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Research Article

Existence and Iterative Approximation Methods for Generalized Mixed Vector Equilibrium Problems with Relaxed Monotone Mappings

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We first consider an auxiliary problem for the generalized mixed vector equilibrium problem with a relaxed monotone mapping and prove the existence and uniqueness of the solution for the auxiliary problem. We then introduce a new iterative scheme for approximating a common element of the set of solutions of a generalized mixed vector equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings. The results presented in this paper can be considered as a generalization of some known results due to Wang et al. (2010).

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let X be a nonempty closed convex subset of H. Let $\varphi : X \times X \to \mathbb{R} = (-\infty, +\infty)$ be a bifunction. The equilibrium problem $\mathrm{EP}(\varphi)$ is to find $x \in X$ such that

$$\varphi(x, y) \geq 0, \quad \forall y \in X. \tag{1}$$

As pointed out by Blum and Oettli [1], $\mathrm{EP}(\varphi)$ provides a unified model of several problems, such as the optimization problem, fixed point problem, variational inequality, and complementarity problem.

A mapping $S : X \to H$ is called nonexpansive, if

$$\|Sz - Sy\| \leq \|z - y\|, \quad \forall z, y \in X. \tag{2}$$

We denote the set of all fixed points of S by $F(S)$, that is, $F(S) = \{z \in X : z = Sz\}$. It is well known that if $X \subset H$ is bounded, closed, convex and S is a nonexpansive mapping of X onto itself, then $F(S)$ is nonempty (see [2]). A mapping $T : C \to H$ is said to be relaxed η-α monotone if there exist a mapping $\eta : C \times C \to H$ and a function $\alpha : H \to \mathbb{R}$ positively homogeneous of degree p, that is, $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in H$ such that

$$\langle Tx - Ty, \eta(x, y)\rangle \geq \alpha(x - y), \quad \forall x, y \in C, \tag{3}$$

where $p > 1$ is a constant; see [3]. In the case of $\eta(x, y) = x - y$ for all $x, y \in C$, T is said to be relaxed α-monotone. In the case of $\eta(x, y) = x - y$ for all $x, y \in C$ and $\alpha(z) = k\|z\|^p$, where $p > 1$ and $k > 0$, T is said to be p-monotone; see [4–6]. In fact, in this case, if $p = 2$, then T is a k-strongly monotone mapping. Moreover, every monotone mapping is relaxed $\eta - \alpha$ monotone with $\eta(x, y) = x - y$ for all $x, y \in C$ and $\alpha = 0$.

In 2000, Moudafi [7] introduced an iterative scheme of finding the solution of nonexpansive mappings and proved a strong convergence theorem. Recently, Huang et al. [8] introduced the approximate method for solving the equilibrium problem and proved the strong convergence theorem.

Let $\varphi : X \times X \to \mathbb{R}$ be a bifunction and $T, A : X \to H$ nonlinear mappings. In 2010, Wang et al. [9] introduced the following generalized mixed equilibrium problem with a relaxed monotone mapping.

Find $z \in C$ such that

$$\varphi(z, y) + \langle Tz, \eta(y, z)\rangle + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{4}$$

Problem (4) is very general setting, and it includes special cases of Nash equilibrium problems, complementarity problems, fixed point problems, optimization problems, and variational inequalities (see, e.g., [8, 10–13] and the references therein). Moreover, Wang et al. [9] studied the existence of solutions for the proposed problem and introduced a new iterative scheme for finding a common element of the set of solutions of a generalized equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings in a Hilbert space.

It is well known that the vector equilibrium problem provides a unified model of several problems, for example, vector optimization, vector variational inequality, vector complementarity problem, and vector saddle point problem [14–16]. In recent years, the vector equilibrium problem has been intensively studied by many authors (see, e.g., [12, 14–19] and the references therein).

