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Research Article

The Schur-Convexity of the Generalized Muirhead-Heronian Means

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We give a unified generalization of the generalized Muirhead means and the generalized Heronian means involving three parameters. The Schur-convexity of the generalized Muirhead-Heronian means is investigated. Our main result implies the sufficient conditions of the Schur-convexity of the generalized Heronian means and the generalized Muirhead means.

1. Introduction

In what follows, we denote the set of real numbers by $\mathbb{R}$, the set of nonnegative real numbers by $\mathbb{R}_+$, and the set of positive real numbers by $\mathbb{R}_{++}$.

Let $(x, y) \in \mathbb{R}_{++}^2$; the classical Heronian means is defined by (see [1])

$$H_e(x, y) = \frac{x + \sqrt{xy} + y}{3}. \tag{1}$$

In 1999, Mao [2] gave the definition of dual Heronian means; that is,

$$\widetilde{H}_e(x, y) = \frac{x + 4\sqrt{xy} + y}{6}. \tag{2}$$

In 2001, Janous [3] considered the unified generalization of Heronian means $H_e(x, y)$ and $\widetilde{H}_e(x, y)$ and presented a weighted generalization of the above-mentioned Heronian-type means, as follows:

$$H_w(x, y) = \begin{cases} \dfrac{x + w\sqrt{xy} + y}{w + 2}, & 0 \le w < \infty, \\ \sqrt{xy}, & w = \infty. \end{cases} \tag{3}$$

Jia and Cao [4] investigated the exponential generalization of Heronian means

$$H_p(x, y) = \begin{cases} \left(\dfrac{x^p + (xy)^{p/2} + y^p}{3}\right)^{1/p}, & p \ne 0, \\ \sqrt{xy}, & p = 0, \end{cases} \tag{4}$$

and they established some related inequalities. The monotonicity and Schur-convexity of the Heronian means $H_p(x, y)$ were discussed by Li et al. in [5].

Shi et al. [6] discussed the Schur-convexity of a further generalization of the Heronian means given by

$$H_{p,w}(x, y) = \begin{cases} \left(\dfrac{x^p + w(xy)^{p/2} + y^p}{w + 2}\right)^{1/p}, & p \ne 0, \\ \sqrt{xy}, & p = 0, \end{cases} \tag{5}$$

and they obtained a significant result asserted by Theorem A below.

Theorem A. *For fixed* $(p, w) \in \mathbb{R}^2$,

(1) *if* $(p, w) \in \{(p, w) \mid p \geq 2, 0 \leq w \leq 2\}$, *then* $H_{p,w}(x, y)$ *is Schur-convex for* $(x, y) \in \mathbb{R}^2_+$.

(2) *If* $(p, w) \in \{(p, w) \mid p \leq 1, w \geq 0\} \cup \{(p, w) \mid 1 < p \leq 3/2, w \geq 1\} \cup \{(p, w) \mid 3/2 < p \leq 2, w \geq 2\}$, *then* $H_{p,w}(x, y)$ *is Schur-concave for* $(x, y) \in \mathbb{R}^2_+$.

As a further investigation of Theorem A, Fu et al. [7] gave the necessary and sufficient condition for the Schur-convexity of the generalized Heronian means $H_{p,w}(x, y)$, which is stated in the following theorem.

Theorem B. *For fixed* $(p, w) \in \mathbb{R}^2$, *the generalized Heronian means* $H_{p,w}(x, y)$ *is Schur-convex for* $(x, y) \in \mathbb{R}^2_{++}$ *if and only if*

$$(p, w) \in \{(p, w) \mid p \geq 2, 0 \leq w \leq 2(p-1)\} \cup \{(p, w) \mid 1 < p \leq 2, w = 0\}. \tag{6}$$

Furthermore, $H_{p,w}(x, y)$ *is Schur-concave for* $(x, y) \in \mathbb{R}^2_{++}$ *if and only if*

$$(p, w) \in \{(p, w) \mid p \leq 2, \max\{0, 2(p-1)\} \leq w\}. \tag{7}$$

Remark 1. It is easy to observe that, for $p = 1$, $w = 0$, $H_{1,0}(x, y) = (x+y)/2$ is Schur-convex and Schur-concave for $(x, y) \in \mathbb{R}^2_{++}$. In addition, we note that $\{(p, w) \mid p = 2, w = 0\} \subset \{(p, w) \mid p \geq 2, 0 \leq w \leq 2(p-1)\}$. Thus, the conditions of Schur-convexity of $H_{p,w}(x, y)$ in Theorem B can be rewritten as

$$(p, w) \in \{(p, w) \mid p \geq 2, 0 \leq w \leq 2(p-1)\} \cup \{(p, w) \mid 1 \leq p < 2, w = 0\}. \tag{8}$$

The Schur-power-convexity of $H_{p,w}(x, y)$ was investigated by Yang [8].

In 2006, Trif [9] considered the following generalized Muirhead means, defined by

$$M(p, q; x, y) = \left(\frac{x^p y^q + x^q y^p}{2}\right)^{1/(p+q)}, \tag{9}$$

where $x, y \in \mathbb{R}_{++}$, $p, q \in \mathbb{R}$, $p + q \neq 0$.

Gong et al. [10] investigated the Schur-convexity of generalized Muirhead means $M(p, q; x, y)$ and obtained the following results.

Theorem C. *For fixed* $(p, q) \in \mathbb{R}^2$, *the generalized Muirhead means* $M(p, q; x, y)$ *is Schur-convex for* $(x, y) \in \mathbb{R}^2_{++}$ *if and only if*

$$(p, q) \in \left\{(p, q) \mid (p-q)^2 \geq p + q > 0, pq \leq 0\right\}. \tag{10}$$

Furthermore, $M(p, q; x, y)$ *is Schur-concave for* $(x, y) \in \mathbb{R}^2_{++}$ *if and only if*

$$(p, q) \in \left\{(p, q) \mid p \geq 0, q \geq 0, (p-q)^2 \leq p + q, (p, q) \neq (0, 0)\right\} \cup \left\{(p, q) \mid p + q < 0\right\}. \tag{11}$$

Remark 2. If we define, for $p = 0$, $q = 0$, the generalized Muirhead means by $M(0, 0; x, y) = \sqrt{xy}$, we can easily find that $M(0, 0; x, y)$ is Schur-concave for $(x, y) \in \mathbb{R}^2_{++}$; thereby, the conditions of Schur-concave of $M(p, q; x, y)$ in Theorem C can be rewritten as

$$(p, q) \in \left\{(p, q) \mid p \geq 0, q \geq 0, (p-q)^2 \leq p + q\right\} \cup \left\{(p, q) \mid p + q < 0\right\}. \tag{12}$$

The Schur-geometric-convexity and Schur-harmonic-convexity of the generalized Muirhead means $M(p, q; x, y)$ were studied by Xia and Chu in [11, 12].

