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Research Article

Strong Convergence Theorems for Mixed Equilibrium Problem and Asymptotically I-Nonexpansive Mapping in Banach Spaces

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This paper aims to use a hybrid algorithm for finding a common element of a fixed point problem for a finite family of asymptotically nonexpansive mappings and the set solutions of mixed equilibrium problem in uniformly smooth and uniformly convex Banach space. Then, we prove some strong convergence theorems of the proposed hybrid algorithm to a common element of the above two sets under some suitable conditions.

1. Introduction

Let E be a Banach space with norm $\|\cdot\|$. Let C be a nonempty closed convex subset of E and E^* denoted the dual space of E. Let $B : C \to E^*$ be a nonlinear mapping and $\mathscr{H}$ a bifunction from $C \times C$ to R, where R denotes the set of numbers. The generalized equilibrium problem is to find $x \in C$ such that

$$\mathscr{H}(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solution of (1) is denoted by $\mathrm{GEP}(\mathscr{H}, B)$, that is,

$$\mathrm{GEP}(\mathscr{H}, B) := \{x \in C, \mathscr{H}(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C\}. \tag{2}$$

In this paper, we are interested in solving the generalized equilibrium problem with those $\mathscr{H}$ given by

$$\mathscr{H}(x, y) = \mathscr{F}(x, y) + \mathscr{G}(x, y), \tag{3}$$

where $\mathscr{F}, \mathscr{G} : C \times C \to R$ are two bifunctions satisfying the following special properties (f_1)–(f_4), (g_1)–(g_3) and (H):

(f_1) $\mathscr{F}(x, x) = 0$, for all $x \in C$;

(f_2) $\mathscr{F}$ is maximal monotone;

(f_3) for all $x, y, z \in C$, we have $\limsup_{t\to 0^+}(\mathscr{F}(tz + (1 - t)x, y)) \leq \mathscr{F}(x, y)$;

(f_4) for all $x \in C$, the function $y \mapsto \mathscr{F}(x, y)$ is convex and weakly lower semicontinuous;

(g_1) $\mathscr{G}(x, x) = 0$, for all $x \in C$;

(g_2) $\mathscr{G}$ is monotone and maximal monotone, and weakly upper semicontinuous in the first variable;

(g_3) $\mathscr{G}$ is convex in the second variable;

(H) for fixed $\lambda > 0$ and $x \in C$, there exist a bounded set $K \subset C$ and $a \in K$ such that

$$-\mathscr{F}(a, z) + \mathscr{G}(z, a) + \frac{1}{\lambda}\langle a - z, z - x\rangle < 0, \quad \forall z \in C \setminus K. \tag{4}$$

This is the well-know generalized mixed equilibrium problem, that is, to find an x in C such that

$$\mathscr{F}(x, y) + \mathscr{G}(x, y) + \langle Bx, y - x\rangle \geq 0, \quad \forall y \in C. \tag{5}$$

The solution set of (5) is denoted by $\mathrm{GMEP}(\mathscr{F}, \mathscr{G}, B)$, that is,

$$\mathrm{GMEP}(\mathscr{F}, \mathscr{G}, B) := \{x \in C, \mathscr{F}(x, y) + \mathscr{G}(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C\}. \tag{6}$$

If $B \equiv 0$, problem (5) reduces into mixed equilibrium problem for $\mathscr{F}$ and $\mathscr{G}$, denoted by $\mathrm{MEP}(\mathscr{F}, \mathscr{G})$, which is to find $x \in C$ such that (3).

If $\mathscr{G} = 0$ and $B \equiv 0$, reduces into equilibrium problem for $\mathscr{F}$, denoted by EP($\mathscr{F}$), which is to find $x \in C$ such that

$$\mathscr{F}(x, y) \geq 0, \quad \forall y \in C. \tag{7}$$

Mixed equilibrium problems are suitable and common format for investigation of various applied problems arising in economics, mathematical physics, transportation, communication systems, engineering, and other fields. Moreover, equilibrium problems are closely related with other general problems in nonlinear analysis, such as fixed points, game theory, variational inequality, and optimization problems. Recently, many authors studied a great number of iterative methods for solving a common element of the set of fixed points for a nonexpansive mapping and the set of solutions to a mixed equilibrium problem in the setting of Hilbert space and uniformly smooth and uniformly convex Banach space, respectively (please see, e.g., [1–11] and the references therein).

Let E be a real Banach space with norm $\|\cdot\|$, let C be a nonempty closed convex subset of E, and let J be the normalized duality mapping from E into E^* given by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\| \|x^*\|, \|x\| = \|x^*\|\}, \quad \forall x \in E, \tag{8}$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ the generalized duality pairing between E and E^*. It is easily known that if E^* is uniformly convex, then J is uniformly continuous on bounded subsets of E.

Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{9}$$

It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{10}$$

On the other hand, in a Hilbert space H, (9) reduced to $\phi(x, y) = \|x - y\|^2$. Following Alber [12], the generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \inf_{y \in C} \phi(y, x), \tag{11}$$

where is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$.

In 2011, Kim [13] considered the following shrinking projection methods to obtain a convergence theorem, and these methods were introduced in [14] for quasi-φ-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space.

Theorem 1 (see [13]). *Let E be a uniformly smooth and strictly convex Banach space which has the Kadec-Klee property and C a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (f_1)–(f_4) and $T : C \to C$ a closed and asymptotically quasi-φ-nonexpansive mapping. Assume that T is asymptotically regular on C and $\digamma = F_{ix}(T) \bigcap EF(f)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{aligned}
&\forall x_0 \in E, \quad C_1 = C, \quad x_1 = \prod_{C_1} x_0, \\
&y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T^n x_n), \\
&u_n \in \text{such that } f(u_n, x) + \frac{1}{r_n} \langle x - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall x \in C, \\
&C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) + (k_n - 1) M_n\}, \\
&x_{n+1} = \prod_{C_{n+1}} x_0,
\end{aligned} \tag{12}$$

where $M_n = \sup\{\phi(z, x_n) : z \in \digamma\}$ for each $n \geq 1$, $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$, and $\{r_n\}$ is a real sequence in $[a, \infty)$, where a is some positive real number and J is the duality mapping on E. Then the sequence $\{x_n\}$ converges strongly to $\prod_\digamma x_0$, where $\prod_\digamma$ is the generalized projection from E onto $\digamma$.

Motivated and inspired by the researches going on in this direction (i.e., [4–11, 13–16]), the purpose of this paper is to use the following hybrid algorithm for finding a common element of the set of solutions to a mixed equilibrium problem and the set of the set of common fixed points for a finite family of asymptotically nonexpansive mappings in a uniformly smooth and uniformly convex Banach space.

Algorithm 2. Let

$$\begin{aligned}
&u_n \in C \text{ such that} \\
&\mathscr{F}(u_n, y) + \mathscr{G}(u_n, y) \\
&\quad \leq \frac{1}{r_n} \langle y - u_n, J u_n - J x_n \rangle, \quad \forall y \in C, \\
&y_n = \beta_n x_n + (1 - \beta_n) T^n u_n, \\
&x_{n+1} = \alpha_n(x_n) + (1 - \alpha_n) I^n y_n, \\
&\qquad \forall n \geq 1.
\end{aligned} \tag{13}$$

Consequently, under suitable conditions, we show that iterative algorithms converge strongly to a solution of some optimization problem. Note that our methods do not use any projection.

2. Preliminaries

Let $T : C \to C$ be a mapping. Denote by $F_{ix}(T)$ the set of fixed points of T, that is, $F_{ix}(T) = \{x \in C : Tx = x\}$. Throughout this paper, we always assume that $F_{ix}(T) \neq \emptyset$. Now we need the following known definitions.