Recently, Li and Wang [18] first studied the viscosity approximation methods for strong vector equilibrium problems and fixed point problems. Very recently, Shan and Huang [20] studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the generalized mixed vector equilibrium problem, and the solution set of a variational inequality problem with a monotone Lipschitz continuous mapping in Hilbert spaces. They first introduced an auxiliary problem for the generalized mixed vector equilibrium problem and proved the existence and uniqueness of the solution for the auxiliary problem. Furthermore, they introduced an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the generalized mixed vector equilibrium problem, and the solution set of a variational inequality problem with a monotone Lipschitz continuous mapping.

Let Y be a Hausdorff topological vector space, and, let C be a closed, convex and pointed cone of Y with $\operatorname{int} C \neq \emptyset$. Let $\varphi : X \times X \to Y$ be a vector-valued bifunction. The strong vector equilibrium problem (for short, SVEP(φ)) is to find $z \in X$ such that

$$\varphi(z, y) \in C, \quad \forall y \in X \tag{5}$$

and the weak vector equilibrium problem (for short, WVEP(φ)) is to find $z \in X$ such that

$$\varphi(z, y) \notin -\operatorname{int} C, \quad \forall y \in X. \tag{6}$$

In this paper, inspired and motivated by the works mentioned previously, we consider the following generalized mixed vector equilibrium problem with a relaxed monotone mapping (for short, GVEPR(φ, T)): find $z \in X$ such that

$$\varphi(z, y) + e\langle Tz, \eta(y, z)\rangle + e\langle Az, y - z\rangle \in C, \quad \forall y \in X, \tag{7}$$

where $e \in \operatorname{int} C$, $\varphi : X \times X \to Y$, and $T, A : X \to H$ are the mappings. The set of all solutions of the generalized mixed vector equilibrium problem with a relaxed monotone mapping is denoted by SGVEPR(φ, T), that is,

$$\begin{aligned} &\operatorname{SGVEPR}(\varphi, T) \\ &\quad = \{z \in X : \varphi(z, y) + e\langle Tz, \eta(y, z)\rangle \\ &\qquad + e\langle Az, y - z\rangle \in C, \forall y \in X\}. \end{aligned} \tag{8}$$

If $A = 0$, we denote the set ASGVEPR(φ, T) by

$$\begin{aligned} &\operatorname{ASGVEPR}(\varphi, T) \\ &\quad = \{z \in X : \varphi(z, y) + e\langle Tz, \eta(y, z)\rangle \in C, \forall y \in X\}. \end{aligned} \tag{9}$$

Some special cases of the problem (7) are as follows.

(1) If $Y = \mathbb{R}$, $C = \mathbb{R}^+$, and $e = 1$, then GVEPR(φ, T) (7) reduces to the generalized mixed equilibrium problem with a relaxed monotone mapping (4).

(2) If $T = 0$ and $A = 0$, then GVEPR(φ, T) (7) reduces to the classic vector equilibrium problem (5).

We consider the auxiliary problem of GVEPR(φ, T) and prove the existence and uniqueness of the solutions of auxiliary problem of GVEPR(φ, T) under some proper conditions. By using the result for the auxiliary problem, we introduce a new iterative scheme for finding a common element of the set of solutions of a generalized mixed vector equilibrium problem with a relaxed monotone mapping and the set of common fixed points of a countable family of nonexpansive mappings and then obtain a strong convergence theorem. The results presented in this paper improve and generalize some known results of Wang et al. [9].

2. Preliminaries

Let $A : X \to H$ be a λ-inverse-strongly monotone mapping of H. For all $z, y \in X$ and $k > 0$, one has [21]

$$\begin{aligned} &\|(I - kA)z - (I - kA)y\|^2 \\ &\quad \leq \|z - y\|^2 + k(k - 2\lambda)\|Az - Ay\|^2. \end{aligned} \tag{10}$$

Hence, if $k \in (0, 2\lambda)$, then $I - kA$ is a nonexpansive mapping of X into H.