In this paper we generalize the generalized Muirhead means $M(p, q; x, y)$ and the generalized Heronian means $H_{p,w}(x, y)$ in a unified form. For this purpose we define a generalized Muirhead-Heronian means $\mathscr{H}_{p,q,w}(x, y)$, as follows:

$$\mathscr{H}_{p,q,w}(x, y) = \begin{cases} \left(\dfrac{x^p y^q + w(xy)^{(p+q)/2} + x^q y^p}{2+w}\right)^{1/(p+q)}, & p + q \neq 0, \\ \sqrt{xy}, & p = q = 0, \end{cases} \tag{13}$$

where $(p, q) \in \mathbb{R}^2$, $(x, y) \in \mathbb{R}^2_{++}$.

The paper is organized as follows. Section 2 introduces several definitions and lemmas; Section 3 discusses the Schur-convexity of the generalized Muirhead-Heronian means; Section 4 provides some remarks on the results given in Theorem 9 and it is shown that the sufficient conditions of Schur-convexity of the generalized Heronian means $H_{p,w}(x, y)$ and the generalized Muirhead means $M(p, q; x, y)$ can be deduced from Theorem 9 as special cases.

2. Definitions and Lemmas

We introduce and establish several definitions and lemmas, which will be used in the proofs of the main results in Section 3.

Definition 3 (see [13, page 7]). For any $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, let $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ denote the components of x and y in decreasing order, respectively.

The n-tuple y is said to majorize x (or x is to be majorized by y), in symbols $x \prec y$, if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{holds for } k = 1, 2, \ldots, n-1, \qquad \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i. \tag{14}$$

Definition 4 (see [13, page 54]). For any $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \Omega\ (\Omega \subset \mathbb{R}^n)$, $\phi : \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $x \prec y$ on Ω implies $\phi(x) \leq \phi(y)$ and ϕ is said to be a Schur-concave function on Ω if and only if $-\phi$ is a Schur-convex function.

Lemma 5 (see [13, page 57]). *Let $\Omega(\subset \mathbb{R}^n)$ be a symmetric convex set with nonempty interior Ω^o and $\phi : \Omega \to R$ a continuous symmetric function on Ω. If ϕ is differentiable on Ω^o, then ϕ is the Schur-convex (Schur-concave) function on Ω if and only if*

$$(x_1 - x_2)\left(\frac{\partial\phi}{\partial x_1} - \frac{\partial\phi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{15}$$

holds for all $(x_1, x_2, \ldots, x_n) \in \Omega^o$.

Lemma 6. *Suppose that $p, q \in \mathbb{R}$, $p > q$, $\lambda \geq 1$ and*

$$\begin{aligned} g(\lambda) &= p\lambda^{p-q} + q + \frac{w(p+q)}{2}\lambda^{(p-q)/2} \\ &\quad - q\lambda^{p-q+1} - p\lambda - \frac{w(p+q)}{2}\lambda^{((p-q)/2)+1}. \end{aligned} \tag{16}$$

Suppose also that

$$\begin{aligned} B_1 &= \{(p,q,w) \mid p+q > 0, p-q-2 \geq 0, w \geq 0\}, \\ B_2 &= \{(p,q,w) \mid p+q > 0, p-q-2 < 0, w \geq 0\}, \\ E_{11} &= B_1 \cap \left\{(p,q,w) \mid q < 0, \right. \\ &\quad \left. 2(p-q)^2 - (2+w)(p+q) \geq 0\right\}, \\ E_{12} &= B_2 \cap \{(p,q,w) \mid q < 0, \\ &\quad 2(p-q)^2 - (2+w)(p+q) \geq 0, \\ &\quad (p+q)(p-q-2)w \\ &\quad -8q(p-q+1) \geq 0\}. \end{aligned} \tag{17}$$

Then $(p+q)g(\lambda) \geq 0$ for $(p,q,w) \in E_{11} \cup E_{12}$.

Proof. Differentiating $g(\lambda)$ with respect to λ gives

$$\begin{aligned} g'(\lambda) &= p(p-q)\lambda^{p-q-1} \\ &\quad + \frac{w(p+q)(p-q)}{4}\lambda^{((p-q)/2)-1} \\ &\quad - q(p-q+1)\lambda^{p-q} - p \\ &\quad - \frac{w(p+q)(p-q+2)}{4}\lambda^{(p-q)/2}, \end{aligned} \tag{18}$$

$$g'(1) = \frac{2(p-q)^2 - (2+w)(p+q)}{2}.$$

Let $f(\lambda) = \lambda^{q-p+2}g''(\lambda)$, then

$$\begin{aligned} f(\lambda) &= p(p-q)(p-q-1) \\ &\quad + \frac{w(p+q)(p-q)(p-q-2)}{8}\lambda^{(q-p)/2} \\ &\quad - q(p-q+1)(p-q)\lambda \\ &\quad - \frac{w(p+q)(p-q+2)(p-q)}{8}\lambda^{((q-p)/2)+1}, \end{aligned} \tag{19}$$

$$f(1) = \frac{p-q}{2}\left(2(p-q)^2 - (2+w)(p+q)\right).$$

Differentiating $f(\lambda)$ with respect to λ yields

$$\begin{aligned} f'(\lambda) &= -\frac{w(p+q)(p-q)^2(p-q-2)}{16}\lambda^{((q-p)/2)-1} \\ &\quad - q(p-q+1)(p-q) \\ &\quad + \frac{w(p+q)(p-q+2)(p-q)(p-q-2)}{16}\lambda^{(q-p)/2}, \\ f'(1) &= \frac{p-q}{8}\left((p+q)(p-q-2)w - 8q(p-q+1)\right), \\ f''(\lambda) &= -\frac{w(p+q)(p-q)^2(p-q-2)(p-q+2)}{32} \\ &\quad \times (\lambda-1)\lambda^{((q-p)/2)-2}. \end{aligned} \tag{20}$$

In order to prove Lemma 6, we need to consider the two cases below.

Case 1 ($(p,q,w) \in E_{11}$). In view of $(p,q,w) \in E_{11}$, we have $f''(\lambda) \leq 0$ for $\lambda \geq 1$. Hence $f'(\lambda)$ is decreasing on $[1, +\infty)$, by which, together with

$$\begin{aligned} &f'(1) > 0, \\ &\lim_{\lambda \to +\infty} f'(\lambda) = -q(p-q+1)(p-q) > 0, \end{aligned} \tag{21}$$

we deduce that $f'(\lambda) > 0$ for $\lambda \geq 1$. This means that $f(\lambda)$ is increasing on $[1, +\infty)$. Thus, we have, for $\lambda \geq 1$,

$$\begin{aligned} &f(\lambda) \geq f(1) \geq 0 \\ &\Longrightarrow g''(\lambda) = \frac{f(\lambda)}{\lambda^{q-p+2}} \geq 0 \\ &\Longrightarrow g'(\lambda) \geq g'(1) \geq 0 \\ &\Longrightarrow g(\lambda) \geq g(1) = 0. \end{aligned} \tag{22}$$

This leads to $(p+q)g(\lambda) \geq 0$.

