

Research Article

Existence and Global Asymptotic Behavior of Positive Solutions for Nonlinear Fractional Dirichlet Problems on the Half-Line

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We are interested in the following fractional boundary value problem: $D^\alpha u(t) + a(t)u^\sigma = 0$, $t \in (0, \infty)$, $\lim_{t \rightarrow 0} t^{2-\alpha}u(t) = 0$, $\lim_{t \rightarrow \infty} t^{1-\alpha}u(t) = 0$, where $1 < \alpha < 2$, $\sigma \in (-1, 1)$, D^α is the standard Riemann-Liouville fractional derivative, and a is a nonnegative continuous function on $(0, \infty)$ satisfying some appropriate assumptions related to Karamata regular variation theory. Using the Schauder fixed point theorem, we prove the existence and the uniqueness of a positive solution. We also give a global behavior of such solution.

1. Introduction

Fractional differential equations arise in various fields of science and engineering such as control, porous media, electrochemistry, viscoelasticity, and electromagnetic. They also serve as an excellent tool for the description of hereditary properties of various materials and processes (see [1–3]). In consequence, the subject of fractional differential equations has been gaining much importance and attention. Most of the related results focused on developing the global existence and uniqueness of solutions on finite intervals (see [4–12]) and the references therein). However, to the best of our knowledge, there exist few articles dealing with the existence of solutions to fractional differential equations on the half-line; see, for instance, [13–21]. In [17], by using the recent Leggett-Williams norm-type theorem due to O'Regan and Zima, the author established the existence of positive solutions for fractional boundary value problems of resonance on infinite intervals. On the other hand, in [20], Su and Zhang studied the following fractional differential problem on the half-line by using Schauder's fixed point theorem:

$$\begin{aligned} D^\alpha u(t) &= f(t, u, D^{\alpha-1}u), \quad t \in (0, \infty), \quad 1 < \alpha \leq 2, \\ u(0) &= 0, \quad \lim_{t \rightarrow \infty} D^{\alpha-1}u(t) = u_\infty, \quad u_\infty \in \mathbb{R}, \end{aligned} \quad (1)$$

where D^α is the standard Riemann-Liouville fractional derivative (see Definition 8 below).

In [21], by means of the Leray-Schauder alternative theorem, Zhao and Ge proved the existence of solutions to the following boundary value problem:

$$\begin{aligned} D^\alpha u(t) + f(t, u) &= 0, \quad t \in (0, \infty), \quad 1 < \alpha < 2, \\ u(0) &= 0, \quad \lim_{t \rightarrow \infty} D^{\alpha-1}u(t) = \beta u(\xi), \end{aligned} \quad (2)$$

where $\beta \in \mathbb{R}$ and $0 < \xi < \infty$.

In this paper, we aim at studying the existence, uniqueness, and the exact asymptotic behavior of a positive solution to the following fractional boundary value problem:

$$\begin{aligned} D^\alpha u(t) + a(t)u^\sigma &= 0, \quad t \in (0, \infty), \\ \lim_{t \rightarrow 0} t^{2-\alpha}u(t) &= 0, \quad \lim_{t \rightarrow \infty} t^{1-\alpha}u(t) = 0, \end{aligned} \quad (3)$$

where $1 < \alpha < 2$, $\sigma \in (-1, 1)$, and a is a nonnegative continuous function on $(0, \infty)$ that may be singular at 0.

To state our result, we need some notations. We first introduce the following Karamata classes.

Definition 1. The class \mathcal{K} is the set of all Karamata functions L defined on $(0, \eta]$ by

$$L(t) := c \exp\left(\int_t^\eta \frac{z(s)}{s} ds\right), \tag{4}$$

for some $\eta > 1$ and where $c > 0$ and $z \in C([0, \eta])$ such that $z(0) = 0$.

Definition 2. The class \mathcal{K}^∞ is the set of all Karamata functions L defined on $[1, \infty)$ by

$$L(t) := c \exp\left(\int_1^t \frac{z(s)}{s} ds\right), \tag{5}$$

where $c > 0$ and $z \in C([1, \infty))$ such that $\lim_{t \rightarrow \infty} z(t) = 0$.

It is easy to verify the following.

Remark 3. (i) A function L is in \mathcal{K} if and only if L is a positive function in $C^1((0, \eta])$, for some $\eta > 1$, such that $\lim_{t \rightarrow 0^+} (tL'(t)/L(t)) = 0$.

(ii) A function L is in \mathcal{K}^∞ if and only if L is a positive function in $C^1([1, \infty))$ such that $\lim_{t \rightarrow \infty} (tL'(t)/L(t)) = 0$.

Remark 4 (see [22]). Let L be a function in \mathcal{K}^∞ , and then there exists $m \geq 0$ such that for every $\beta > 0$ and $t \geq 1$ we have

$$(1 + \beta)^{-m} L(t) \leq L(\beta + t) \leq (1 + \beta)^m L(t). \tag{6}$$

As a typical example of function belonging to the class \mathcal{K} , we quote

$$L(t) = \prod_{k=1}^m \left(\log_k \left(\frac{\omega}{t}\right)\right)^{\xi_k}, \tag{7}$$

where ξ_k are real numbers, $\log_k x = \log \circ \log \circ \dots \circ \log x$ (k times), and ω is a sufficiently large positive real number such that L is defined and positive on $(0, \eta]$, for some $\eta > 1$.

In the sequel, we denote by $B^+((0, \infty))$ the set of nonnegative Borel measurable functions in $(0, \infty)$ and by $C_{2-\alpha}([0, \infty))$ the set of all functions f such that $t \rightarrow t^{2-\alpha} f(t)$ is continuous on $[0, \infty)$.