For each point $z \in H$, there exists a unique nearest point of X, denoted by $P_X z$, such that

$$\|z - P_X z\| \leq \|z - y\|, \tag{11}$$

for all $y \in X$. Such a P_X is called the metric projection from H onto X. The well-known Browder's characterization of P_X ensures that P_X is a firmly nonexpansive mapping from H onto X, that is,

$$\|P_X z - P_X y\|^2 \leq \langle P_X z - P_X y, z - y\rangle, \quad \forall z, y \in H. \tag{12}$$

Further, we know that for any $z \in H$ and $x \in X$, $x = P_X z$ if and only if

$$\langle z - x, x - y \rangle \geq 0, \quad \forall y \in X. \tag{13}$$

Let S be a nonexpansive mapping of X into itself such that $F(S) \neq \emptyset$. Then we have

$$\widehat{x} \in F(S) \Longleftrightarrow \|Sx - x\|^2 \leq 2\langle x - Sx, x - \widehat{x} \rangle, \quad \forall x \in X, \tag{14}$$

which is obtained directly from the following:

$$\begin{aligned} &\|x - \widehat{x}\|^2 \\ &\quad \geq \|Sx - S\widehat{x}\|^2 = \|Sx - \widehat{x}\|^2 = \|Sx - x + (x - \widehat{x})\|^2 \\ &\quad = \|Sx - x\|^2 + \|x - \widehat{x}\|^2 + 2\langle Sx - x, x - \widehat{x} \rangle. \end{aligned} \tag{15}$$

This inequality is a very useful characterization of $F(S)$. Observe what is more that it immediately yields that $\mathrm{Fix}(S)$ is a convex closed set.

Definition 1 (see [6, 22]). Let X and Y be two Hausdorff topological vector spaces, E a nonempty, convex, subset of X and C a closed, convex and pointed cone of Y with $\mathrm{int}C \neq \emptyset$. Let θ be the zero point of Y, $\mathbb{U}(\theta)$ the neighborhood set of θ, $\mathbb{U}(x_0)$ be the neighborhood set of x_0, and $f : E \to Y$ a mapping.

(1) If for any $V \in \mathbb{U}(\theta)$ in Y, and there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + C, \quad \forall x \in U \cap E, \tag{16}$$

then f is called *upper C-continuous* on x_0. If f is upper C-continuous for all $x \in E$, then f is called upper C-continuous on E.

(2) If for any $V \in \mathbb{U}(\theta)$ in Y, and there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - C, \quad \forall x \in U \cap E, \tag{17}$$

then f is called *lower C-continuous* on x_0. If f is lower C-continuous for all $x \in E$, then f is called lower C-continuous on E.

(3) f is called C-continuous if f is upper C-continuous and lower C-continuous.

(4) If for any $x, y \in E$ and $t \in [0,1]$, and the mapping f satisfies

$$\begin{gathered} f(x) \in f(tx + (1-t)y) + C \\ \text{or } f(y) \in f(tx + (1-t)y) + C, \end{gathered} \tag{18}$$

then f is called *proper C-quasiconvex*.

(5) If for any $x_1, x_2 \in E$ and $t \in [0,1]$, and the mapping f satisfies

$$tf(x_1) + (1-t)f(x_2) \in f(tx_1 - (1-t)x_2) + C, \tag{19}$$

then f is called *C-convex*.

Lemma 2 (see [19]). *Let X and Y be two real Hausdorff topological vector spaces, E is a nonempty, compact, convex subset of X, and C is a closed, convex, and pointed cone of Y. Assume that $f : E \times E \to Y$ and $\psi : E \to Y$ are two vector valued mappings. Suppose that f and ψ satisfy the following:*

(i) *$f(x,x) \in C$, for all $x \in E$;*

(ii) *ψ is upper C-continuous on E;*

(iii) *$f(\cdot, y)$ is lower C-continuous for all $y \in E$;*

(iv) *$f(x, \cdot) + \psi(\cdot)$ is proper C-quasiconvex for all $x \in E$.*

Then there exists a point $x \in E$ satisfying

$$F(x,y) \in C \setminus \{0\}, \quad \forall y \in E, \tag{20}$$

where

$$F(x,y) = f(x,y) + \psi(y) - \psi(x), \quad \forall x, y \in E. \tag{21}$$

Definition 3 (see [3]). Let E be a Banach space with the dual space E^* and let K be a nonempty subset of E. Let $T : K \to E^*$, and $\eta : K \times K \to E$ be two mappings. The mapping $T : K \to E^*$ is said to be η-hemicontinuous, if for any fixed $x, y \in K$, the function $f : [0,1] \to (-\infty, \infty)$ defined by

$$f(t) = \langle T((1-t)x + ty), \eta(x,y) \rangle \tag{22}$$

is continuous at 0^+.