Case 2 ($(p,q,w)\in E_{12}$). By $(p,q,w)\in E_{12}$, we have $f''(\lambda)\ge 0$ for $\lambda\ge 1$. Thus $f'(\lambda)$ is increasing on $[1,+\infty)$, by which, together with

$$f'(1)>0,\qquad \lim_{\lambda\to+\infty} f'(\lambda)=-q(p-q+1)(p-q)>0, \tag{23}$$

we obtain $f'(\lambda)>0$ for $\lambda\ge 1$. It follows that $f(\lambda)$ is increasing on $[1,+\infty)$. Thus, we have, for $\lambda\ge 1$,

$$\begin{aligned} f(\lambda)&\ge f(1)\ge 0\\ &\Longrightarrow g''(\lambda)=\frac{f(\lambda)}{\lambda^{q-p+2}}\ge 0\\ &\Longrightarrow g'(\lambda)\ge g'(1)\ge 0\\ &\Longrightarrow g(\lambda)\ge g(1)=0.\end{aligned} \tag{24}$$

This implies that $(p+q)g(\lambda)\ge 0$. The proof of Lemma 6 is complete. □

Lemma 7. *Suppose that $p,q\in\mathbb{R}$, $p>q$, $\lambda\ge 1$ and*

$$\begin{aligned} g(\lambda)&=p\lambda^{p-q}+q+\frac{w(p+q)}{2}\lambda^{(p-q)/2}\\ &\quad-q\lambda^{p-q+1}-p\lambda-\frac{w(p+q)}{2}\lambda^{((p-q)/2)+1}.\end{aligned} \tag{25}$$

Suppose also that

$$\begin{aligned} B_1&=\{(p,q,w)\mid p+q>0,\ p-q-2\ge 0,\ w\ge 0\},\\ B_2&=\{(p,q,w)\mid p+q>0,\ p-q-2<0,\ w\ge 0\},\\ B_3&=\{(p,q,w)\mid p+q<0,\ p-q-2\ge 0,\ w\ge 0\},\\ B_4&=\{(p,q,w)\mid p+q<0,\ p-q-2<0,\ w\ge 0\},\\ E_{21}&=B_1\cap\{(p,q,w)\mid q>0,\\ &\qquad 2(p-q)^2-(2+w)(p+q)\le 0,\\ &\qquad (p+q)(p-q-2)w\\ &\qquad -8q(p-q+1)\le 0\},\\ E_{22}&=B_2\cap\{(p,q,w)\mid q>0,\\ &\qquad 2(p-q)^2-(2+w)(p+q)\le 0\},\\ E_{23}&=B_3\cap\{(p,q,w)\mid q<0,\\ &\qquad 2(p-q)^2-(2+w)(p+q)>0,\\ &\qquad (p+q)(p-q-2)w\\ &\qquad -8q(p-q+1)\ge 0\},\\ E_{24}&=B_4\cap\{(p,q,w)\mid q<0,\\ &\qquad 2(p-q)^2-(2+w)(p+q)>0\}.\end{aligned} \tag{26}$$

Then $(p+q)g(\lambda)\le 0$ for $(p,q,w)\in E_{21}\cup E_{22}\cup E_{23}\cup E_{24}$.

Proof. Using the differential expressions obtained in Lemma 6, one has

$$\begin{aligned} g'(\lambda)&=p(p-q)\lambda^{p-q-1}+\frac{w(p+q)(p-q)}{4}\lambda^{((p-q)/2)-1}\\ &\quad-q(p-q+1)\lambda^{p-q}\\ &\quad-p-\frac{w(p+q)(p-q+2)}{4}\lambda^{(p-q)/2},\\ g'(1)&=\frac{2(p-q)^2-(2+w)(p+q)}{2},\\ f(\lambda)&=\lambda^{q-p+2}g''(\lambda),\\ f(\lambda)&=p(p-q)(p-q-1)\\ &\quad+\frac{w(p+q)(p-q)(p-q-2)}{8}\lambda^{(q-p)/2}\\ &\quad-q(p-q+1)(p-q)\lambda\\ &\quad-\frac{w(p+q)(p-q+2)(p-q)}{8}\lambda^{((q-p)/2)+1},\\ f(1)&=\frac{p-q}{2}\left(2(p-q)^2-(2+w)(p+q)\right),\\ f'(\lambda)&=-\frac{w(p+q)(p-q)^2(p-q-2)}{16}\lambda^{((q-p)/2)-1}\\ &\quad-q(p-q+1)(p-q)\\ &\quad+\frac{w(p+q)(p-q+2)(p-q)(p-q-2)}{16}\lambda^{(q-p)/2},\\ f'(1)&=\frac{p-q}{8}\left((p+q)(p-q-2)w-8q(p-q+1)\right),\\ f''(\lambda)&=-\frac{w(p+q)(p-q)^2(p-q-2)(p-q+2)}{32}\\ &\quad\times(\lambda-1)\lambda^{((q-p)/2)-2}.\end{aligned} \tag{27}$$

We divide the proof of Lemma 7 into four cases.

Case 1. If $(p,q,w)\in E_{21}$, then

$$f'(1)\le 0,\qquad f(1)\le 0,\qquad g'(1)\le 0,\qquad f''(\lambda)\le 0. \tag{28}$$

Thus we have, for $\lambda\in[1,+\infty)$,

$$\begin{aligned} &f''(\lambda)\le 0\\ &\quad\Longrightarrow f'(\lambda)\ \text{is decreasing}\\ &\quad\Longrightarrow f'(\lambda)\le 0\\ &\quad\Longrightarrow f(\lambda)\ \text{is decreasing}\Longrightarrow f(\lambda)\le 0\\ &\quad\Longrightarrow g''(\lambda)\le 0\\ &\quad\Longrightarrow g'(\lambda)\ \text{is decreasing}\Longrightarrow g'(\lambda)\le 0\\ &\quad\Longrightarrow g(\lambda)\ \text{is decreasing}\Longrightarrow g(\lambda)\le g(1)=0\\ &\quad\Longrightarrow (p+q)g(\lambda)\le 0.\end{aligned} \tag{29}$$

