t,0)| + |g(t,0)| \\
&\le \phi(\|x\|) + K \le \ell r_0 + K,
\end{aligned} \tag{36}$$

we have

$$\begin{aligned}
&\left|\mathscr{P}_2(x)\right| \\
&\le |B| (\ell r_0 + K) \\
&\quad \times \left\{\frac{1}{\Gamma(q+r+1)} \right. \\
&\qquad + \frac{\left|\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)\right| \theta^{q+r}}{|Q| \Gamma(q+r+1)} \\
&\qquad + \frac{|\alpha\beta| \theta}{|Q|} \left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \\
&\qquad + \frac{|\beta| \theta \xi^{q+r}}{|Q| \Gamma(q+r+1)} + \frac{1}{|Q|} \\
&\qquad \times \left[\left|\alpha(1-\beta)\right| \left(\frac{1}{\Gamma(q+r+2)} + \frac{\eta^{q+r+1}}{\Gamma(q+r+2)}\right) \right. \\
&\qquad\quad \left. \left. + \frac{\left|1-\beta\right| \xi^{q+r}}{\Gamma(q+r+1)} + \frac{\left|\beta(1-\alpha(1-\eta))\right| \theta^{q+r}}{\Gamma(q+r+1)}\right]\right\},
\end{aligned} \tag{37}$$

for any $x \in \overline{\Omega}_{r_0}$. This, with the boundedness of the set $\mathscr{P}_1(\overline{\Omega}_{r_0})$, implies that the set $\mathscr{P}(\overline{\Omega}_{r_0})$ is bounded.

Step 4 (finally, it will be shown that the case (C2) in Lemma 10 does not hold). On the contrary, we suppose that (C2) holds. Then, we have that there exist $\lambda \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathscr{P} x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned}
x(t) &= \lambda A \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\
&\quad + \lambda A \frac{\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)}{Q} \\
&\quad \times \int_0^\theta \frac{(\theta - s)^{q-1}}{\Gamma(q)} f(s, x(s))\, ds \\
&\quad + \lambda A \frac{\alpha\beta\theta}{Q} \left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s, x(s))\, ds \right. \\
&\qquad \left. - \int_0^\eta \frac{(\eta - s)^q}{\Gamma(q+1)} f(s, x(s))\, ds\right) \\
&\quad - \lambda A \frac{\beta\theta}{Q} \int_0^\xi \frac{(\xi - s)^q}{\Gamma(q)} f(s, x(s))\, ds + \frac{\lambda A t}{Q}
\end{aligned}$$

$$\times \left[\alpha(1-\beta)\left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} f(s,x(s))\,ds - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} f(s,x(s))\,ds\right) - (1-\beta)\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q)} f(s,x(s))\,ds - \beta(1-\alpha(1-\eta))\int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \times f(s,x(s))\,ds\right]$$

$$+ \lambda B \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} g(u,x(u))\,du\,ds$$

$$+ \lambda B \frac{\beta\left(\xi - (\alpha/2)\left(1-\eta^2\right)\right)}{Q} \times \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} g(u,x(u))\,du\,ds$$

$$+ \lambda B \frac{\alpha\beta\theta}{Q}\left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \times g(u,x(u))\,du\,ds - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \times g(u,x(u))\,du\,ds\right)$$

$$- \lambda B \frac{\beta\theta}{Q} \int_0^\xi \frac{(\xi-s)^q}{\Gamma(q)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} g(u,x(u))\,du\,ds$$

$$+ \frac{\lambda B t}{Q}\left[\alpha(1-\beta) \times \left(\int_0^1 \frac{(1-s)^q}{\Gamma(q+1)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \times g(u,x(u))\,du\,ds - \int_0^\eta \frac{(\eta-s)^q}{\Gamma(q+1)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \times g(u,x(u))\,du\,ds\right) - (1-\beta)\int_0^\xi \frac{(\xi-s)^q}{\Gamma(q)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \times g(u,x(u))\,du\,ds - \beta(1-\alpha(1-\eta)) \times \int_0^\theta \frac{(\theta-s)^{q-1}}{\Gamma(q)} \int_0^s \frac{(s-u)^{r-1}}{\Gamma(r)} \times g(u,x(u))\,du\,ds\right]. \tag{38}$$

Using the assumptions (A_1) and (A_2), we get

$$r_0 \le k_0 K + p_0 \zeta(r_0)\|p\| + k_0 \ell r_0, \tag{39}$$

which leads to a contradiction:

$$\frac{r_0}{k_0 K + p_0 \zeta(r_0)\|p\|} \le \frac{1}{1-k_0\ell}. \tag{40}$$

Thus the operators $\mathscr{P}_1$ and $\mathscr{P}_2$ satisfy all the conditions of Lemma 10. Hence, the operator $\mathscr{P}$ has at least one fixed point $x \in \overline{\Omega}_{r_0}$, which is the solution of the problem (1). This completes the proof. □

Example 12. Consider a nonlocal integral boundary value problem of fractional integrodifferential equations given by

$$D^{3/2}x(t) = f(t,x(t)) + I^{3/4}g(t,x(t)), \quad t \in [0,1], \qquad x(0) = \frac{1}{2}x\left(\frac{1}{4}\right), \quad x\left(\frac{1}{3}\right) = \int_{2/3}^1 x(s)\,ds, \tag{41}$$

where $q = 3/2, A = B = 1, r = 3/4, \theta = 1/4, \xi = 1/3, \eta = 2/3, \alpha = 1, \beta = 1/2$, $f(t,x) = (1/27)(2\sqrt{(1+t)} - 1)\sin x$, and $g(t,x) = (1/(2+t)^3)(3 + (|x|/(1+|x|)))$.

Observe that $|f(t,x)| \le (1/27)(2\sqrt{(1+t)} - 1)x$, $|g(t,x) - g(t,y)| \le (1/8)|x-y|$, and $\sup_{t\in[0,1]} g(t,0) = 3/8$. Further, we set $p(t) = (1/3)(2\sqrt{(1+t)} - 1), \zeta(x) = x/9, \ell = 1/8$, and $K = 3/8$. With the given data, it is found that $k_0 \simeq 1.495606$, $p_0 \simeq 4.178968$, $|Q| = 1/9$, $\|p\| = (2\sqrt{2} - 1)/3$, and

$$\sup_{r\in(0,\infty)}\left\{\frac{r}{k_0 K + p_0\zeta(r)\|p\|}\right\} \simeq 3.533597, \qquad \frac{1}{1-k_0\ell} \simeq 1.229953. \tag{42}$$

Clearly, all the conditions of Theorem 11 are satisfied and hence there exists a solution for the problem (41).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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