Definition 3. A mapping $T : C \to C$ is said to be

(1) nonexpansive, if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$;

(2) asymptotically nonexpansive, if there exists a sequence $\{\lambda_n\} \subset [1, \infty)$ with $\lim_{n\to\infty}\lambda_n = 1$ such that $\|T^n x - T^n y\| \leq \lambda_n \|x - y\|$, for all $x, y \in C$ and $n \in N$;

(3) quasi-nonexpansive, $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F_{ix}(T)$;

(4) asymptotically quasi-nonexpansive, if there exists a sequence $\{\mu_n\} \subset [1, \infty)$ with $\lim_{n\to\infty}\mu_n = 1$ such that $\|T^n x - p\| \leq \mu_n \|x - p\|$, for all $x, y \in C$, $p \in F_{ix}(T)$ and $n \in N$.

There are many concepts which generalize a notion of nonexpansive mapping. In 2004, Shahzad [17] introduced the following concepts about I-nonexpansivity of a mapping T.

Definition 4. Let $T : C \to C$ and $I : C \to C$ be two mappings of a nonempty subset C, a real normal linear space E. Then T is said to be

(i) I-nonexpansive, if $\|Tx - Ty\| \leq \|Ix - Iy\|$, for all x, $y \in C$;

(ii) asymptotically I-nonexpansive, if there exists a sequence $\{\lambda_n\} \subset [1, \infty)$ with $\lim_{n\to\infty}\lambda_n = 1$ such that $\|T^n x - T^n y\| \leq \lambda_n \|I^n x - I^n y\|$, for all $x, y \in C$ and $n \geq 1$;

(iii) asymptotically quasi-I-nonexpansive, if there exists a sequence $\{\mu_n\} \subset [1, \infty)$ with $\lim_{n\to\infty}\mu_n = 1$ such that $\|T^n x - p\| \leq \mu_n \|I^n x - p\|$, for all $x, y \in C$, $p \in F_{ix}(T) \bigcap F_{ix}(I)$ and $n \geq 1$.

Lemma 5 (see [4]). *Assume that $\psi : K \to R$ is convex, $x_0 \in \operatorname{core}_K C$, $\psi(x_0) \leq 0$, and $\psi(y) \geq 0$, for all $y \in C$. Then $\psi(y) \geq 0$, for all $y \in K$.*

Lemma 6 (see [18]). *Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, and let T be a relatively nonexpansive mapping from C into itself. Then $F_{ix}(T)$ is closed and convex.*

Lemma 7 (see [19]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\sigma_n\}$ be sequences of nonnegative real sequences satisfying the following conditions: for all $n \geq 1$*

(1) $a_n \leq a_n + b_n$,

(2) $a_n \leq (1 + \sigma_n)a_n + b_n$,

where $\sum_{n=1}^{\infty} \sigma_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then $\lim_{n\to\infty}a_n$ exists.

Lemma 8 (see [20]). *Let E be a uniformly convex Banach space. Then, for each $r > 0$, there exists a strictly increasing, continuous, and convex function $h : [0, 2r] \to R$ such that $h(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)h\left(\|x - y\|^2\right), \tag{14}$$

for $\forall x, y \in B_r$, $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 9 (see [21]). *Let E be a uniformly convex Banach space and let b, c be two constants with $0 < b < c < 1$. Suppose that $\{t_n\}$ is a sequence in $[b, c]$ and $\{x_n\}$, $\{y_n\}$ are two sequence in E such that*

$$\lim_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = d, \qquad \limsup_{n\to\infty}\|x_n\| \leq d, \qquad \limsup_{n\to\infty}\|y_n\| \leq d \tag{15}$$

holds some $d \geq 0$. Then $\lim\|x_n - y_n\| = 0$.

Definition 10 (see [22]). The mappings $T, I : C \to C$ are said to be satisfying condition (A) if there is a nondecreasing function $\mathfrak{f} : [0, \infty) \to [0, \infty)$ with $\mathfrak{f}(0) = 0$, $\mathfrak{f}(r) > 0$ for each $r \in [0, \infty)$ such that $(1/2)(\|x - Tx\| + \|x - Ix\|) \geq \mathfrak{f}(d(x, \Omega))$ for all $x \in C$, where $d(x, \Omega) = \inf\{\|x - p\| : p \in \Omega = F_{ix}(T) \bigcap F_{ix}(I)\}$.

Lemma 11 (see [23]). *Let E be a uniformly convex Banach space satisfying the Opial's condition, C a nonempty closed subset of E, and $T : C \to C$ an asymptotically nonexpansive mapping. If the sequence $\{x_n\} \subset C$ is a weakly convergent sequence with the weak limit p and if $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $Tp = p$.*

3. Main Results

Theorem 12. *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}, \mathcal{G} : C \times C \to R$ be two bifunctions which satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H). Then for every $x^* \in E^*$, there exists a unique point $z \in C$ such that*

$$0 \leq \mathcal{F}(z, y) + \mathcal{G}(z, y) + \frac{1}{r}\langle y - z, Jz - Jx^*\rangle, \quad \forall y \in C. \tag{16}$$

The proof goes over the following three steps.

Proof.
Step 1. There exists point $z \in C$ such that

$$\mathcal{F}(y, z) \leq \mathcal{G}(z, y) + \frac{1}{r}\langle y - z, Jz - Jx^*\rangle, \quad \forall y \in C. \tag{17}$$

Consider the closed sets

$$T_r(y) = \left\{z \in C \mid \mathcal{F}(y, z) \leq \mathcal{G}(z, y) + \frac{1}{r}\langle y - z, Jz - Jx^*\rangle, y \in C\right\}. \tag{18}$$

We will show that $\bigcap_{y\in C} T_r(y) \neq \emptyset$. Let y_i, $i \in N$, be a finite subset of C. Let $I \subset N$ be nonempty. Let for all $\xi \in \operatorname{conv}\{y_i \mid i \in N\}$. Then

$$\xi = \sum_{i\in I}\mu_i y_i \quad \text{with } \mu_i \geq 0 \ (i \in I), \ \sum_{i\in I}\mu_i = 1. \tag{19}$$

Assume, for contradiction, that

$$-\mathcal{F}(y_i, \xi) + \mathcal{G}(\xi, y_i) + \frac{1}{r}\langle y_i - \xi, J\xi - Jx^*\rangle < 0, \quad \forall i \in N. \tag{20}$$

By the convexity of $\mathcal{F}$ and $\mathcal{G}$ and the monotonicity of $\mathcal{F}$, we obtain that

$$\begin{aligned}0 &= \mathcal{F}(\xi,\xi) + \mathcal{G}(\xi,\xi) + \frac{1}{r}\langle \xi - \xi, J\xi - Jx^*\rangle \\ &\le \sum_{i\in I}\mu_i \mathcal{F}(\xi, y_i) + \sum_{i\in I}\mu_i \mathcal{G}(\xi, y_i) \\ &\quad + \frac{1}{r}\sum_{i\in I}\mu_i \langle y_i - \xi, J\xi - Jx^*\rangle \\ &\le -\sum_{i\in I}\mu_i \mathcal{F}(y_i, \xi) + \sum_{i\in I}\mu_i \mathcal{G}(\xi, y_i) \\ &\quad + \frac{1}{r}\sum_{i\in I}\mu_i \langle y_i - \xi, J\xi - Jx^*\rangle \\ &= \sum_{i\in I}\mu_i \Big[-\mathcal{F}(y_i,\xi) + \mathcal{G}(\xi, y_i) \\ &\qquad + \frac{1}{r}\langle y_i - \xi, J\xi - Jx^*\rangle \Big] < 0,\end{aligned} \tag{21}$$

and that is absurd. Hence (20) cannot be true. and we have $\mathcal{F}(y_i,\xi) \le \mathcal{G}(\xi, y_i) + (1/r)\langle y_i - \xi, J\xi - Jx^*\rangle$ for some $i \in I$. Thus $\xi \in \bigcap_{y\in C} T_r(y_i)$ for some $i \in N$. Since for all $\xi \in \operatorname{conv}\{y_i \mid i \in N\}$, it follows that

$$\operatorname{conv}\{y_i \mid i \in N\} \subset \{T_r(y_i) \mid i \in N\}. \tag{22}$$