Case 2. If $(p,q,w) \in E_{22}$, then

$$\begin{gathered} f'(1) < 0, \\ \lim_{\lambda\to+\infty} f'(\lambda) = -q(p-q+1)(p-q) < 0, \\ f(1) \le 0, \qquad g'(1) \le 0, \\ f''(\lambda) \ge 0. \end{gathered} \tag{30}$$

Thus we have, for $\lambda \in [1,+\infty)$,

$$\begin{aligned} f''(\lambda) &\ge 0 \\ &\Longrightarrow f'(\lambda) \text{ is increasing} \\ &\Longrightarrow f'(\lambda) \le 0 \\ &\Longrightarrow f(\lambda) \text{ is decreasing} \Longrightarrow f(\lambda) \le 0 \\ &\Longrightarrow g''(\lambda) \le 0 \\ &\Longrightarrow g'(\lambda) \text{ is decreasing} \Longrightarrow g'(\lambda) \le 0 \\ &\Longrightarrow g(\lambda) \text{ is decreasing} \Longrightarrow g(\lambda) \le g(1) = 0 \\ &\Longrightarrow (p+q)\, g(\lambda) \le 0. \end{aligned} \tag{31}$$

Case 3. If $(p,q,w) \in E_{23}$, then

$$f'(1) \ge 0, \qquad f(1) > 0, \qquad g'(1) > 0, \qquad f''(\lambda) \ge 0. \tag{32}$$

Thus we have, for $\lambda \in [1,+\infty)$,

$$\begin{aligned} f''(\lambda) &\ge 0 \\ &\Longrightarrow f'(\lambda) \text{ is increasing} \\ &\Longrightarrow f'(\lambda) \ge 0 \Longrightarrow f(\lambda) \text{ is increasing} \\ &\Longrightarrow f(\lambda) > 0 \Longrightarrow g''(\lambda) > 0 \\ &\Longrightarrow g'(\lambda) \text{ is increasing} \Longrightarrow g'(\lambda) > 0 \\ &\Longrightarrow g(\lambda) \text{ is increasing} \Longrightarrow g(\lambda) \ge g(1) = 0 \\ &\Longrightarrow (p+q)\, g(\lambda) \le 0. \end{aligned} \tag{33}$$

Case 4. If $(p,q,w) \in E_{24}$, then

$$\begin{gathered} f'(1) > 0, \qquad f(1) > 0, \\ \lim_{\lambda\to+\infty} f'(\lambda) = -q(p-q+1)(p-q) > 0, \\ g'(1) > 0, \qquad f''(\lambda) < 0. \end{gathered} \tag{34}$$

Thus we have, for $\lambda \in [1,+\infty)$,

$$\begin{aligned} f''(\lambda) &< 0 \\ &\Longrightarrow f'(\lambda) \text{ is decreasing} \Longrightarrow f'(\lambda) > 0 \\ &\Longrightarrow f(\lambda) \text{ is increasing} \Longrightarrow f(\lambda) > 0 \\ &\Longrightarrow g''(\lambda) > 0 \Longrightarrow g'(\lambda) \text{ is increasing} \\ &\Longrightarrow g'(\lambda) > 0 \Longrightarrow g(\lambda) \text{ is increasing} \\ &\Longrightarrow g(\lambda) \ge g(1) = 0 \Longrightarrow (p+q)\, g(\lambda) \le 0. \end{aligned} \tag{35}$$

This completes the proof of Lemma 7. □

Lemma 8. *Suppose that $p, q \in \mathbb{R}$, $p > q$, $\lambda \ge 1$ and*

$$\begin{aligned} g(\lambda) = {} & p\lambda^{p-q} + q + \frac{w(p+q)}{2}\lambda^{(p-q)/2} \\ & - q\lambda^{p-q+1} - p\lambda - \frac{w(p+q)}{2}\lambda^{((p-q)/2)+1}. \end{aligned} \tag{36}$$

Suppose also that

$$\begin{aligned} B_1 &= \{(p,q,w) \mid p+q > 0, p-q-2 \ge 0, w \ge 0\}, \\ B_3 &= \{(p,q,w) \mid p+q < 0, p-q-2 \ge 0, w \ge 0\}, \\ E_{31} &= B_1 \cap \{(p,q,w) \mid q > 0, \\ &\qquad (p+q)(p-q-2)w - 8q(p-q+1) > 0, \\ &\qquad 2(p-q)^2 - (2+w)(p+q) < 0\} \\ &\qquad \cap \{(p,q,w) \mid (p-q)^2 - 3(p+q) + 2 \le 0\}, \\ E_{32} &= B_3 \cap \{(p,q,w) \mid q < 0, (p+q)(p-q-2)w \\ &\qquad -8q(p-q+1) < 0\}. \end{aligned} \tag{37}$$

Then $(p+q)g(\lambda) \le 0$ for $(p,q,w) \in E_{31} \cup E_{32}$.

Proof. Based on the differential expressions $g'(\lambda)$, $g''(\lambda)$, $g'(1)$, $f'(\lambda)$, $f''(\lambda)$, $f'(1)$ obtained in the proof of Lemma 6, in order to prove Lemma 8, we need to consider the two cases below.

Case 1. If $(p,q,w) \in E_{31}$, then

$$\begin{gathered} f''(\lambda) \le 0, \qquad f'(1) > 0, \\ \lim_{\lambda\to+\infty} f'(\lambda) = -q(p-q+1)(p-q) < 0. \end{gathered} \tag{38}$$

Hence, we deduce that there exists $\lambda_1 \in (1,+\infty)$ such that $f'(\lambda_1) = 0$, satisfying $f'(\lambda) > 0$ for $\lambda \in [1,\lambda_1)$ and $f'(\lambda) < 0$ for $\lambda \in (\lambda_1,+\infty)$.

Further, we conclude that $f(\lambda)$ is increasing on $[1,\lambda_1)$ and decreasing on $(\lambda_1,+\infty)$; thereby, we get $f(\lambda)\le f(\lambda_1)$ for $\lambda\in[1,+\infty)$.