We also denote by $C_0([0, \infty))$ the set of continuous functions v on $[0, \infty)$ such that $\lim_{t \rightarrow \infty} v(t) = 0$. It is easy to see that $C_0([0, \infty))$ is a Banach space with the uniform norm $\|v\|_\infty = \sup_{t>0} |v(t)|$.

For two nonnegative functions f and g defined on a set S , the notation $f(t) \approx g(t)$, $t \in S$ means that there exists $c > 0$ such that $(1/c)f(t) \leq g(t) \leq cf(t)$, for all $t \in S$.

Finally, for $\lambda \in \mathbb{R}$, we put $\lambda^+ = \max(\lambda, 0)$.

Throughout this paper we assume that the function a is nonnegative on $(0, \infty)$ and satisfies the following condition:

(H) $a \in C((0, \infty))$ such that

$$a(t) \approx t^{-\lambda} (1+t)^{\lambda-\mu} L_1(\min(t, 1)) L_2(\max(t, 1)), \quad t > 0, \tag{8}$$

where $\lambda \leq 2 + (\alpha - 2)\sigma$, $\mu \geq 1 + (\alpha - 1)\sigma$, $L_1 \in \mathcal{K}$ defined on $(0, \eta]$, for some $\eta > 1$ and $L_2 \in \mathcal{K}^\infty$ satisfying

$$\int_0^\eta \frac{L_1(s)}{s^{\lambda-(\alpha-2)\sigma-1}} ds < \infty, \quad \int_1^\infty \frac{L_2(s)}{s^{\mu-(\alpha-1)\sigma}} ds < \infty. \tag{9}$$

In what follows, we put

$$\begin{aligned} \nu &= \min\left(1, \frac{2 - \lambda + (\alpha - 2)\sigma}{1 - \sigma}\right), \\ \zeta &= \max\left(0, \frac{2 - \mu + (\alpha - 2)\sigma}{1 - \sigma}\right), \end{aligned} \tag{10}$$

and we define the function θ on $(0, \infty)$ by

$$\begin{aligned} \theta(t) &= t^\nu (1+t)^{\zeta-\nu} (\tilde{L}_1(\min(t, 1)))^{1/(1-\sigma)} \\ &\quad \times (\tilde{L}_2(\max(t, 1)))^{1/(1-\sigma)}, \end{aligned} \tag{11}$$

where, for $t \in (0, \eta)$,

$$\tilde{L}_1(t) = \begin{cases} \int_0^t \frac{L_1(s)}{s} ds & \text{if } \lambda = 2 + (\alpha - 2)\sigma, \\ L_1(t) & \text{if } 1 + (\alpha - 1)\sigma < \lambda < 2 + (\alpha - 2)\sigma, \\ \int_t^\eta \frac{L_1(s)}{s} ds & \text{if } \lambda = 1 + (\alpha - 1)\sigma, \\ 1 & \text{if } \lambda < 1 + (\alpha - 1)\sigma \end{cases} \tag{12}$$

and, for $t \geq 1$,

$$\tilde{L}_2(t) = \begin{cases} \int_t^\infty \frac{L_2(s)}{s} ds & \text{if } \mu = 1 + (\alpha - 1)\sigma, \\ L_2(t) & \text{if } 1 + (\alpha - 1)\sigma < \mu < 2 + (\alpha - 2)\sigma, \\ \int_1^{t+1} \frac{L_2(s)}{s} ds & \text{if } \mu = 2 + (\alpha - 2)\sigma, \\ 1 & \text{if } \mu > 2 + (\alpha - 2)\sigma. \end{cases} \tag{13}$$

Our main result is the following.

Theorem 5. *Let $1 < \alpha < 2$, $\sigma \in (-1, 1)$ and assume (H). Then problem (3) has a unique positive solution $u \in C_{2-\alpha}([0, \infty))$ satisfying, for $t \in (0, \infty)$,*

$$u(t) \approx t^{\alpha-2} \theta(t). \tag{14}$$

Remark 6. The conclusion of Theorem 5 remains valid for the case $\alpha = 2$ and $\sigma < 1$ (see [23]).

The content of this paper is organized as follows. In Section 2, we present some properties of the Green function $G_\alpha(t, s)$ of the operator $u \rightarrow -D^\alpha u$ on $(0, \infty)$ with Dirichlet conditions $\lim_{t \rightarrow 0} t^{2-\alpha} u(t) = 0$ and $\lim_{t \rightarrow \infty} t^{1-\alpha} u(t) = 0$. Next, we give some fundamental properties of the two Karamata classes \mathcal{K} and \mathcal{K}^∞ and we establish sharp estimates on some potential functions. In Section 3, exploiting the results of the previous section and using the Schauder fixed point theorem, we prove Theorem 5.

2. Preliminaries

2.1. Fractional Calculus and Green Function. For the convenience of the reader, we recall in this section some basic definitions on fractional calculus (see [2, 24, 25]) and we give some properties of the Green function $G_\alpha(t, s)$.

Definition 7. The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $h : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I^\beta h(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s) ds, \quad t > 0, \quad (15)$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 8. The Riemann-Liouville fractional derivative of order $\beta > 0$ of a function $h : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D^\beta h(t) = \frac{1}{\Gamma(n-\beta)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\beta-1} h(s) ds, \quad t > 0, \quad (16)$$

where $n = [\beta] + 1$ provided that the right-hand side is pointwise defined on $(0, \infty)$. Here $[\beta]$ means the integer part of the number β .

So we have the following properties (see [2]).