By the sets $T_r(y_i)$ being closed, it follows form the standard version of the KKM-Theorem that

$$\bigcap_{i\in N} T_r(y_i) \ne \emptyset. \tag{23}$$

In other words, any finite subfamily of the family $T_r(y)_{y\in C}$ has nonempty intersection. Since these sets are closed subsets of the compact set C, it follows that the entire family has nonempty intersection. Hence

$$\bigcap_{y\in C} T_r(y) \ne \emptyset. \tag{24}$$

Step 2. For every $x^* \in E^*$, the following statement are equivalent:

(i) $z \in C$, $\mathcal{F}(y,z) \le \mathcal{G}(z,y) + \langle y - z, Jz - Jx^*\rangle$, for all $y \in C$,

(ii) $z \in C$, $0 \le \mathcal{F}(z,y) + \mathcal{G}(z,y) + \langle y - z, Jz - Jx^*\rangle$, for all $y \in C$.

Case 1. Let (ii) hold; since $\mathcal{F}$ is monotone, one has

$$\mathcal{F}(z,y) \le -\mathcal{F}(y,z). \tag{25}$$

Hence (i) follows.

Case 2. Let (i) hold, for t with $0 < t \le 1$ and $y \in C$, and let

$$x_t = ty + (1-t)z. \tag{26}$$

Then $x_t \in C$, and from (i), $\mathcal{F}(x_t, z) \le \mathcal{G}(z, x_t) + \langle x_t - z, Jz - Jx^*\rangle$. By the properties of $\mathcal{F}$ and $\mathcal{G}$, it follows then, for all $0 < t \le 1$,

$$\begin{aligned}0 &= \mathcal{F}(x_t, x_t) + \mathcal{G}(x_t, x_t) + \langle x_t - x_t, Jz - Jx^*\rangle \\ &\le t\mathcal{F}(x_t, y) + (1-t)\mathcal{F}(x_t, z) \\ &\quad + t\mathcal{G}(x_t, y) + (1-t)\mathcal{G}(x_t, z) \\ &\le \mathcal{F}(x_t, y) + \mathcal{G}(x_t, y).\end{aligned} \tag{27}$$

Let $t \to 0$ and thereby $x_t \to z$ and using the hemicontinuity of $\mathcal{F}$ we obtain in the limit

$$0 \le \mathcal{F}(z,y) + \mathcal{G}(z,y) + \langle y - z, Jz - Jx^*\rangle. \tag{28}$$

Step 3. Take $\psi(\cdot) = \mathcal{F}(z,\cdot) + \mathcal{G}(z,\cdot) + \langle \cdot - z, Jz - Jx^*\rangle$. Then the function $\psi(\cdot)$ is convex and $\psi(y) \ge 0$, for all $y \in C$. If $z \in \operatorname{core}_K C$, then set $x_0 = z$. If $z \in C \setminus \operatorname{core}_K C$, then set $x_0 = a$, where a is as in assumption H for $x = z$. In both cases $x_0 \in \operatorname{core}_K C$, and $\psi(x_0) \le 0$. Hence it follows from the Lemma 5 that

$$\begin{aligned}&\psi(y) \ge 0 \quad \forall y \in C, \\ &\text{that is, } \mathcal{F}(z,y) + \mathcal{G}(z,y) + \langle y - z, Jz - Jx^*\rangle \ge 0, \\ &\qquad \forall y \in K.\end{aligned} \tag{29}$$

□

Corollary 13. *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}, \mathcal{G} : C \times C \to R$ be two bifunctions which satisfy the following conditions: (f_1)–(f_4), (g_1)–(g_3), and (H) in Theorem 12. There for every $x^* \in E$ and $r > 0$, there exists a unique point $z_r \in C$ such that*

$$0 \le \mathcal{F}(z_r, y) + \mathcal{G}(z_r, y) + \frac{1}{r}\langle y - z_r, Jz_r - Jx^*\rangle, \quad \forall y \in C. \tag{30}$$

Proof. Let $x \in E$ and $r > 0$ be given. Note that functions $r\mathcal{F}$ and $r\mathcal{G}$ also satisfy the conditions (f_1)–(f_4) and (g_1)–(g_3). Therefore, for $Jx^* \in E^*$, there exists a unique point $z_r \in C$ such that

$$r\mathcal{F}(z_r, y) + r\mathcal{G}(z_r, y) + \langle y - z_r, Jz_r - Jx^*\rangle \ge 0, \quad \forall y \in C. \tag{31}$$

This completes the proof. □

Under the same assumptions in Corollary 13, for every $r > 0$, we may define a single-valued mapping $S_r : E \to C$ as follows:

$$\begin{aligned}S_r(x) = \Big\{ z \in C \mid 0 \le \mathcal{F}(z,y) + \mathcal{G}(z,y) \\ + \frac{1}{r}\langle y - z, Jz - Jx\rangle, \forall y \in C \Big\},\end{aligned} \tag{32}$$

for $x \in E$, which is called the resolvent of $\mathcal{F}$ and $\mathcal{G}$ for r.

Theorem 14. *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Let $\mathscr{F}, \mathscr{G} : C \times C \rightarrow R$ be two bifunctions which satisfy conditions (f_1)–(f_4), (g_1)–(g_3), and (H). For $r > 0$ and $x \in E$, define a mapping S_r in (32). Then, the following hold:*

(a) S_r *is single-valued;*

(b) S_r *is a firmly nonexpansive mapping, that is,*

$$\langle S_r x - S_r y, JS_r x - JS_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle, \quad \forall x, y \in E; \tag{33}$$

(c) $F_{ix}(S_r) = \mathrm{MEP}(\mathscr{F}, \mathscr{G})$;

(d) $\mathrm{MEP}(\mathscr{F}, \mathscr{G})$ *is closed and convex;*

(e) $\phi(p, S_r x) + \phi(S_r x, x) \leq \phi(p, x)$.

Proof. We divide the proof into several steps.