From $f'(\lambda_1)=0$ we have

$$\left(\frac{w(p+q)(p-q+2)(p-q)(p-q-2)}{16}-\frac{w(p+q)(p-q)^2(p-q-2)}{16\lambda_1}\right)\lambda_1^{(q-p)/2}=q(p-q+1)(p-q); \tag{39}$$

this yields

$$\lambda_1^{(q-p)/2}=\frac{16q(p-q+1)\lambda_1}{(p+q)((p-q+2)\lambda_1+q-p)(p-q-2)w}; \tag{40}$$

we thus have

$$\begin{aligned}f(\lambda_1)&=p(p-q)(p-q-1)-q(p-q+1)(p-q)\lambda_1\\&\quad+\left(\frac{w(p+q)(p-q)(p-q-2)}{8}-\frac{w(p+q)(p-q+2)(p-q)\lambda_1}{8}\right)\lambda_1^{(q-p)/2}\\&=\frac{(p-q)^2G_1(\lambda_1)}{(p-q-2)((p-q+2)\lambda_1+q-p)},\end{aligned} \tag{41}$$

where

$$\begin{aligned}G_1(\lambda_1)&=-q(p-q+2)(p-q+1)(\lambda_1-1)^2\\&\quad+\left((p-q)^2-3(p+q)+2\right)\\&\quad\times(2+(p-q+2)(\lambda_1-1))<0.\end{aligned} \tag{42}$$

Note that $(p,q,w)\in E_{31}$ implies $p-q-2>0$ and

$$\begin{aligned}&(p-q+2)\lambda_1+q-p\\&\quad>(p-q+2)+q-p=2;\end{aligned} \tag{43}$$

we conclude that $f(\lambda)\le f(\lambda_1)<0$ for $\lambda\in[1,+\infty)$.

Hence, from $g'(1)<0$, one has, for $\lambda\in[1,+\infty)$,

$$\begin{aligned}&f(\lambda)<0\\&\Longrightarrow g''(\lambda)<0\Longrightarrow g'(\lambda)\text{ is decreasing}\\&\Longrightarrow g'(\lambda)<0\Longrightarrow g(\lambda)\text{ is decreasing}\\&\Longrightarrow g(\lambda)\le g(1)=0\Longrightarrow(p+q)g(\lambda)\le 0.\end{aligned} \tag{44}$$

Case 2. If $(p,q,w)\in E_{32}$, then

$$\begin{gathered}f''(\lambda)\ge 0,\qquad f'(1)<0,\\ \lim_{\lambda\to+\infty}f'(\lambda)=-q(p-q+1)(p-q)>0.\end{gathered} \tag{45}$$

Thus, we deduce that there exists $\lambda_2\in(1,+\infty)$ such that $f'(\lambda_2)=0$, satisfying $f'(\lambda)<0$ for $\lambda\in[1,\lambda_2)$ and $f'(\lambda)>0$ for $\lambda\in(\lambda_2,+\infty)$.

It follows that $f(\lambda)$ is decreasing on $[1,\lambda_2)$ and increasing on $(\lambda_2,+\infty)$, therefore, we obtain

$$f(\lambda)\ge f(\lambda_2)\quad\text{for }\lambda\in[1,+\infty). \tag{46}$$

From $f'(\lambda_2)=0$, we have

$$\left(\frac{w(p+q)(p-q+2)(p-q)(p-q-2)}{16}-\frac{w(p+q)(p-q)^2(p-q-2)}{16\lambda_2}\right)\lambda_2^{(q-p)/2}=q(p-q+1)(p-q); \tag{47}$$

that is,

$$\lambda_2^{(q-p)/2}=\frac{16q(p-q+1)\lambda_2}{(p+q)((p-q+2)\lambda_2+q-p)(p-q-2)w}; \tag{48}$$

we thus have

$$\begin{aligned}f(\lambda_2)&=p(p-q)(p-q-1)-q(p-q+1)(p-q)\lambda_2\\&\quad+\left(\frac{w(p+q)(p-q)(p-q-2)}{8}-\frac{w(p+q)(p-q+2)(p-q)\lambda_2}{8}\right)\lambda_2^{(q-p)/2}\\&=\frac{(p-q)^2G_2(\lambda_2)}{(p-q-2)((p-q+2)\lambda_2+q-p)},\end{aligned} \tag{49}$$

where

$$\begin{aligned}G_2(\lambda_2)&=-q(p-q+2)(p-q+1)(\lambda_2-1)^2\\&\quad+\left((p-q)^2-3(p+q)+2\right)\\&\quad\times(2+(p-q+2)(\lambda_2-1))>0.\end{aligned} \tag{50}$$

Note that $(p,q,w)\in E_{32}$ implies $p-q-2>0$ and

$$\begin{aligned}&(p-q+2)\lambda_2+q-p\\&\quad>(p-q+2)+q-p=2,\end{aligned} \tag{51}$$

which yields that $f(\lambda)\ge f(\lambda_2)>0$ for $\lambda\in[1,+\infty)$.

Therefore, from $g'(1) > 0$, one has, for $\lambda \in [1, +\infty)$,

$$f(\lambda) > 0$$
$$\begin{aligned} &\Longrightarrow g''(\lambda) > 0 \Longrightarrow g'(\lambda) \text{ is increasing} \\ &\Longrightarrow g'(\lambda) > 0 \Longrightarrow g(\lambda) \text{ is increasing} \\ &\Longrightarrow g(\lambda) \geq g(1) = 0 \Longrightarrow (p+q)\, g(\lambda) \leq 0. \end{aligned} \tag{52}$$

The proof of Lemma 8 is completed. □

3. Main Result

The main result of this paper is given by Theorem 9 below.

Theorem 9. *For fixed $(p, q, w) \in \mathbb{R}^3$, let*

$$\begin{aligned}
A_1 = {}& \{(p,q,w) \mid p-q-2 \geq 0, q \leq 0\} \\
& \cup \{(p,q,w) \mid p-q-2 < 0, q \leq 0, \\
& (p+q)(p-q-2)w \\
& -8q(p-q+1) \geq 0\}, \\
A_2 = {}& \{(p,q,w) \mid q-p-2 \geq 0, p \leq 0\} \\
& \cup \{(p,q,w) \mid q-p-2 < 0, p \leq 0, \\
& (p+q)(q-p-2)w \\
& -8p(q-p+1) \geq 0\}, \\
A_3 = {}& \{(p,q,w) \mid p-q-2 \geq 0, q > 0, \\
& (p+q)(p-q-2)w \\
& -8q(p-q+1) \leq 0\} \\
& \cup \{(p,q,w) \mid p > q, p-q-2 < 0, q > 0\}, \\
A_4 = {}& \{(p,q,w) \mid q-p-2 \geq 0, p > 0, \\
& (p+q)(q-p-2)w \\
& -8p(q-p+1) \leq 0\} \\
& \cup \{(p,q,w) \mid q > p, q-p-2 < 0, p > 0\}, \\
A_5 = {}& \{(p,q,w) \mid p-q-2 \geq 0, q < 0, \\
& (p+q)(p-q-2)w \\
& -8q(p-q+1) \geq 0\} \\
& \cup \{(p,q,w) \mid p > q, p-q-2 < 0, q < 0\}, \\
A_6 = {}& \{(p,q,w) \mid q-p-2 \geq 0, p < 0, \\
& (p+q)(q-p-2)w \\
& -8p(q-p+1) \geq 0\} \\
& \cup \{(p,q,w) \mid q > p, q-p-2 < 0, p < 0\}, \\
A_7 = {}& \{(p,q,w) \mid p-q-2 \geq 0, q > 0, \\
& (p+q)(p-q-2)w \\
& -8q(p-q+1) > 0\}, \\
A_8 = {}& \{(p,q,w) \mid q-p-2 \geq 0, p > 0, \\
& (p+q)(q-p-2)w \\
& -8p(q-p+1) > 0\}, \\
A_9 = {}& \{(p,q,w) \mid p-q-2 \geq 0, q < 0, \\
& (p+q)(p-q-2)w \\
& -8q(p-q+1) < 0\}, \\
A_{10} = {}& \{(p,q,w) \mid q-p-2 \geq 0, p < 0, \\
& (p+q)(q-p-2)w \\
& -8p(q-p+1) < 0\},
\end{aligned} \tag{53}$$