Proposition 9. (1) Let $\beta > 0$ and let $h \in L^1(0, \infty)$, and then one has

$$D^\beta I^\beta h(t) = h(t), \quad \text{for } t > 0. \quad (17)$$

(2) Let $\beta > 0$, and then

$$D^\beta h(t) = 0 \quad \text{iff } h(t) = \sum_{j=1}^m c_j t^{\beta-j}, \quad (18)$$

where m is the smallest integer greater than or equal to β and $(c_1, c_2, \dots, c_m) \in \mathbb{R}^m$.

Corollary 10. Let $\beta > 0$ and assume that $D^\beta h \in L^1(0, \infty)$. Then,

$$I^\beta D^\beta h(t) = h(t) + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \dots + c_m t^{\beta-m}, \quad t > 0, \quad (19)$$

where m is the smallest integer greater than or equal to β and $(c_1, c_2, \dots, c_m) \in \mathbb{R}^m$.

Lemma 11. Let $1 < \alpha < 2$ and $h \in L^1(0, \infty)$. The unique solution of

$$\begin{aligned} D^\alpha u(t) + h(t) &= 0, \quad t > 0 \\ \lim_{t \rightarrow 0} t^{2-\alpha} u(t) &= 0, \quad \lim_{t \rightarrow \infty} t^{1-\alpha} u(t) = 0 \end{aligned} \quad (20)$$

is given by

$$u(t) = \int_0^\infty G_\alpha(t, s) h(s) ds, \quad (21)$$

where

$$G_\alpha(t, s) = \frac{1}{\Gamma(\alpha)} \left[t^{\alpha-1} - ((t-s)^+)^{\alpha-1} \right] \quad (22)$$

is Green's function for the boundary value problem (20).

Proof. We may apply Corollary 10 and Proposition 9 to reduce equation $D^\alpha u(t) + h(t) = 0$ to an equivalent integral equation

$$u(t) = -I^\alpha h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, \quad (23)$$

where $(c_1, c_2) \in \mathbb{R}^2$. Hence the general solution of $D^\alpha u(t) + h(t) = 0$ is

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds. \quad (24)$$

By using $\lim_{t \rightarrow 0} t^{2-\alpha} u(t) = 0$ and $\lim_{t \rightarrow \infty} t^{1-\alpha} u(t) = 0$, we get

$$c_2 = 0, \quad c_1 = \frac{1}{\Gamma(\alpha)} \int_0^\infty h(s) ds. \quad (25)$$

Therefore, the unique solution of problem (20) is

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty h(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \\ &= \int_0^\infty G_\alpha(t, s) h(s) ds. \end{aligned} \quad (26)$$

The proof is complete. \square

Next we give sharp estimates on the Green function $G_\alpha(t, s)$. To this end, we need the following lemma.

Lemma 12. (i) For $\lambda, \mu \in (0, \infty)$ and $x \in [0, 1]$, one has

$$\min\left(1, \frac{\mu}{\lambda}\right) (1-x^\lambda) \leq 1-x^\mu \leq \max\left(1, \frac{\mu}{\lambda}\right) (1-x^\lambda). \quad (27)$$

(ii) For $(t, s) \in (0, \infty) \times (0, \infty)$ one has

$$\min(1, t) \min(1, s) \leq \min(t, s) \leq \max(1, t) \min(1, s). \quad (28)$$

Proposition 13. The Green function $G_\alpha(t, s)$ defined by (22) satisfies

$$G_\alpha(t, s) \approx t^{\alpha-2} \min(t, s), \quad \text{for } (t, s) \in (0, \infty) \times (0, \infty). \quad (29)$$

Proof. For $(t, s) \in (0, \infty) \times (0, \infty)$ we have

$$G_\alpha(t, s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left[1 - \left(\frac{(t-s)^+}{t}\right)^{\alpha-1} \right]. \quad (30)$$

Since $((t-s)^+/t) \in [0, 1]$ for $(t, s) \in (0, \infty) \times (0, \infty)$, then by applying Lemma 12 (i) with $\mu = \alpha - 1$ and $\lambda = 1$, we obtain

$$G_\alpha(t, s) \approx t^{\alpha-1} \left[1 - \left(\frac{(t-s)^+}{t}\right) \right] = t^{\alpha-2} \min(t, s). \quad (31) \quad \square$$

From here on, we define the potential kernel G_α on $B^+((0, \infty))$ by

$$G_\alpha h(t) := \int_0^\infty G_\alpha(t, s) h(s) ds, \quad \text{for } t > 0. \quad (32)$$

Using Proposition 13 and Lemma 12 (ii), we deduce the following.

Corollary 14. *Let $1 < \alpha < 2$ and $h \in B^+((0, \infty))$, and then the function $t \rightarrow G_\alpha h(t)$ belongs to $C_{2-\alpha}([0, \infty))$ if and only if the integral $\int_0^\infty \min(1, s)h(s)ds$ converges.*

Proposition 15. *Let $1 < \alpha < 2$ and f be a function such that the map $s \rightarrow \min(1, s)f(s)$ is continuous and integrable on $(0, \infty)$. Then $G_\alpha f$ is the unique solution in $C_{2-\alpha}([0, \infty))$ of the boundary value problem*

$$D^\alpha u(t) + f(t) = 0, \quad t > 0$$

$$\lim_{t \rightarrow 0} t^{2-\alpha} u(t) = 0, \quad \lim_{t \rightarrow \infty} t^{1-\alpha} u(t) = 0. \quad (33)$$

Proof. From Corollary 14, the function $G_\alpha f$ is well defined in $(0, \infty)$. Using Proposition 13 and Lemma 12 (ii), we get

$$G_\alpha |f|(t) \leq c \left(t^{\alpha-2} (t+1) \int_0^\infty \min(1, s) |f(s)| ds \right). \quad (34)$$