Step 1 (S_r is single-valued). Indeed, for $x \in C$ and $r > 0$, let $z_1, z_2 \in S_r x$. Then

$$\begin{gathered} \mathscr{F}(z_1, z_2) + \mathscr{G}(z_1, z_2) + \frac{1}{r}\langle z_2 - z_1, Jz_1 - Jx \rangle \geq 0, \\ \mathscr{F}(z_2, z_1) + \mathscr{G}(z_2, z_1) + \frac{1}{r}\langle z_1 - z_2, Jz_1 - Jx \rangle \geq 0. \end{gathered} \tag{34}$$

Adding the two inequalities, we obtain

$$\begin{aligned} &\mathscr{F}(z_1, z_2) + \mathscr{F}(z_2, z_1) + \mathscr{G}(z_1, z_2) + \mathscr{G}(z_2, z_1) \\ &\quad + \frac{1}{r}\langle z_1 - z_2, Jz_1 - Jz_2 \rangle \geq 0. \end{aligned} \tag{35}$$

From (f_2), (g_2), and $r > 0$, we obtain

$$\frac{1}{r}\langle z_1 - z_2, Jz_1 - Jz_2 \rangle \geq 0. \tag{36}$$

Since E is strictly convex, we obtain

$$z_1 = z_2. \tag{37}$$

Step 2 (S_r is a firmly nonexpansive mapping). For $x, y \in C$, we obtain

$$\begin{gathered} \mathscr{F}(S_r x, S_r y) + \mathscr{G}(S_r x, S_r y) + \frac{1}{r}\langle S_r y - S_r x, JS_r x - Jx \rangle \geq 0, \\ \mathscr{F}(S_r y, S_r x) + \mathscr{G}(S_r y, S_r x) + \frac{1}{r}\langle S_r x - S_r y, JS_r y - Jy \rangle \geq 0. \end{gathered} \tag{38}$$

Adding the two inequalities, we obtain

$$\begin{aligned} &\mathscr{F}(S_r x, S_r y) + \mathscr{F}(S_r y, S_r x) + \mathscr{G}(S_r x, S_r y) + \mathscr{G}(S_r y, S_r x) \\ &\quad + \frac{1}{r}\langle S_r y - S_r x, JS_r x - JS_r y - Jx + Jy \rangle \geq 0. \end{aligned} \tag{39}$$

From (f_2), (g_2), and $r > 0$, we obtain

$$\langle S_r y - S_r x, JS_r x - JS_r y - Jx + Jy \rangle \geq 0. \tag{40}$$

Therefore, we have

$$\langle S_r x - S_r y, JS_r x - JS_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle. \tag{41}$$

Step 3 ($F_{ix}(S_r) = \mathrm{MEP}(\mathscr{F}, \mathscr{G})$). Indeed, we obtain the following equation:

$$\begin{aligned} u \in F_{ix}(S_r) &\Longleftrightarrow u = S_r u \\ &\Longleftrightarrow \mathscr{F}(u, y) + \mathscr{G}(u, y) \\ &\qquad + \frac{1}{r}\langle y - u, Ju - Ju \rangle \geq 0, \quad \forall y \in C, \\ &\Longleftrightarrow \mathscr{F}(u, y) + \mathscr{G}(u, y), \quad \forall y \in C, \\ &\Longleftrightarrow u \in \mathrm{MEP}(\mathscr{F}, \mathscr{G}). \end{aligned} \tag{42}$$

Step 4 ($\mathrm{MEP}(\mathscr{F}, \mathscr{G})$ is closed and convex). From (c), we have $\mathrm{MEP}(\mathscr{F}, \mathscr{G}) = F_{ix}(S_r)$, and from (b), we obtain

$$\langle S_r x - S_r y, JS_r x - JS_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle, \quad x, y \in C. \tag{43}$$

Moreover, we obtain

$$\begin{aligned} &\phi(S_r x, S_r y) + \phi(S_r y, S_r x) \\ &\quad = 2\|S_r x\|^2 - 2\langle S_r x, JS_r y \rangle \\ &\qquad - 2\langle S_r y, JS_r x \rangle + 2\|S_r y\|^2 \\ &\quad = 2\langle S_r x, S_r x - JS_r y \rangle \\ &\qquad + 2\langle S_r y, S_r y - JS_r x \rangle \\ &\quad = 2\langle S_r x - S_r y, S_r x - JS_r y \rangle, \\ &\phi(S_r x, y) + \phi(S_r y, x) - \phi(S_r x, x) - \phi(S_r y, y) \\ &\quad = \|S_r x\|^2 - 2\langle S_r x, Jy \rangle + \|y\|^2 \\ &\qquad + \|S_r y\|^2 - 2\langle S_r y, Jx \rangle + \|x\|^2 \\ &\qquad - \|S_r x\|^2 + 2\langle S_r x, Jx \rangle - \|y\|^2 \\ &\qquad - \|S_r y\|^2 + 2\langle S_r y, Jy \rangle - \|x\|^2 \\ &\quad = 2\langle S_r x, Jx - Jy \rangle + 2\langle S_r y, Jy - Jx \rangle \\ &\quad = 2\langle S_r x - S_r y, Jx - Jy \rangle. \end{aligned} \tag{44}$$

Hence, we obtain

$$\begin{aligned} &\phi(S_r x, S_r y) + \phi(S_r y, S_r x) \\ &\quad \leq \phi(S_r x, y) + \phi(S_r y, x) - \phi(S_r x, x) - \phi(S_r y, y). \end{aligned} \tag{45}$$

So we get

$$\phi(S_r x, S_r y) + \phi(S_r y, S_r x) \leq \phi(S_r x, y) + \phi(S_r y, x). \quad (46)$$

Taking $y = u \in F_{ix}(S_r)$, we obtain

$$\phi(u, S_r x) \leq \phi(u, x). \quad (47)$$

Next, we show that $\widehat{F}_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G})$. Let $p \in \widehat{F}_{ix}(S_r)$. Then, there exists the sequence of $\{z_n \in E\}$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty}(z_n - S_r z_n) = 0$. Moreover, we obtain $S_r z_n \rightharpoonup p$. Hence we have $p \in C$. Since J is uniformly continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Jz_n - JS_r z_n\| = 0. \quad (48)$$

Form the definition of S_r, we obtain

$$\mathcal{F}(S_r z_n, y) + \mathcal{G}(S_r z_n, y) + \frac{1}{r}\langle y - S_r z_n, JS_r z_n - Jz_n\rangle \geq 0. \quad (49)$$

Since the monotone of the $\mathcal{F}$, we have

$$\mathcal{G}(S_r z_n, y) + \frac{1}{r}\langle y - S_r z_n, JS_r z_n - Jz_n\rangle \geq -\mathcal{F}(S_r z_n, y) = \mathcal{F}(y, S_r z_n). \quad (50)$$

According to (48) and $z_n \rightharpoonup p$ and form (f_3) and (g_2), we obtain

$$\mathcal{F}(y, p) \leq \mathcal{G}(p, y), \quad \forall y \in C. \quad (51)$$

For t with $0 < t \leq 1$ and $y \in H$, let $x_t = ty + (1-t)p$; then by the convexity of $\mathcal{F}$ and $\mathcal{G}$ we have

$$\begin{aligned} 0 &= \mathcal{F}(x_t, x_t) + \mathcal{G}(x_t, x_t) \\ &\leq t\mathcal{F}(x_t, y) + (1-t)\mathcal{F}(x_t, p) \\ &\quad + t\mathcal{G}(x_t, y) + (1-t)\mathcal{G}(x_t, \omega) \\ &\leq t\mathcal{F}(x_t, y) + t\mathcal{G}(x_t, y). \end{aligned} \quad (52)$$

Passing $t \to 0^+$ and by (f_1) and (g_1), we have $0 \leq \mathcal{F}(p, y) + \mathcal{G}(p, y)$ for all $y \in H$. Therefore, $p \in \mathrm{MEP}(\mathcal{F}, \mathcal{G})$. So, we get $F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G}) = \widehat{F}_{ix}(S_r)$. Therefore, we have that S_r is a relatively nonexpansive mapping. From Lemma 6, then $F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G})$ is closed and convex.

Step 5 ($\phi(p, S_r x) + \phi(S_r x, x) \leq \phi(p, x)$). From (b) and (45), for each x, $y \in E$, we obtain

$$\phi(S_r x, S_r y) + \phi(S_r y, S_r x) \leq \phi(S_r x, y) + \phi(S_r y, x) - \phi(S_r x, x) - \phi(S_r y, y). \quad (53)$$

Letting $y = p \in F_{ix}(S_r)$, we obtain

$$\phi(p, S_r x) + \phi(S_r x, x) \leq \phi(p, x). \quad (54)$$

□

If $\mathcal{G}(x, y) = \psi(x) - \psi(y)$ and form Theorems 12 and 14, we obtain the following corollary.