and let

$$\begin{aligned}
S_1 = {}& \left\{(p,q,w) \mid p+q > 0, 2(p-q)^2 \right. \\
& \left. -(2+w)(p+q) \geq 0, w \geq 0\right\} \\
& \cap (A_1 \cup A_2), \\
S_2 = {}& \{(p,q,w) \mid p = q, w \geq 0\} \\
& \cup \{(p,q,w) \mid p \leq 2, q = 0, \max\{0, 2(p-1)\} \leq w\} \\
& \cup \{(p,q,w) \mid q \leq 2, p = 0, \max\{0, 2(q-1)\} \leq w\} \\
& \cup \left[\left\{(p,q,w) \mid p+q > 0, w \geq 0, \right.\right. \\
& \left.\left. 2(p-q)^2 - (2+w)(p+q) \leq 0\right\} \cap (A_3 \cup A_4)\right] \\
& \cup \left[\left\{(p,q,w) \mid p+q < 0, w \geq 0, \right.\right. \\
& \left.\left. 2(p-q)^2 - (2+w)(p+q) > 0\right\} \cap (A_5 \cup A_6)\right] \\
& \cup \left[\left\{(p,q,w) \mid p+q > 0, w \geq 0, \right.\right. \\
& (p-q)^2 - 3(p+q) + 2 \leq 0, \\
& \left. 2(p-q)^2 - (2+w)(p+q) < 0\right\} \\
& \left. \cap (A_7 \cup A_8)\right] \\
& \cup \left[\{(p,q,w) \mid p+q < 0, w \geq 0\} \cap (A_9 \cup A_{10})\right].
\end{aligned} \tag{54}$$

The following assertions holds true.

(1) *If $(p, q, w) \in S_1$, then the generalized Muirhead-Heronian means $\mathscr{H}_{p,q,w}(x, y)$ is Schur-convex for $(x, y) \in \mathbb{R}^2_{++}$.*

(2) *If* $(p,q,w) \in S_2$, *then the generalized Muirhead-Heronian means* $\mathscr{H}_{p,q,w}(x,y)$ *is Schur-concave for* $(x,y) \in \mathbb{R}^2_{++}$.

Proof. Note that the expression $\mathscr{H}_{p,q,w}(x,y)$ is of symmetry between x and y and without loss of generality we assume that $x \geq y$.

Case 1. If $p = q$, then $\mathscr{H}_{p,q,w}(x,y) = \sqrt{xy}$. Define

$$(x-y)\left(\frac{\partial \mathscr{H}}{\partial x} - \frac{\partial \mathscr{H}}{\partial y}\right) = \frac{-(x-y)^2}{2\sqrt{xy}} \leq 0. \tag{55}$$

Hence, $\mathscr{H}_{p,q,w}(x,y)$ is Schur-concave for $(x,y) \in \mathbb{R}^2_{++}$.

Case 2. If $q = 0$, we have the following known results (see Theorem B and Remark 1 in Section 1).

$\mathscr{H}_{p,0,w}(x,y)$ is Schur-convex for $(x,y) \in \mathbb{R}^2_{++}$ if and only if

$$\begin{aligned}(p,w) \in &\{(p,w) \mid p \geq 2, 0 \leq w \leq 2(p-1)\}\\ &\cup \{(p,w) \mid 1 \leq p < 2, w = 0\};\end{aligned} \tag{56}$$

$\mathscr{H}_{p,0,w}(x,y)$ is Schur-concave for $(x,y) \in \mathbb{R}^2_{++}$ if and only if

$$(p,w) \in \{(p,w) \mid p \leq 2, \max\{0, 2(p-1)\} \leq w\}. \tag{57}$$

Case 3. If $q \neq 0$, then

$$\begin{aligned}(x-y)&\left(\frac{\partial \mathscr{H}}{\partial x} - \frac{\partial \mathscr{H}}{\partial y}\right)\\ &= \frac{(x-y)\mathscr{H}_{p,q,w}(x,y)F(x,y)}{x^p y^q + w(xy)^{(p+q)/2} + x^q y^p},\end{aligned} \tag{58}$$

where

$$\begin{aligned}F&(x,y)\\ &= \frac{y^{p+q-1}}{p+q}\left(p\left(\frac{x}{y}\right)^{p-1} + q\left(\frac{x}{y}\right)^{q-1}\right.\\ &\qquad + \frac{w(p+q)}{2}\left(\frac{x}{y}\right)^{((p+q)/2)-1}\\ &\qquad - q\left(\frac{x}{y}\right)^{p} - p\left(\frac{x}{y}\right)^{q}\\ &\qquad \left. - \frac{w(p+q)}{2}\left(\frac{x}{y}\right)^{(p+q)/2}\right)\\ &= \frac{y^{p+q-1}}{p+q}\left(p\lambda^{p-1} + q\lambda^{q-1}\right.\\ &\qquad + \frac{w(p+q)}{2}\lambda^{((p+q)/2)-1} - q\lambda^p - p\lambda^q\\ &\qquad \left. - \frac{w(p+q)}{2}\lambda^{(p+q)/2}\right)\\ &= \frac{\lambda^{q-1}y^{p+q-1}}{p+q}\left(p\lambda^{p-q} + q\right.\\ &\qquad + \frac{w(p+q)}{2}\lambda^{(p-q)/2} - q\lambda^{p-q+1}\\ &\qquad \left. - p\lambda - \frac{w(p+q)}{2}\lambda^{((p-q)/2)+1}\right)\\ &= \frac{\lambda^{q-1}y^{p+q-1}}{p+q}g(\lambda),\end{aligned} \tag{59}$$

where $\lambda = x/y \geq 1$; in addition, the definition of $\mathscr{H}_{p,q,w}(x,y)$ implies that $p + q \neq 0$.