This implies that $I^{2-\alpha}(G_\alpha |f|)$ is finite on $(0, \infty)$. So by using Fubini's theorem, we obtain

$$I^{2-\alpha}(G_\alpha f)(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} G_\alpha f(s) ds$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^\infty \left(\int_0^t (t-s)^{1-\alpha} G_\alpha(s, r) ds \right) f(r) dr. \quad (35)$$

Observe that by considering the substitution $s = r + (t-r)\theta$, we obtain

$$\int_r^t (t-s)^{1-\alpha} (s-r)^{\alpha-1} ds = \Gamma(\alpha) \Gamma(2-\alpha) (t-r). \quad (36)$$

Using this fact and (22) we deduce that

$$\frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} G_\alpha(s, r) ds = \frac{1}{\Gamma(2-\alpha) \Gamma(\alpha)}$$

$$\times \left[\int_0^t (t-s)^{1-\alpha} s^{\alpha-1} ds - \int_0^t (t-s)^{1-\alpha} ((s-r)^+)^{\alpha-1} ds \right]$$

$$= t - \frac{1}{\Gamma(2-\alpha) \Gamma(\alpha)} \int_0^t (t-s)^{1-\alpha} ((s-r)^+)^{\alpha-1} ds. \quad (37)$$

Now, assume that $r \leq t$, and then by (36) we have

$$\int_0^t (t-s)^{1-\alpha} ((s-r)^+)^{\alpha-1} ds = \int_r^t (t-s)^{1-\alpha} (s-r)^{\alpha-1} ds = \Gamma(\alpha) \Gamma(2-\alpha) (t-r). \quad (38)$$

On the other hand, if $t \leq r$ and $s \in (0, t)$, we have

$$\int_0^t (t-s)^{1-\alpha} ((s-r)^+)^{\alpha-1} ds = 0. \quad (39)$$

So combining (38) and (39), we obtain

$$\frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} G_\alpha(s, r) ds = t - (t-r)^+ = \min(t, r). \quad (40)$$

This implies that

$$I^{2-\alpha}(G_\alpha f)(t) = \int_0^\infty \min(t, r) f(r) dr$$

$$= \int_0^t r f(r) dr + t \int_t^\infty f(r) dr, \quad (41)$$

and $D^\alpha(G_\alpha f)(t) = (d^2/dt^2)(I^{2-\alpha}(G_\alpha f))(t) = -f(t)$, for $t > 0$.

Moreover, using Proposition 13 and the dominated convergence theorem, we deduce that

$$\lim_{t \rightarrow 0} t^{2-\alpha} G_\alpha f(t) = 0, \quad \lim_{t \rightarrow \infty} t^{1-\alpha} G_\alpha f(t) = 0. \quad (42)$$

Finally, we need to prove the uniqueness. Let $u, v \in C_{2-\alpha}([0, \infty))$ be two solutions of (33) and put $\omega = u - v$. Then $\omega \in C_{2-\alpha}([0, \infty))$ and $D^\alpha \omega = 0$. Hence, it follows from Corollary 10 that $\omega(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$. Using the fact that $\lim_{t \rightarrow 0} t^{2-\alpha} \omega(t) = \lim_{t \rightarrow \infty} t^{1-\alpha} \omega(t) = 0$, we deduce that $\omega = 0$ and therefore $u = v$. The proof is complete. \square

2.2. Sharp Estimates on the Potential of Some Karamata Functions. We collect in this paragraph some properties of functions belonging to the Karamata class \mathcal{K} (resp., \mathcal{K}^∞) and we give estimates on some potential functions.

Proposition 16 (see [26, 27]). (i) *Let $L_1, L_2 \in \mathcal{K}$ (resp., \mathcal{K}^∞) and $p \in \mathbb{R}$. Then the functions*

$$L_1 + L_2, \quad L_1 L_2 \text{ and } L_1^p \text{ belong to the class } \mathcal{K} \text{ (resp., } \mathcal{K}^\infty). \quad (43)$$

(ii) *Let L be a function in \mathcal{K} (resp., \mathcal{K}^∞) and $\varepsilon > 0$. Then one has*

$$\lim_{t \rightarrow 0^+} t^\varepsilon L(t) = 0, \quad \left(\text{resp. } \lim_{t \rightarrow \infty} t^{-\varepsilon} L(t) = 0 \right). \quad (44)$$

Theorem 17 (see [26, 27]). (a) *Let $\gamma \in \mathbb{R}$ and L be a function in \mathcal{K} defined on $(0, \eta]$. One has the following.*

(i) *If $\gamma < -1$, then $\int_0^\eta s^\gamma L(s) ds$ diverges and $\int_t^\eta s^\gamma L(s) ds \sim -t^{\gamma+1} L(t) / (\gamma + 1)$.*

(ii) *If $\gamma > -1$, then $\int_0^\eta s^\gamma L(s) ds$ converges and $\int_0^t s^\gamma L(s) ds \sim t^{\gamma+1} L(t) / (\gamma + 1)$.*

(b) *Let $\gamma \in \mathbb{R}$ and L be a function in \mathcal{K}^∞ . One has the following.*

- (i) If $\gamma > -1$, then $\int_1^\infty s^\gamma L(s) ds$ diverges and $\int_1^t s^\gamma L(s) ds \underset{t \rightarrow \infty}{\sim} t^{\gamma+1} L(t) / (\gamma + 1)$.
- (ii) If $\gamma < -1$, then $\int_1^\infty s^\gamma L(s) ds$ converges and $\int_t^\infty s^\gamma L(s) ds \underset{t \rightarrow \infty}{\sim} -t^{\gamma+1} L(t) / (\gamma + 1)$.