3. The Existence of Solutions for the Generalized Mixed Vector Equilibrium Problem with a Relaxed Monotone Mapping

For solving the generalized mixed vector equilibrium problem with a relaxed monotone mapping, we give the following assumptions. Let H be a real Hilbert space with inner $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Assume that $X \subseteq H$ is nonempty, compact, convex subset, Y is real Hausdorff topological vector space, and $C \subseteq Y$ is a closed, convex, and pointed cone. Let $\varphi : X \times X \to Y$, $T : X \to H$ be two mappings. For any $x \in H$, define a mapping $\Phi_x : X \times X \to Y$ as follows:

$$\begin{aligned} \Phi_x(z,y) = {} & \varphi(z,y) + e\langle Tz, \eta(y,z) \rangle \\ & + \frac{e}{r}\langle y - z, z - x \rangle, \end{aligned} \tag{23}$$

where r is a positive number in $\mathbb{R}$ and $e \in C \setminus \{0\}$. Let Φ_x, φ, and T satisfy the following conditions:

(A_1) for all $z \in X$, $\varphi(z,z) = \theta$;

(A_2) φ is monotone, that is, $\varphi(z,y) + \varphi(y,z) \in -C$ for all $z, y \in X$;

(A_3) $\varphi(\cdot, y)$ is C-continuous for all $y \in X$;

(A_4) $\varphi(z, \cdot)$ is C-convex, that is,

$$t\varphi(z,y_1) + (1-t)\varphi(z,y_2) \in \varphi(z, ty_1 + (1-t)y_2) + C, \quad \forall z, y_1, y_2 \in X, \ \forall t \in [0,1], \tag{24}$$

(A_5) for all $y \in X$, $z \mapsto \langle Tz, \eta(y,z)\rangle$ is continuous, and for any $u, v \in X$,

$$y \longmapsto \langle Tu, \eta(y,v)\rangle \text{ is convex and lower semicontinuous;} \tag{25}$$

(A_6) $\Phi_x(z,\cdot)$ is proper C-quasiconvex for all $z \in X$ and $x \in H$.

Remark 4. Let $Y = \mathbb{R}$, $C = \mathbb{R}^+$, and $e = 1$. For any $y \in X$, if $\varphi(\cdot, y)$ is upper semicontinuous and $z \mapsto \langle Tz, \eta(y,z)\rangle$ is continuous, then $\Phi_x(\cdot, y)$ is lower C-continuous. In fact, since $\varphi(\cdot, y)$ is upper semicontinuous and $z \mapsto \langle Tz, \eta(y,z)\rangle$ is continuous, for any $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $z \in \{z \in X, \|z - z_0\| < \delta\}$, we have

$$\Phi_x(z,y) < \Phi_x(z_0,y) + \epsilon, \tag{26}$$

where z_0 is a point in X. This means that $\Phi_x(\cdot, y)$ is lower C-continuous.

Remark 5. Let $Y = \mathbb{R}$, $C = \mathbb{R}^+$ and $e = 1$. Assume that $\varphi(z,\cdot)$ is a convex mapping for all $z \in X$. Then for any $y_1, y_2 \in X$ and $t \in [0,1]$, we have