Using Lemma 6 gives

$$(p+q)g(\lambda) \geq 0 \quad \text{for } (p,q,w) \in E_{11} \cup E_{12}, \tag{60}$$

where

$$\begin{aligned}E_{11} &= B_1 \cap \{(p,q,w) \mid q < 0,\\ &\qquad 2(p-q)^2 - (2+w)(p+q) \geq 0\},\\ E_{12} &= B_2 \cap \{(p,q,w) \mid q < 0,\\ &\qquad 2(p-q)^2 - (2+w)(p+q) \geq 0,\\ &\qquad (p+q)(p-q-2)w\\ &\qquad - 8q(p-q+1) \geq 0\}\\ B_1 &= \{(p,q,w) \mid p+q > 0, p-q-2 \geq 0, w \geq 0\},\\ B_2 &= \{(p,q,w) \mid p+q > 0, p-q-2 < 0, w \geq 0\}.\end{aligned} \tag{61}$$

On the other hand, we deduce from the symmetry of $\mathscr{H}_{p,q,w}(x,y)$ with respect to p and q that

$$(p+q)g(\lambda) \geq 0 \quad \text{for } (p,q,w) \in E'_{11} \cup E'_{12}, \tag{62}$$

where

$$\begin{aligned}E'_{11}&\\ &= B'_1 \cap \{(p,q,w) \mid p < 0,\\ &\qquad 2(p-q)^2 - (2+w)(p+q) \geq 0\},\end{aligned}$$

$$\begin{aligned}
&E'_{12}\\
&\quad = B'_2 \cap \{(p,q,w) \mid p < 0,\\
&\qquad 2(p-q)^2 - (2+w)(p+q) \geq 0,\\
&\qquad (p+q)(q-p-2)w - 8p(q-p+1) \geq 0\},\\
&B'_1 = \{(p,q,w) \mid p+q > 0,\\
&\qquad q-p-2 \geq 0, w \geq 0\},\\
&B'_2 = \{(p,q,w) \mid p+q > 0,\\
&\qquad q-p-2 < 0, w \geq 0\}.
\end{aligned} \tag{63}$$

Now, by using Lemma 5 and combining the result stated in Case 2, we deduce that $\mathscr{H}_{p,q,w}(x,y)$ is Schur-convex under the conditions below:

$$\begin{aligned}
&(p,q,w)\\
&\quad \in E_{11} \cup E_{12} \cup E'_{11} \cup E'_{12}\\
&\qquad \cup \{(p,q,w) \mid q = 0, p \geq 2, 0 \leq w \leq 2(p-1)\}\\
&\qquad \cup \{(p,q,w) \mid q = 0, 1 \leq p < 2, w = 0\}\\
&\qquad \cup \{(p,q,w) \mid p = 0, q \geq 2, 0 \leq w \leq 2(q-1)\}\\
&\qquad \cup \{(p,q,w) \mid p = 0, 1 \leq q < 2, w = 0\}\\
&\quad = \{(p,q,w) \mid p+q > 0,\\
&\qquad 2(p-q)^2 - (2+w)(p+q) \geq 0, w \geq 0\}\\
&\qquad \cap (A_1 \cup A_2) = S_1.
\end{aligned} \tag{64}$$

This proves the validity of the first assertion in Theorem 9.

It is easy to find that

$$\begin{aligned}
&E_{21} \cup E_{22}\\
&\quad = A_3 \cap \{(p,q,w) \mid p+q > 0, w \geq 0,\\
&\qquad 2(p-q)^2 - (2+w)(p+q) \leq 0\}.
\end{aligned} \tag{65}$$

In view of the symmetry of $\mathscr{H}_{p,q,w}(x,y)$ between p and q, utilizing a positional exchange between p and q in the above expression gives

$$\begin{aligned}
&E'_{21} \cup E'_{22}\\
&\quad = A_4 \cap \{(p,q,w) \mid p+q > 0, w \geq 0,\\
&\qquad 2(p-q)^2 - (2+w)(p+q) \leq 0\}.
\end{aligned} \tag{66}$$

Hence, we deduce from Lemma 7 that

$$(p+q)\,g(\lambda) \leq 0 \quad \text{for } E_{21} \cup E_{22} \cup E'_{21} \cup E'_{22}, \tag{67}$$

where

$$\begin{aligned}
&E_{21} \cup E_{22} \cup E'_{21} \cup E'_{22}\\
&\quad = \{(p,q,w) \mid p+q > 0, w \geq 0,\\
&\qquad 2(p-q)^2 - (2+w)(p+q) \leq 0\}\\
&\qquad \cap (A_3 \cup A_4).
\end{aligned} \tag{68}$$

By using the same method as above, we can deduce that

$$\begin{aligned}
&(p+q)\,g(\lambda) \leq 0\\
&\text{for } (E_{23} \cup E_{24} \cup E'_{23} \cup E'_{24})\\
&\cup (E_{31} \cup E'_{31}) \cup (E_{32} \cup E'_{32}),
\end{aligned} \tag{69}$$

where

$$\begin{aligned}
&E_{23} \cup E_{24} \cup E'_{23} \cup E'_{24}\\
&\quad = \{(p,q,w) \mid p+q < 0, w \geq 0,\\
&\qquad 2(p-q)^2 - (2+w)(p+q) > 0\}\\
&\qquad \cap (A_5 \cup A_6),\\
&E_{31} \cup E'_{31}\\
&\quad = \{(p,q,w) \mid p+q > 0, w \geq 0,\\
&\qquad (p-q)^2 - 3(p+q) + 2 \leq 0,\\
&\qquad 2(p-q)^2 - (2+w)(p+q) < 0\}\\
&\qquad \cap (A_7 \cup A_8),\\
&E_{32} \cup E'_{32}\\
&\quad = \{(p,q,w) \mid p+q < 0, w \geq 0\}\\
&\qquad \cap (A_9 \cup A_{10}).
\end{aligned} \tag{70}$$

Therefore, we deduce from Lemma 5 and the results stated in Cases 1 and 2 that $\mathscr{H}_{p,q,w}(x,y)$ is Schur-concave under the following conditions:

$$\begin{aligned}
&(p,q,w)\\
&\quad \in (E_{21} \cup E_{22} \cup E'_{21} \cup E'_{22})\\
&\qquad \cup (E_{23} \cup E_{24} \cup E'_{23} \cup E'_{24})\\
&\qquad \cup (E_{31} \cup E'_{31}) \cup (E_{32} \cup E'_{32})\\
&\qquad \cup \{(p,q,w) \mid p = q, w \geq 0\}\\
&\qquad \cup \{(p,q,w) \mid p \leq 2, q = 0, \max\{0, 2(p-1)\} \leq w\}\\
&\qquad \cup \{(p,q,w) \mid q \leq 2, p = 0, \max\{0, 2(q-1)\} \leq w\}\\
&\quad = S_2.
\end{aligned} \tag{71}$$

This proves the validity of the second assertion in Theorem 9. The proof of Theorem 9 is thus completed. □

4. Some Remarks on the Results of Theorem 9

In this section, we provide some remarks on the results given in Theorem 9; we show that the sufficient conditions of Schur-convexity of the generalized Heronian means $H_{p,w}(x, y)$ and the generalized Muirhead means $M(p, q; x, y)$ can be deduced from Theorem 9 as special cases.