Corollary 15 (see [24]). *Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty closed convex subset of E. Let $\mathcal{F} : C \times C \to R$ be a bifunctions which satisfy conditions (f_1)–(f_4). Let $\psi : C \to R$ be a lower semi-continuous and convex function. For $r > 0$ and $x \in E$. Then, the following hold:*

(i) $0 \leq \mathcal{F}(z, y) + \psi(y) - \psi(z) + (1/r)\langle y - z, Jz - Jx^*\rangle$, *for all $y \in C$.*

(ii) *If we define a mapping $S_r : E \to C$ as follows:*

$$S_r(x) = \Big\{x \in C \mid \leq 0 \leq F(z, y) + \psi(y) - \psi(z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle, y \in C\Big\}, \quad (55)$$

and the mapping S_r has the following properties:

(a) *S_r is single-valued;*

(b) *S_r is a firmly nonexpansive mapping, that is,*

$$\langle S_r z - S_r y, JS_r z - JS_r y\rangle \leq \langle S_r z - S_r y, Jz - Jy\rangle, \quad \forall z, y \in E; \quad (56)$$

(c) *$F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \psi)$;*

(d) *$\mathrm{MEP}(\mathcal{F}, \psi)$ is closed and convex;*

(e) *$\phi(p, S_r z) + \phi(S_r z, z) \leq \phi(p, z)$.*

4. Strong Convergence Theorems

In this section, we introduce a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problems and the set of fixed points for I-asymptotically nonexpansive mapping in Banach spaces.

Theorem 16. *Let E be uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}$, $\mathcal{G} : C \times C \to R$ be two bifunctions which satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H), and let T be I-asymptotically nonexpansive self-mapping of C with sequences $\{s_n\} \subset [0, \infty\}$ such that $\sum_{n=1}^{\infty} s_n < \infty$, and let I be asymptotically nonexpansive self-mapping of C with sequences $\{t_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} t_n < \infty$, and $\Omega = F_{ix}(I) \bigcap F_{ix}(T) \bigcap \mathrm{MEP}(\mathcal{F}, \mathcal{G}) \neq \emptyset$. For an initial point $x_0 \in C$, generate a sequence $\{x_n\}$ by*

$$\begin{aligned} &u_n \in C \\ &\text{such that } \mathcal{F}(u_n, y) + \mathcal{G}(u_n, y) \leq \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle, \\ &\qquad \forall y \in C, \\ &y_n = \beta_n x_n + (1-\beta_n) T^n u_n, \\ &x_{n+1} = \alpha_n(x_n) + (1-\alpha_n) I^n y_n, \quad \forall n \geq 1, \end{aligned} \quad (57)$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$, $\{\beta_n\} \subset [a,b]$ *for some* a, $b \in (0,1)$ *and* $\{r_n\} \subset [d,+\infty)$ *for* $d > 0$. *If the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\sum_{n=0}^{\infty} \beta_n < \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$,

then the sequence $\{x_n\}$ *generated by* (57) *converges strongly to a fixed point in* Ω *if and only if*

$$\liminf_{n\to\infty} d(x_n,\Omega) = 0. \tag{58}$$

Proof. We divide the proof into several steps.

Step 1 (The sequence $\{x_n\}$ is bounded). Let $u_n = T_{r_n}x_n$. Since T is a I-asymptotically nonexpansive mapping, it follows from and Theorem 14 that $\Omega := F_{ix}(T)\bigcap F_{ix}(I)\bigcap \mathrm{MEP}(\mathcal{F},\mathcal{G})$ is nonempty closed convex subset E and for each $p \in \Omega$.

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\quad = \|\alpha_n(x_n)+(1-\alpha_n)I^n y_n - p\| \\
&\quad = \alpha_n\|x_n-p\| + (1-\alpha_n)\|I^n y_n - p\| \\
&\quad \le \alpha_n\|x_n-p\| + (1-\alpha_n)(1+t_n)\|y_n-p\|.
\end{aligned} \tag{59}$$

Again from (57), we obtain that

$$\begin{aligned}
\|y_n-p\| &= \|\beta_n x_n + (1-\beta_n)T^n u_n - p\| \\
&\le \beta_n\|x_n-p\| + (1-\beta_n)\|T^n u_n - p\| \\
&\le \beta_n\|x_n-p\| + (1-\beta_n) \\
&\quad \times (1+s_n)\|I^n u_n - p\| \\
&\le \beta_n\|x_n-p\| + (1-\beta_n) \\
&\quad \times (1+s_n)(1+t_n)\|T_r x_n - p\| \\
&\le \beta_n\|x_n-p\| + (1-\beta_n) \\
&\quad \times (1+s_n)(1+t_n)\|x_n-p\| \\
&= [1+(1-\beta_n)(s_n+t_n+s_nt_n)]\|x_n-p\|.
\end{aligned} \tag{60}$$

From (59) and (60), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\| \\
&\quad \le \alpha_n\|x_n-p\| + (1-\alpha_n)(1+t_n) \\
&\qquad \times [1+(1-\beta_n)(s_n+t_n+s_nt_n)]\|x_n-p\| \\
&\quad \le \alpha_n\|x_n-p\| + (1-\alpha_n)(1+t_n) \\
&\qquad \times [1+(1-\beta_n)(s_n+t_n+s_nt_n)]\|x_n-p\| \\
&\quad \le (1+\rho_n)\|x_n-p\|,
\end{aligned} \tag{61}$$

where

$$\begin{aligned}
\rho_n &= (1-\alpha_n)(1-\beta_n)(s_n+t_n+s_nt_n) \\
&\quad + (1-\alpha_n)t_n + (1-\alpha_n)(1-\beta_n) \\
&\quad \times t_n(s_n+t_n+s_nt_n).
\end{aligned} \tag{62}$$

Moreover since $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [a,b]$ for some a, $b \in (0,1)$, $\sum_{n=1}^{\infty} s_n < \infty$, and $\sum_{n=1}^{\infty} t_n < \infty$, it follow that $\sum_{n=1}^{\infty}\rho_n < \infty$. Form (60) and, by Lemma 7, we obtain that the limit of $\{\|x_n-p\|\}$ exists for each $p \in \Omega$. This implies that $\{\|x_n-p\|\}$ is bounded and so are $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{I^n y_n\}$, and $\{T^n u_n\}$; on the other hand, we obtain that $d(x_{n+1},\Omega) \le (1+\rho_n)d(x_n,\Omega)$. Then by Lemma 7, $\lim_{n\to\infty} d(x_n,\Omega)$ exists and, by assumption $\liminf_{n\to\infty} d(x_n,\Omega) = 0$, we obtain

$$\lim_{n\to\infty} d(x_n,\Omega) = 0. \tag{63}$$

Step 2 ($\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$). Taking lim sup on both sides in the above inequality,

$$\lim_{n\to\infty}\|y_n-p\| = d. \tag{64}$$

Since I^n is asymptotically nonexpansive self-mappings of C, we can get that $\|I^n y_n - p\| \le (1+t_n)\|y_n-p\|$, which on taking $\limsup_{n\to\infty}$ and using (64), we obtain

$$\limsup_{n\to\infty}\|I^n y_n - p\| \le d. \tag{65}$$

Further,

$$\lim_{n\to\infty}\|x_{n+1}-p\| \le d. \tag{66}$$

That means that

$$\begin{aligned}
&\lim_{n\to\infty}\|\alpha_n(x_n)+(1-\alpha_n)I^n y_n - p\| \le d, \\
&\lim_{n\to\infty}\alpha_n\|(x_n)-p\| + (1-\alpha_n)\|I^n y_n - p\| \le d.
\end{aligned} \tag{67}$$