The proof of the next lemma can be found in [11].

Lemma 18. Let L be a function in \mathcal{K} defined on $(0, \eta]$. Then one has

$$\lim_{t \rightarrow 0^+} \frac{L(t)}{\int_t^\eta (L(s)/s) ds} = 0. \tag{45}$$

In particular,

$$t \longrightarrow \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}. \tag{46}$$

If further $\int_0^\eta (L(s)/s) ds$ converges, then one has $\lim_{t \rightarrow 0^+} (L(t) / \int_0^t (L(s)/s) ds) = 0$.

In particular,

$$t \longrightarrow \int_0^t \frac{L(s)}{s} ds \in \mathcal{K}. \tag{47}$$

In the next lemma, we have the following properties related to the class \mathcal{K}^∞ . For the proof we refer to [22].

Lemma 19. Let L be a function in \mathcal{K}^∞ . Then one has

$$\lim_{t \rightarrow \infty} \frac{L(t)}{\int_1^t (L(s)/s) ds} = 0. \tag{48}$$

In particular,

$$t \longrightarrow \int_1^{t+1} \frac{L(s)}{s} ds \in \mathcal{K}^\infty. \tag{49}$$

If further $\int_1^\infty (L(s)/s) ds$ converges, then one has

$$\lim_{t \rightarrow \infty} \frac{L(t)}{\int_t^\infty (L(s)/s) ds} = 0. \tag{50}$$

In particular,

$$t \longrightarrow \int_t^\infty \frac{L(s)}{s} ds \in \mathcal{K}^\infty. \tag{51}$$

Now, we put

$$b(t) = t^{-\beta} (1+t)^{\beta-\gamma} L_3(\min(t, 1)) L_4(\max(t, 1)), \quad t > 0, \tag{52}$$

where $L_3 \in \mathcal{K}$ and $L_4 \in \mathcal{K}^\infty$. We aim at giving sharp estimates on the potential function $G_\alpha b(t)$.

Proposition 20. Assume that $L_3 \in \mathcal{K}$ defined on $(0, \eta]$, for some $\eta > 1$ and $L_4 \in \mathcal{K}^\infty$. Let $\beta \leq 2$ and $\gamma \geq 1$ such that

$$\int_0^\eta s^{1-\beta} L_3(s) ds < \infty, \quad \int_1^\infty s^{-\gamma} L_4(s) ds < \infty. \tag{53}$$

Then for $t > 0$

$$G_\alpha b(t) \approx t^{\alpha-2} \psi_\beta(\min(t, 1)) \phi_\gamma(\max(t, 1)), \tag{54}$$

where, for $t \in (0, 1]$,

$$\psi_\beta(t) = \begin{cases} \int_0^t \frac{L_3(s)}{s} ds & \text{if } \beta = 2, \\ t^{2-\beta} L_3(t) & \text{if } 1 < \beta < 2, \\ \int_t^\eta \frac{L_3(s)}{s} ds & \text{if } \beta = 1, \\ t & \text{if } \beta < 1 \end{cases} \tag{55}$$

and, for $t \geq 1$,

$$\phi_\gamma(t) = \begin{cases} \int_t^\infty \frac{L_4(s)}{s} ds & \text{if } \gamma = 1, \\ t^{2-\gamma} L_4(t) & \text{if } 1 < \gamma < 2, \\ \int_1^{t+1} \frac{L_4(s)}{s} ds & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2. \end{cases} \tag{56}$$

Proof. Using Proposition 13 and Remark 4, we have

$$\begin{aligned} t^{2-\alpha} G_\alpha b(t) &\approx \int_0^\eta \min(t, s) s^{-\beta} L_3(s) ds \\ &\quad + \int_\eta^\infty \min(t, s) s^{-\gamma} L_4(s) ds \\ &= I(t) + J(t). \end{aligned} \tag{57}$$

Case 1. Assume that $0 < t \leq 1$.

By using (53), we deduce that

$$J(t) \approx t. \tag{58}$$

On the other hand,

$$\begin{aligned} I(t) &= \int_0^t s^{1-\beta} L_3(s) ds + t \int_t^\eta s^{-\beta} L_3(s) ds \\ &= I_1(t) + I_2(t). \end{aligned} \tag{59}$$

Using Theorem 17 and hypothesis (53), we deduce that

$$\begin{aligned} I_1(t) &\approx \begin{cases} t^{2-\beta} L_3(t) & \text{if } \beta < 2, \\ \int_0^t \frac{L_3(s)}{s} ds & \text{if } \beta = 2, \end{cases} \\ I_2(t) &\approx \begin{cases} t^{2-\beta} L_3(t) & \text{if } 1 < \beta \leq 2, \\ \int_t^\eta s^{-\beta} L_3(s) ds & \text{if } \beta \leq 1. \end{cases} \end{aligned} \tag{60}$$

Hence, it follows by Lemma 18, Proposition 16, and hypothesis (53) that

$$I(t) \approx \begin{cases} \int_0^t \frac{L_3(s)}{s} ds & \text{if } \beta = 2, \\ t^{2-\beta} L_3(t) & \text{if } 1 < \beta < 2, \\ t \int_t^\eta \frac{L_3(s)}{s} ds & \text{if } \beta = 1, \\ t & \text{if } \beta < 1. \end{cases} \quad (61)$$

Combining (58) and (61) and using Proposition 16 and hypothesis (53), we deduce that, for $0 < t \leq 1$,

$$t^{2-\alpha} G_\alpha b(t) \approx \psi_\beta(t). \quad (62)$$

Case 2. Assume that $t > \eta + 1$.