Remark 10. If we take $q = 0$ in Theorem 9, we have $\mathscr{H}_{p,q,w}(x, y) = H_{p,w}(x, y)$. Furthermore, we have

$$\begin{aligned} S_1 &= \{(p,q,w) \mid p > 0, q = 0, 0 \le w \le 2(p-1)\} \\ &\quad \cap [\{(p,q,w) \mid p \ge 2, q = 0, w \ge 0\} \\ &\quad \cup \{(p,q,w) \mid p < 2, q = 0, w = 0\}] \\ &= \{(p,q,w) \mid p \ge 2, q = 0, 0 \le w \le 2(p-1)\} \\ &\quad \cup \{(p,q,w) \mid 1 \le p < 2, q = 0, w = 0\}, \end{aligned} \tag{72}$$

which are the sufficient conditions of Schur-convex of the generalized Heronian means $H_{p,w}(x, y)$ asserted by Theorem B.

On the other hand, we note that, for $q = 0$,

$$\begin{aligned} &\{(p,q,w) \mid p+q > 0, w \ge 0, \\ &\quad 2(p-q)^2 - (2+w)(p+q) \le 0\} \\ &\quad \cap (A_3 \cup A_4) = \emptyset, \\ &\{(p,q,w) \mid p+q < 0, w \ge 0, \\ &\quad 2(p-q)^2 - (2+w)(p+q) > 0\} \\ &\quad \cap (A_5 \cup A_6) \\ &\quad = \{(p,q,w) \mid p < 0, q = 0, w \ge 0\} \\ &\quad\quad \cap [\{(p,q,w) \mid p \le -2, q = 0, \\ &\quad\quad\quad (-p-2)w \le 8(-p+1)\} \\ &\quad\quad \cup \{(p,q,w) \mid -2 < p < 0, q = 0, w \ge 0\}], \\ &\{(p,q,w) \mid p+q > 0, (p-q)^2 - 3(p+q) + 2 \le 0, \\ &\quad 2(p-q)^2 - (2+w)(p+q) < 0\} \\ &\quad \cap (A_7 \cup A_8) = \emptyset, \\ &\{(p,q,w) \mid p+q < 0, w \ge 0\} \\ &\quad \cap (A_9 \cup A_{10}) \\ &\quad = \{(p,q,w) \mid p < 0, q = 0, w \ge 0\} \\ &\quad\quad \cap \{(p,q,w) \mid p \le -2, q = 0, \\ &\quad\quad (-p-2)w > 8(-p+1)\}. \end{aligned} \tag{73}$$

Hence, the set S_2 given in Theorem 9 reduces to the following form:

$$\begin{aligned} S_2 &= \{(p,q,w) \mid p = 0, q = 0, w \ge 0\} \\ &\quad \cup \{(p,q,w) \mid p \le 2, q = 0, \max\{0, 2(p-1)\} \le w\} \\ &\quad \cup \{(p,q,w) \mid q = 0, p = 0, w \ge 0\} \\ &\quad \cup \{(p,q,w) \mid p < 0, q = 0, w \ge 0\} \\ &= \{(p,q,w) \mid p \le 2, q = 0, \max\{0, 2(p-1)\} \le w\}. \end{aligned} \tag{74}$$

These are the sufficient conditions of Schur-concave of the generalized Heronian means $H_{p,w}(x, y)$ stated by Theorem B.

Remark 11. If we put $w = 0$ in Theorem 9, we have $\mathscr{H}_{p,q,w}(x, y) = M(p, q; x, y)$. Furthermore, we have

$$\begin{aligned} S_1 &= \{(p,q,w) \mid p+q > 0, (p-q)^2 \ge p+q, w = 0\} \\ &\quad \cap (\{(p,q,w) \mid q \le 0, w = 0\} \\ &\quad \cup \{(p,q,w) \mid p \le 0, w = 0\}) \\ &= \{(p,q,w) \mid p+q > 0, pq \le 0, \\ &\quad (p-q)^2 \ge p+q, w = 0\}, \end{aligned} \tag{75}$$

which are the sufficient conditions of Schur-convex of the generalized Muirhead means $M(p, q; x, y)$ given by Theorem C.

In addition, for $w = 0$, we have

$$\begin{aligned} &\{(p,q,w) \mid p+q > 0, w \ge 0, \\ &\quad 2(p-q)^2 - (2+w)(p+q) \le 0\} \\ &\quad \cap (A_3 \cup A_4) \\ &\quad = \{(p,q,w) \mid p > 0, q > 0, p \ne q, \\ &\quad\quad (p-q)^2 \le p+q, w = 0\}, \end{aligned}$$

$$\begin{aligned}
&\{(p,q,w) \mid p+q<0, w\geq 0,\\
&\quad 2(p-q)^2-(2+w)(p+q)>0\}\\
&\quad \cap (A_5 \cup A_6)\\
&\qquad = \{(p,q,w) \mid p+q<0, p\neq q, w=0\},\\
&\{(p,q,w) \mid p+q>0, (p-q)^2-3(p+q)+2\leq 0,\\
&\quad 2(p-q)^2-(2+w)(p+q)<0\}\\
&\quad \cap (A_7 \cup A_8) = \emptyset,\\
&\{(p,q,w) \mid p+q<0, w\geq 0\} \cap (A_9 \cup A_{10}) = \emptyset.
\end{aligned} \tag{76}$$

Thus, the set S_2 given in Theorem 9 reduces to the following form:

$$\begin{aligned}
S_2 &= \{(p,q,w) \mid p=q, w=0\}\\
&\quad \cup \{(p,q,w) \mid p\leq 1, q=0, w=0\}\\
&\quad \cup \{(p,q,w) \mid q\leq 1, p=0, w=0\}\\
&\quad \cup \{(p,q,w) \mid p>0, q>0, p\neq q,\\
&\quad (p-q)^2 \leq p+q, w=0\}\\
&\quad \cup \{(p,q,w) \mid p+q<0, p\neq q, w=0\}\\
&= \{(p,q,w) \mid p\geq 0, q\geq 0, (p-q)^2\leq p+q, w=0\}\\
&\quad \cup \{(p,q,w) \mid p+q<0, w=0\}.
\end{aligned} \tag{77}$$

These are the sufficient conditions of Schur-concave of the generalized Muirhead means $M(p,q;x,y)$ asserted by Theorem C.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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