It follows from Lemma 9 that

$$\lim_{n\to\infty}\|I^n y_n - x_n\| = 0. \tag{68}$$

Moveover,

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|(1-\alpha_n)(I^n y_n - x_n)\| \\
&= (1-\alpha_n)\|I^n y_n - x_n\|.
\end{aligned} \tag{69}$$

Thus, from (68), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{70}$$

Step 3 ($\lim_{n\to\infty}\|x_n - T^n u_n\| = 0$). Use (57) again, and Lemma 8 that for $r = \sup_{n\ge1}\{\|x_n\|, \|u_n\|, \|T^n u_n\|\}$, there exists a strictly increasing, continuous and convex function $h : [1,2] \to R$ that $h(0)=0$ and

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|I^n y_n - p\|^2 \\
&\qquad - \alpha_n(1-\alpha_n)h\left(\|x_n - I^n y_n\|^2\right) \\
&\quad \le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)(1+t_n)\|y_n-p\|^2 \\
&\qquad - \alpha_n(1-\alpha_n)h\left(\|x_n - I^n y_n\|^2\right)
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(1+t_n) \\
&\quad \times \left(\beta_n\|x_n - p\|^2 + (1-\beta_n)\|T^n u_n - p\|^2 \right.\\
&\qquad \left. -\beta_n(1-\beta_n)h\left(\|x_n - T^n u_n\|^2\right)\right) \\
&\quad - \alpha_n(1-\alpha_n)h\left(\|x_n - I^n y_n\|^2\right) \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(1+t_n) \\
&\quad \times \left(\beta_n\|x_n - p\|^2 + (1-\beta_n)(1+s_n)\|u_n - p\|^2 \right.\\
&\qquad \left. -\beta_n(1-\beta_n)h\left(\|x_n - T^n u_n\|^2\right)\right) \\
&\quad - \alpha_n(1-\alpha_n)h\left(\|x_n - I^n y_n\|^2\right) \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n) \\
&\quad \times (1+t_n)\left(\beta_n\|x_n - p\|^2 \right.\\
&\qquad + (1-\beta_n)(1+s_n)\|x_n - p\|^2 \\
&\qquad \left. -\beta_n(1-\beta_n)h\left(\|x_n - T^n u_n\|^2\right)\right) \\
&\quad - \alpha_n(1-\alpha_n)h\left(\|x_n - I^n y_n\|^2\right) \\
&\leq (1+\rho_n)\|x_n - p\|^2 - (1-\alpha_n) \\
&\quad \times (1+t_n)\beta_n(1-\beta_n)h\left(\|x_n - T^n u_n\|^2\right) \\
&\quad - \alpha_n(1-\alpha_n)h\left(\|x_n - I^n y_n\|^2\right),
\end{aligned} \tag{71}$$

where

$$\begin{aligned}
\rho_n &= (1-\alpha_n)(1-\beta_n) \\
&\quad \times (s_n + t_n + s_n t_n) + (1-\alpha_n)t_n + (1-\alpha_n) \\
&\quad \times (1-\beta_n)t_n(s_n + t_n + s_n t_n).
\end{aligned} \tag{72}$$

From the discuss of the Step 1, we can easily know that $\sum_{n=1}^{\infty}\rho_n < \infty$. On the other hand, by (71) and the bounded sequence of $\{x_n\}$, we obtain that

$$\begin{aligned}
&(1-\alpha_n)(1+t_n)\beta_n(1-\beta_n)h(\|x_n - T^n u_n\|) \\
&\quad \leq \phi(x_n, p) - \phi(x_{n+1}, p) + \rho_n\phi(x_n, p).
\end{aligned} \tag{73}$$

From $\lim_{n\to\infty}h(\|x_n - T^n u_n\|) = 0$, (73) and the property of h, we have

$$\lim_{n\to\infty}\|x_n - T^n u_n\| = 0. \tag{74}$$

The same as the proof of (74), we can easily obtain that

$$\lim_{n\to\infty}\|x_n - I^n y_n\| = 0. \tag{75}$$

From (57), we obtain that

$$\|y_n - x_n\| \leq (1-\beta_n)\|x_n - T^n u_n\|. \tag{76}$$

It follows that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{77}$$

Step 4 ($\lim_{n\to\infty}\|x_n - u_n\| = 0$). Let $p \in \Omega = F_{ix}(I)\bigcap F_{ix}(T)\bigcap \mathrm{MEP}(\mathcal{F},\mathcal{G})$. Then, from (59) and (60), it follows that

$$\begin{aligned}
\|u_{n+1} - p\| &= \left\|T_{r_{n+1}}x_{n+1} - p\right\| \\
&\leq \|x_{n+1} - p\| \\
&\leq \|\alpha_n x_n + (1-\alpha_n)I^n y_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n) \\
&\quad \times (1+t_n)\|y_n - p\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)(1+t_n) \\
&\quad \times [\beta_n\|x_n - p\| + (1-\beta_n) \\
&\qquad \times (1+s_n)(1+t_n)\|u_n - p\|] \\
&\leq [\alpha_n + (1-\alpha_n)(1+t_n)\beta_n] \\
&\quad \times \|x_n - p\| + (1-\alpha_n)(1-\beta_n) \\
&\quad \times (1+s_n)(1+t_n)^2\|u_n - p\| \\
&\leq M_1\|x_n - p\| + (1+M_2)\|u_n - p\|,
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
M_1 &= \alpha_n + (1-\alpha_n)(1+t_n)\beta_n, \\
M_2 &= [t_n(2+t_n)(1+s_n) + s_n](\alpha_n\beta_n - \alpha_n - \beta_n) \\
&\quad + (t_n(2+t_n)(1+s_n) + s_n) \\
&\quad + (\alpha_n\beta_n - \alpha_n - \beta_n).
\end{aligned} \tag{79}$$

Moreover since $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [a,b]$ for some $a, b \in (0,1)$, $\sum_{n=1}^{\infty}s_n < \infty$ and $\sum_{n=1}^{\infty}t_n < \infty$, we can easily claim that $\sum_{n=1}^{\infty}M_1 < \infty$ and $\sum_{n=1}^{\infty}M_2 < \infty$. By Lemma 7, we obtain that $\lim_{n\to\infty}\|u_n - p\|$ exists and from Theorem 14(b) and (78), we have

$$\begin{aligned}
\|x_n - u_n\| &\leq \|x_n - p\| - \left\|T_{r_n}x_n - p\right\| \\
&= \|x_n - p\| - \|u_n - p\| \\
&\leq M_1\|x_{n-1} - p\| + (1+M_2) \\
&\quad \times \|u_{n-1} - p\| - \|u_n - p\| \\
&\leq M_1\|x_{n-1} - p\| + M_2\|u_{n-1} - p\| \\
&\quad + \|u_{n-1} - p\| - \|u_n - p\|.
\end{aligned} \tag{80}$$

Thus, since $\{u_n\}$ converges, $\sum_{n=1}^{\infty}M_1 < \infty$ and $\sum_{n=1}^{\infty}M_2 < \infty$ and $\{x_n\}$ is bounded, it follows form Lemma 7 that