By using (53), we deduce that

$$I(t) \approx 1. \quad (63)$$

On the other hand,

$$\begin{aligned} J(t) &= \int_t^\eta s^{1-\gamma} L_4(s) ds + t \int_t^\infty s^{-\gamma} L_4(s) ds \\ &= J_1(t) + J_2(t). \end{aligned} \quad (64)$$

Using again Theorem 17 and hypothesis (53), we deduce that

$$\begin{aligned} J_1(t) &\approx \begin{cases} t^{2-\gamma} L_4(t) & \text{if } 1 \leq \gamma < 2, \\ \int_t^\eta s^{1-\gamma} L_4(s) ds & \text{if } \gamma \geq 2, \end{cases} \\ J_2(t) &\approx \begin{cases} t^{2-\gamma} L_4(t) & \text{if } \gamma > 1, \\ t \int_t^\infty \frac{L_4(s)}{s} ds & \text{if } \gamma = 1. \end{cases} \end{aligned} \quad (65)$$

Hence, it follows from Lemma 19 and hypothesis (53) that

$$J(t) \approx \begin{cases} t \int_t^\infty \frac{L_4(s)}{s} ds & \text{if } \gamma = 1, \\ t \int_t^\eta \frac{L_4(s)}{s} ds & \text{if } 1 < \gamma < 2, \\ \int_t^\eta \frac{L_4(s)}{s} ds & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2. \end{cases} \quad (66)$$

Combining (63) and (66) and using Proposition 16, hypothesis (53), and Remark 4, we deduce that, for $t > \eta + 1$,

$$\begin{aligned} t^{2-\alpha} G_\alpha b(t) &\approx \begin{cases} t \int_t^\infty \frac{L_4(s)}{s} ds & \text{if } \gamma = 1, \\ t^{2-\gamma} L_4(t) & \text{if } 1 < \gamma < 2, \\ \int_t^\eta \frac{L_4(s)}{s} ds & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2. \end{cases} \\ &\approx \phi_\gamma(t). \end{aligned} \quad (67)$$

Now since the functions $t \rightarrow t^{2-\alpha} G_\alpha b(t)$ and $t \rightarrow \phi_\gamma(t)$ are positive and continuous on $[1, \eta + 1]$, we deduce that, for $t \in [1, \eta + 1]$,

$$t^{2-\alpha} G_\alpha b(t) \approx \phi_\gamma(t). \quad (68)$$

Finally, using (62), (67), and (68), we obtain the required result. \square

3. Proof of the Main Result

The next lemma will play a crucial role in the proof of Theorem 5

Lemma 21. Assume that the function a satisfies (H) and put $\omega(t) = a(t)t^{(\alpha-2)\sigma}\theta^\sigma(t)$ for $t > 0$. Then one has for $t \in (0, \infty)$

$$t^{2-\alpha} G_\alpha \omega(t) \approx \theta(t). \quad (69)$$

Proof. We recall that

$$\begin{aligned} \nu &= \min\left(1, \frac{2 - \lambda + (\alpha - 2)\sigma}{1 - \sigma}\right), \\ \zeta &= \max\left(0, \frac{2 - \mu + (\alpha - 2)\sigma}{1 - \sigma}\right), \\ \theta(t) &= t^\nu(1+t)^{\zeta-\nu}(\tilde{L}_1(\min(t, 1)))^{1/(1-\sigma)} \\ &\quad \times (\tilde{L}_2(\max(t, 1)))^{1/(1-\sigma)}, \end{aligned} \quad (70)$$

where, for $t \in (0, 1]$,

$$\tilde{L}_1(t) = \begin{cases} \int_0^t \frac{L_1(s)}{s} ds & \text{if } \lambda = 2 + (\alpha - 2)\sigma, \\ L_1(t) & \text{if } 1 + (\alpha - 1)\sigma < \lambda < 2 + (\alpha - 2)\sigma, \\ \int_t^\eta \frac{L_1(s)}{s} ds & \text{if } \lambda = 1 + (\alpha - 1)\sigma, \\ 1 & \text{if } \lambda < 1 + (\alpha - 1)\sigma \end{cases} \quad (71)$$

and, for $t \geq 1$,

$$\tilde{L}_2(t) = \begin{cases} \int_t^\infty \frac{L_2(s)}{s} ds & \text{if } \mu = 1 + (\alpha - 1)\sigma, \\ \frac{L_2(t)}{t} & \text{if } 1 + (\alpha - 1)\sigma < \mu < 2 + (\alpha - 2)\sigma, \\ \int_1^{t+1} \frac{L_2(s)}{s} ds & \text{if } \mu = 2 + (\alpha - 2)\sigma, \\ 1 & \text{if } \mu > 2 + (\alpha - 2)\sigma. \end{cases} \quad (72)$$

For $t > 0$, we have

$$\begin{aligned} \omega(t) &\approx t^{-\lambda+\nu\sigma+(\alpha-2)\sigma}(1+t)^{\lambda-\mu+(\zeta-\nu)\sigma} \\ &\quad \times L_1(\min(t, 1))(\tilde{L}_1(\min(t, 1)))^{\sigma/(1-\sigma)} \\ &\quad \times L_2(\max(t, 1))(\tilde{L}_2(\max(t, 1)))^{\sigma/(1-\sigma)}. \end{aligned} \quad (73)$$