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{81}$$

Step 5 ($\lim_{n\to\infty}\|u_n - T^n u_n\| = 0$). By using the triangle inequality, we have

$$\|T^n u_n - u_n\| \le \|T^n u_n - x_n\| + \|x_n - u_n\|. \quad (82)$$

Thus, from (74) and (81), we obtain that

$$\lim_{n\to\infty} \|u_n - T^n u_n\| = 0. \quad (83)$$

Step 6 ($\lim_{n\to\infty}\|x_n - T^n x_n\| = \lim_{n\to\infty}\|x_n - I^n x_n\| = 0$). By using the triangle inequality again, we obtain

$$\begin{aligned}\|x_n - T^n x_n\| &\le \|x_n - T^n u_n\| + \|T^n x_n - T^n u_n\| \\ &\le \|x_n - T^n u_n\| + (1 + s_n)\|x_n - u_n\|.\end{aligned} \quad (84)$$

From (74) and (81), we have

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \quad (85)$$

From (57), we have

$$\begin{aligned}\|x_n - I^n x_n\| &\le \|x_n - I^n y_n\| + \|I^n x_n - I^n y_n\| \\ &\le \|x_n - I^n y_n\| + (1 + t_n)\|x_n - y_n\| \\ &\le \|x_n - I^n y_n\| + (1 + t_n) \\ &\quad \times (1 - \alpha_n)\|x_n - I^n y_n\|.\end{aligned} \quad (86)$$

From $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty} t_n < \infty$, and (68), we obtain

$$\lim_{n\to\infty} \|x_n - I^n x_n\| = 0. \quad (87)$$

Step 7 ($x^* \in \Omega = F_{ix}(I) \bigcap F_{ix}(T) \bigcap \mathrm{MEP}(\mathcal{F}, \mathcal{G})$). Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $x^* \in C$ when $x^* = J^{-1}p^*$ for some $p^* \in J(C)$. From (61), we have that $\{x_{n_k}\}$ converges weakly to $x^* \in C$ and, by (77), we also have that $\{y_{n_k}\}$ converges weakly to $x^* \in C$. Also, by (85), (87), and Lemma 11, we obtain that $x^* \in F_{ix}(I) \bigcap F_{ix}(T)$.

Next, we show that $x^* \in \mathrm{MEP}(\mathcal{F}, \mathcal{G})$; that is, $Jx^* = p \in J(\mathrm{MEP}(\mathcal{F}, \mathcal{G}))$. Since J is uniformly norm-to-norm continuous on bounded subset of E, it follows from (61) that

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \quad (88)$$

From the assumption $r_n \in [d, \infty)$, one sees

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0. \quad (89)$$

Since $\{x_n\}$ is bounded and so is $\{Jx_n\}$, there exists a subsequence $\{Jx_{n_k}\}$ of $\{Jx_n\}$ such that $\{Jx_n \rightharpoonup p^*\}$. Since $\{u_n\}$ is bounded, by (89), we also obtain $\{Ju_n \rightharpoonup p^*\}$. Noticing that $u_n = T_{r_n}x_n$, we obtain

$$\begin{aligned}&\mathcal{F}(u_n, y) \le \mathcal{G}(y, u_n) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle, \quad y \in C, \\ &\mathcal{F}(u_{n_k}, y) \le \mathcal{G}(y, u_{n_k}) + \left\langle y - u_{n_k}, \frac{Ju_{n_k} - Jx_{n_k}}{r_n}\right\rangle, \\ &\qquad\qquad y \in C.\end{aligned} \quad (90)$$

According to (89), we obtain $\lim_{k\to\infty}(\|Jx_{n_k} - Ju_{n_k}\|/r_{n_k}) = 0$. Then, by the conditions of (f_2) and (h_2), we obtain

$$\begin{aligned}\frac{1}{r_n}\|y - u_n\|\|Jx_n - Ju_n\| &\ge \langle y - u_n, Ju_n - Jx_n\rangle \\ &\ge -\mathcal{F}(u_n, y) + \mathcal{G}(y, u_n) \\ &\ge \mathcal{F}(y, u_n) + \mathcal{G}(y, u_n).\end{aligned} \quad (91)$$

Since $(1/r_n)\|Jx_n - Ju_n\| \to 0$ and $\{Ju_n \rightharpoonup p^*\}$, we obtain

$$\mathcal{F}(y, p^*) + \mathcal{G}(y, p^*) \le 0. \quad (92)$$

For t with $0 < t \le 1$ and $y \in E$, let $y_t = ty + (1 - t)p^*$, we obtain

$$\mathcal{F}(y_t, p^*) + \mathcal{G}(y_t, p^*) \le 0. \quad (93)$$

So, from the conditions of (f_1), (f_3), (h_1), and (h_3), we have

$$\begin{aligned}0 &= \mathcal{F}(y_t, y_t) + \mathcal{G}(y_t, y_t) \\ &\le t\mathcal{F}(y_t, y) + (1 - t)\mathcal{F}(y_t, p^*) \\ &\quad + t\mathcal{G}(y_t, y) + (1 - t)\mathcal{G}(y_t, p^*) \\ &\le \mathcal{F}(y_t, y) + \mathcal{G}(y_t, y).\end{aligned} \quad (94)$$

Consequently

$$\mathcal{F}(y_t, y) + \mathcal{G}(y_t, y) \ge 0 \quad (95)$$

by (f_2) and (h_2), as $t \to 0$, and we obtain $p^* \in \mathrm{MEP}(\mathcal{F}, \mathcal{G})$.

Step 8 (The sequence of $\{x_n\}$ converges strongly to a common Ω). From Step 1 and (61), for all $p \in \Omega$, $\|x_{n+1} - p\| \le (1 + \rho_n)\|x_n - p\|$ for $n \ge 1$ with $\sum_{n=1}^{\infty} \rho_n < \infty$. This implies that $d(x_{n+1} - \Omega) \le (1 + \rho_n)d(x_n - \Omega)$. Then by Lemma 7, $\lim_{n\to\infty} d(x_{n+1} - \Omega)$ exists. Also by Step 6, $\lim_{n\to\infty}\|x_n - T^n x_n\| = \|x_n - I^n x_n\| = 0$, and by the condition (A) in Definition 10 which guarantees that $\lim_{n\to\infty}\mathfrak{f}(d(x_{n+1} - \Omega)) = 0$. Since $\mathfrak{f}$ is a nondecreasing function and $\mathfrak{f}(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n - \Omega)$. Form (81), we obtain

$$\|x_n - x_{n+m}\| \le \|x_n - u_n\| + \|x_{n+m} - u_{n+m}\|. \quad (96)$$

We know that $\{x_n\}$ is Cauchy sequence in C for all numbers m, n. This implies that $\{x_n\}$ converges strongly to $p \in \Omega$. This completes the proof. □

If T is an asymptotically quasi-nonexpansive self-mapping in Theorem 16, we easily obtain the following corollary.

Corollary 17. *Let E be uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let $\mathcal{F}, \mathcal{G} : C \times C \to R$ be two bifunctions which satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H), and let T be asymptotically quasi-nonexpansive self-mapping of C with sequences $\{s_n\} \subset [0, \infty\}$ such that $\sum_{n=1}^{\infty} s_n < \infty$, and let I be* *identity self-mapping of* C, *and* $\Omega = F_{ix}(T) \bigcap \mathrm{MEP}(\mathcal{F}, \mathcal{G}) \neq \emptyset$. *For an initial point* $x_0 \in C$, *generate a sequence* $\{x_n\}$ *by*

$$\begin{aligned} & u_n \in C \\ & \text{such that} \\ & \mathcal{F}(u_n, y) + \mathcal{G}(u_n, y) \\ & \quad \leq \frac{1}{r_n}\langle y - u_n, Ju_n - Jx_n\rangle, \quad \forall y \in C, \\ & y_n = \beta_n x_n + (1-\beta_n) T^n u_n, \\ & x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad \forall n \geq 1, \end{aligned} \tag{97}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$, $\{\beta_n\} \subset [a,b]$ *for some* $a, b \in (0,1)$ *and* $\{r_n\} \subset [d, +\infty)$ *for* $d > 0$. *If the following conditions are satisfied:*

(i) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(ii) $\sum_{n=0}^{\infty} \beta_n < \infty$;

(iii) $\liminf_{n\to\infty} r_n > 0$,

then the sequence $\{x_n\}$ *generated by* (97) *converges strongly to a fixed point in* Ω *if and only if* $\liminf_{n\to\infty} d(x_n, \Omega) = 0$.

5. Numerical Example

In this section, we introduce an example of numerical test to illustrate the algorithms given in Corollary 17.

Example 1. Let $E = R$, $C = [-2000, 2000]$. The mixed equilibrium problem is to find $x \in C$ such that

$$\mathcal{F}(x,y) + \mathcal{G}(x,y) \geq 0, \quad \forall y \in C, \tag{98}$$

where we define $\mathcal{F}(x,y) = -3x^2 + 2xy + y^2$ and $\mathcal{G}(x,y) = x^2 + 3xy - 4y^2$.

Now, we can easily know that $\mathcal{F}$ and $\mathcal{G}$ satisfy the conditions (f_1)–(f_4), (g_1)–(g_3), and (H) as follows:

(f_1) $\mathcal{F}(x,x) = -3x^2 + 2xx + x^2 = 0$ for all $x \in [-2000, 2000]$;

(f_2) $\mathcal{F}(x,y) + \mathcal{F}(y,x) = -2(x-y)^2 \leq 0$ for all $x, y \in [-2000, 2000]$;

(f_3) for all $x, y, z \in [-2000, 2000]$,

$$\begin{aligned} & \limsup_{t\to 0^+} \mathcal{F}(x + t(z-x), y) \\ & \quad = \limsup_{t\to 0^+} -3(x + t(z-x))^2 \\ & \qquad + 2x + t(z-x) y + y^2 \\ & \quad = -3x^2 + 2xy + y^2 \\ & \quad \leq \mathcal{F}(x,y); \end{aligned} \tag{99}$$

(f_4) for each $x \in [-2000, 2000]$, $\theta(y) = \mathcal{F}(x,y) = -3x^2 + 2xy + y^2$ is convex and weakly lower semicontinuous.

(g_1) $\mathcal{G}(x,x) = x^2 + 3xx - 4x^2 = 0$ for each $x \in [-2000, 2000]$;

(g_2) $\mathcal{G}(x,y) + \mathcal{G}(y,x) = -3(x-y)^2 \leq 0$ for all $x, y \in [-2000, 2000]$, and weakly upper semicontinuous in first variable;

(g_3) for each $x \in [-2000, 2000]$, $\theta(y) = \mathcal{G}(x,y) = x^2 + 3xy - 4y^2$ is convex.

Next, we find the formula of $S_r x$. From Theorem 14, we can claim that $S_r x$ is single-valued, for any $y \in C$, $r > 0$,

$$\begin{aligned} & \mathcal{F}(x,y) + \mathcal{G}(x,y) + \frac{1}{r}\langle x - z, y - x\rangle \\ & \quad \Longleftrightarrow -3ry^2 + (5rx + x - z) y \\ & \qquad + xz - 2rx^2 - x^2 \geq 0. \end{aligned} \tag{100}$$

Let $M(y) = -3ry^2 + (5rx + x - z)y + xz - 2rx^2 - x^2$. Then $M(y)$ is a quadratic function of y with coefficients $a = -3r$, $b = 5rx + x - z$, and $c = xz - 2rx^2 - x^2$. So its discriminant $\Delta = b^2 - 4ac$ is

$$\begin{aligned} \Delta &= (5rx + x - z)^2 - 4(-3r)\left(xz - 2rx^2 - x^2\right) \\ &= ((r+1)x - z)^2. \end{aligned} \tag{101}$$

According to $M(y) \geq 0$ for all $y \in C$, form $\Delta \leq 0$, that is

$$((r+1)x - z)^2 \leq 0. \tag{102}$$

Therefore, it follows that

$$x = \frac{z}{r+1} \tag{103}$$

and so

$$S_r z = \frac{z}{r+1}. \tag{104}$$

Now, let $C = [-1/\pi, 1/\pi]$ and $|k| < 1$, and define a mapping $T : C \to C$ by

$$T(x) = \begin{cases} kx \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{105}$$

for all $x \in C$. From the example in [25–27], we can easily know that T is an asymptotically quasi-nonexpansive mapping; furthermore $F_{ix}(T) = \{0\}$.

According to Theorem 14, we obtain

$$F_{ix}(S_r) = \mathrm{MEP}(\mathcal{F}, \mathcal{G}) = 0, \qquad F_{ix}(T) = 0, \tag{106}$$

and so $\Omega = 0$. Therefore, all the assumptions in Corollary 17 are satisfied. we can obtain the following numerical algorithms.

Algorithm 18. Let $r_n = 1$, $\alpha_n = 1/n^2$, and $\beta_n = 1/2n^2$. It is claim to check that

$$\begin{aligned} & \sum_{n=0}^{\infty} \alpha_n < \infty, \qquad \sum_{n=0}^{\infty} \beta_n < \infty, \\ & \liminf_{n\to\infty} r_n = 1. \end{aligned} \tag{107}$$

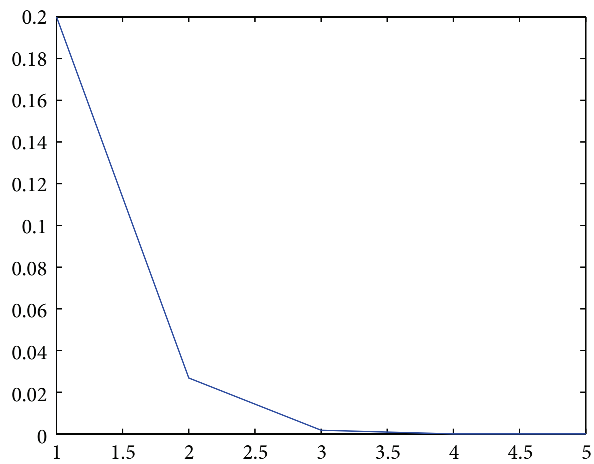

FIGURE 1: Convergence of iterative sequence $\{x_n\}$.

For an initial value $x_0 = 0.2$ and $k = 0.5$, let the sequences $\{u_n\}$ and $\{x_n\}$ generate by

$$\begin{aligned} T(x) &= \frac{1}{2} x \sin \frac{1}{x}, \\ u_n &= S_r(x_n) = \frac{1}{2} x_n, \\ x_{n+1} &= \frac{1+n^2}{2n^4} x_n + \frac{(1-n)(1-2n^2)}{4n^4} T^n x_n, \\ & \qquad \forall n \geq 1. \end{aligned} \tag{108}$$

Then, by the Corollary 17, the sequence $\{x_n\}$ converges to a solution of Example 1. Let $\|x_{n+1} - x_n\| \leq 10^{-5}$ and x^* be the fixed point of the Algorithm 18. Using the software of MATLAB, we generated a sequence $\{x_n\}$ convergence to $x^* = x_7 = 0$ as shown in Figure 1.

Hence the sequence x_n converges strongly to solve Example 1.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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