Using Proposition 20 with $\beta = \lambda - \nu\sigma - (\alpha - 2)\sigma$ and $\gamma = \mu - \zeta\sigma - (\alpha - 2)\sigma$, $L_3(t) = L_1(t)(\tilde{L}_1(t))^{\sigma/(1-\sigma)}$ and $L_4(t) = L_2(t)(\tilde{L}_2(t))^{\sigma/(1-\sigma)}$, we obtain for $t \in (0, 1]$

$$\begin{aligned}
 t^{2-\alpha}G_\alpha\omega(t) &\approx \begin{cases} \int_0^t \frac{L_1(s)}{s} \left(\int_0^s \frac{L_1(r)}{r} dr \right)^{\sigma/(1-\sigma)} ds & \text{if } \lambda = 2 + (\alpha - 2)\sigma, \\ t^{(2-\lambda+(\alpha-2)\sigma)/(1-\sigma)} L_1(t) (L_1(t))^{\sigma/(1-\sigma)} & \text{if } 1 + (\alpha - 1)\sigma < \lambda < 2 + (\alpha - 2)\sigma, \\ t \int_t^\eta \frac{L_1(s)}{s} \left(\int_s^\eta \frac{L_1(r)}{r} dr \right)^{\sigma/(1-\sigma)} ds & \text{if } \lambda = 1 + (\alpha - 1)\sigma, \\ t & \text{if } \lambda < 1 + (\alpha - 1)\sigma, \end{cases} \\
 &\approx \begin{cases} \int_0^t \left(\frac{L_1(s)}{s} ds \right)^{1/(1-\sigma)} & \text{if } \lambda = 2 + (\alpha - 2)\sigma, \\ t^{(2-\lambda+(\alpha-2)\sigma)/(1-\sigma)} (L_1(t))^{1/(1-\sigma)} & \text{if } 1 + (\alpha - 1)\sigma < \lambda < 2 + (\alpha - 2)\sigma, \\ t \left(\int_t^\eta \frac{L_1(s)}{s} ds \right)^{1/(1-\sigma)} & \text{if } \lambda = 1 + (\alpha - 1)\sigma, \\ t & \text{if } \lambda < 1 + (\alpha - 1)\sigma, \end{cases} \\
 &\approx \theta(t).
 \end{aligned} \tag{74}$$

On the other hand, using again Proposition 20 and Remark 4, we get, for $t \geq 1$,

$$\begin{aligned}
 t^{2-\alpha}G_\alpha\omega(t) &\approx \begin{cases} t \int_t^\infty \frac{L_2(s)}{s} \left(\int_s^\infty \frac{L_2(r)}{r} dr \right)^{\sigma/(1-\sigma)} ds & \text{if } \mu = 1 + (\alpha - 1)\sigma, \\ t^\zeta L_2(t) (L_2(t))^{\sigma/(1-\sigma)} & \text{if } 1 + (\alpha - 1)\sigma < \mu < 2 + (\alpha - 2)\sigma, \\ \int_1^{t+1} \frac{L_2(s)}{s} \left(\int_1^{s+1} \frac{L_2(r)}{r} dr \right)^{\sigma/(1-\sigma)} ds & \text{if } \mu = 2 + (\alpha - 2)\sigma, \\ 1 & \text{if } \mu > 2 + (\alpha - 2)\sigma. \end{cases} \\
 &\approx \begin{cases} t \left(\int_t^\infty \frac{L_2(s)}{s} ds \right)^{1/(1-\sigma)} & \text{if } \mu = 1 + (\alpha - 1)\sigma, \\ t^\zeta (L_2(t))^{1/(1-\sigma)} & \text{if } 1 + (\alpha - 1)\sigma < \mu < 2 + (\alpha - 2)\sigma, \\ \left(\int_1^{t+1} \frac{L_2(s)}{s} ds \right)^{1/(1-\sigma)} & \text{if } \mu = 2 + (\alpha - 2)\sigma, \\ 1 & \text{if } \mu > 2 + (\alpha - 2)\sigma. \end{cases} \\
 &\approx \theta(t).
 \end{aligned} \tag{75}$$

This completes the proof. \square

In order to use a fixed point theorem, we define the operator T on Λ by

Proof of Theorem 5. From Lemma 21, there exists $M > 1$ such that for each $t > 0$

$$\frac{1}{M}\theta(t) \leq t^{2-\alpha}G_\alpha\omega(t) \leq M\theta(t), \tag{76}$$

$$Tv(t) = \frac{t^{2-\alpha}}{1+t} \int_0^\infty G_\alpha(t,s) a(s) s^{(\alpha-2)\sigma} (1+s)^\sigma v^\sigma(s) ds. \tag{78}$$

where $\omega(t) = a(t)t^{(\alpha-2)\sigma}\theta^\sigma(t)$. Put $c_0 = M^{1/(1-\sigma)}$ and let

For this choice of c_0 and using (76), we easily prove that for all $v \in \Lambda$ and $t > 0$

$$\Lambda = \left\{ v \in C_0([0, \infty)) : \frac{\theta(t)}{c_0(1+t)} \leq v(t) \leq \frac{c_0\theta(t)}{1+t}, t > 0 \right\}. \tag{77}$$

$$Tv(t) \leq \frac{c_0\theta(t)}{1+t}, \quad Tv(t) \geq \frac{\theta(t)}{c_0(1+t)}. \tag{79}$$

On the other hand, using Proposition 13 and Lemma 12 (ii), there exists $c > 0$ such that for all $t, s > 0$, we have

$$\frac{t^{2-\alpha}G_\alpha(t, s)}{1+t} \leq c \min(1, s). \tag{80}$$

This implies that there exists $c > 0$ such that, for each $v \in \Lambda$ and $t > 0$,

$$|Tv(t)| \leq c \int_0^\infty \min(1, s) \omega(s) ds. \tag{81}$$

Now by hypothesis (H) and Theorem 17, the function $t \rightarrow \min(1, t)\omega(t)$ is in $L^1(0, \infty)$, which implies that the family $\{Tv(t), v \in \Lambda\}$ is uniformly bounded.

Using (80) and the fact that, for each $s > 0$, the function $t \rightarrow (t^{2-\alpha}G_\alpha(t, s))/(1+t)$ is in $C_0([0, \infty))$, we deduce that the family $\{Tv(t), v \in \Lambda\}$ is equicontinuous in $[0, \infty)$.

Hence, it follows by Ascoli’s theorem that $T(\Lambda)$ is relatively compact in $C_0([0, \infty))$ and therefore $T(\Lambda) \subset \Lambda$.

Next, we will prove the continuity of T in the supremum norm. Let $(v_k)_k$ be a sequence in Λ which converges to v in Λ . Using again (80) and Lebesgue’s theorem, we deduce that $Tv_k(t) \rightarrow Tv(t)$ as $k \rightarrow \infty$, for $t > 0$.

Since $T(\Lambda)$ is relatively compact in $C_0([0, \infty))$, then the pointwise convergence implies the uniform convergence. Thus we have proved that T is a compact mapping from Λ to itself.

Now, the Schauder fixed point theorem implies the existence of $v \in \Lambda$ such that

$$v(t) = \frac{t^{2-\alpha}}{1+t} \int_0^\infty G_\alpha(t, s) a(s) s^{(\alpha-2)\sigma} (1+s)^\sigma v^\sigma(s) ds. \tag{82}$$

Put $u(t) = t^{\alpha-2}(1+t)v(t)$. Then $u \in C_{2-\alpha}([0, \infty))$ and u satisfies the equation

$$u(t) = G_\alpha(au^\sigma)(t). \tag{83}$$

Since the function $s \rightarrow \min(1, s)a(s)u^\sigma(s)$ is continuous and integrable on $(0, \infty)$, then by Proposition 15, the function u is a positive solution in $C_{2-\alpha}([0, \infty))$ of problem (3).

Finally, it remains to prove that u is the unique positive solution in $C_{2-\alpha}([0, \infty))$ satisfying (14). To this end, assume that problem (3) has two positive solutions $u, v \in C_{2-\alpha}([0, \infty))$ satisfying (14). Then there exists a constant $m > 1$ such that

$$\frac{1}{m} \leq \frac{u}{v} \leq m. \tag{84}$$

This implies that the set

$$J = \left\{ m \geq 1 : \frac{1}{m} \leq \frac{u}{v} \leq m \right\} \tag{85}$$

is not empty. Let $c = \inf J$. Then $c \geq 1$ and we have $(1/c)v \leq u \leq cv$. It follows that $u^\sigma \leq c^{|\sigma|}v^\sigma$ and consequently

$$-D^\alpha(c^{|\sigma|}v - u) = a(c^{|\sigma|}v^\sigma - u^\sigma) \geq 0, \tag{86}$$

$$\lim_{t \rightarrow 0} t^{2-\alpha}(c^{|\sigma|}v - u)(t) = 0, \quad \lim_{t \rightarrow \infty} t^{1-\alpha}(c^{|\sigma|}v - u)(t) = 0. \tag{87}$$

This implies by Proposition 15 that $c^{|\sigma|}v - u = G_\alpha(a(c^{|\sigma|}v^\sigma - u^\sigma)) \geq 0$. By symmetry, we obtain also $v \leq c^{|\sigma|}u$. Hence $c^{|\sigma|} \in J$ and $c \leq c^{|\sigma|}$. Since $|\sigma| < 1$, then $c = 1$ and therefore $u = v$. \square

Example 22. Let $1 < \alpha < 2$, $\sigma \in (-1, 1)$, and a be a positive continuous function on $(0, \infty)$ such that

$$a(t) \approx t^{-\lambda}(1+t)^{\lambda-\mu} \log\left(\frac{2}{\min(t, 1)}\right), \quad t > 0, \tag{88}$$

where $\lambda < 2+(\alpha-2)\sigma$ and $\mu > 1+(\alpha-1)\sigma$. Then by Theorem 5, problem (3) has a unique positive solution $u \in C_{2-\alpha}([0, \infty))$ satisfying, for $t > 0$,

$$u(t) \approx t^{\alpha-2+\nu}(1+t)^{\zeta-\nu} \left(\tilde{L}_1(\min(t, 1))\right)^{1/(1-\sigma)} \times \left(\tilde{L}_2(\max(t, 1))\right)^{1/(1-\sigma)}, \tag{89}$$

where $\nu = \min(1, (2-\lambda+(\alpha-2)\sigma)/(1-\sigma))$, $\zeta = \max(0, (2-\mu+(\alpha-2)\sigma)/(1-\sigma))$,

$$\tilde{L}_1(t) = \begin{cases} \log\left(\frac{2}{t}\right) & \text{if } 1+(\alpha-1)\sigma < \lambda < 2+(\alpha-2)\sigma, \\ \left(\log\left(\frac{2}{t}\right)\right)^2 & \text{if } \lambda = 1+(\alpha-1)\sigma, \\ 1 & \text{if } \lambda < 1+(\alpha-1)\sigma, \end{cases}$$

$$\tilde{L}_2(t) = \begin{cases} \log(1+t) & \text{if } \mu = 2+(\alpha-2)\sigma, \\ 1 & \text{if } \mu \neq 2+(\alpha-2)\sigma. \end{cases} \tag{90}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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