$$\begin{aligned}
&\Phi_x(z, ty_1 + (1-t)y_2) \\
&= \varphi(z, ty_1 + (1-t)y_2) \\
&\quad + \langle Tz, \eta(ty_1 + (1-t)y_2, z)\rangle \\
&\quad + \frac{1}{r}\langle ty_1 + (1-t)y_2 - z, z - x\rangle \\
&\leq t\varphi(z,y_1) + (1-t)\varphi(z,y_2) \\
&\quad + t\langle Tz, \eta(y_1,z)\rangle + (1-t)\langle Tz, \eta(y_2,z)\rangle \\
&\quad + \frac{t}{r}\langle y_1 - z, z - x\rangle + \frac{1-t}{r}\langle y_2 - z, z - x\rangle \\
&= t\left(\varphi(z,y_1) + \langle Tz, \eta(y_1,z)\rangle + \frac{1}{r}\langle y_1 - z, z - x\rangle\right) \\
&\quad + (1-t)\Big(\varphi(z,y_2) + \langle Tz, \eta(y_2,z)\rangle \\
&\qquad\qquad + \frac{1}{r}\langle y_2 - z, z - x\rangle\Big) \\
&= t\Phi_x(z,y_1) + (1-t)\Phi_x(z,y_2) \\
&\leq \max\{\Phi_x(z,y_1), \Phi_x(z,y_2)\},
\end{aligned} \tag{27}$$

which implies that $\Phi_x(z,\cdot)$ is proper C-quasiconvex.

Now we are in the position to state and prove the existence of solutions for the generalized mixed vector equilibrium problem with a relaxed monotone mapping.

Theorem 6. *Let X be a nonempty, compact, convex subset of a real Hilbert space H. Let C be a closed, convex, and pointed cone of a Hausdorff topological vector space Y. Let $T : X \to H$ be an η-hemicontinuous and relaxed η-α-monotone mapping. Let $\varphi : X \times X \to Y$ be a vector-valued bifunction. Suppose that all the conditions (A_1)–(A_6) are satisfied. Let $r > 0$ and define a mapping $B_r : H \to X$ as follows:*

$$\begin{aligned}
B_r(x) = \Big\{ &z \in X : \varphi(z,y) + e\langle Tz, \eta(y,z)\rangle \\
&+ \frac{e}{r}\langle y - z, z - x\rangle \in C, \ \forall y \in X \Big\}
\end{aligned} \tag{28}$$

for all $x \in H$. Assume that

(i) *$\eta(x,y) + \eta(y,x) = 0$, for all $x, y \in X$;*

(ii) *for any $x, y \in X$, $\alpha(x-y) + \alpha(y-x) \geq 0$.*

Then, the following holds.

(1) *$B_r(z) \neq \emptyset$ for all $z \in X$.*

(2) *B_r is single-value.*

(3) *B_r is a firmly nonexpansive mapping, that is, for all $x, y \in X$,*

$$\|B_r x - B_r y\|^2 \leq \langle B_r x - B_r y, x - y\rangle, \tag{29}$$

(4) *$F(B_r) = ASGVEPR(\varphi, T)$,*

(5) *$ASGVEPR(\varphi, T)$ is closed and convex.*

Proof. (1) In Lemma 2, let $f(z,y) = \Phi_x(z,y)$, and, let $\psi(z) = \theta$ for all $z, y \in X$ and $x \in H$. Then it is easy to check that f and Φ satisfy all the conditions of Lemma 2. Thus, there exists a point $z \in X$ such that

$$f(z,y) + \psi(z) - \psi(y) \in C, \quad \forall y \in X, x \in H, \tag{30}$$

which gives that, for any $x \in H$,

$$\varphi(z,y) + e\langle Tz, \eta(y,z)\rangle + \frac{e}{r}\langle y - z, z - x\rangle \in C, \quad \forall y \in X. \tag{31}$$

Therefor we conclude that $B_r(x) \neq \emptyset$ for all $x \in H$.

(2) For $x \in H$ and $r > 0$, let $z_1, z_2 \in B_r(x)$. Then

$$\begin{aligned}
&\varphi(z_1,y) + e\langle Tz_1, \eta(y,z_1)\rangle \\
&\quad + \frac{e}{r}\langle y - z_1, z_1 - x\rangle \in C, \quad \forall y \in X,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\varphi(z_2,y) + e\langle Tz_2, \eta(y,z_2)\rangle \\
&\quad + \frac{e}{r}\langle y - z_2, z_2 - x\rangle \in C, \quad \forall y \in X.
\end{aligned} \tag{33}$$

Letting $y = z_2$ in (32) and $y = z_1$ in (33), adding (32) and (33), we have

$$\begin{aligned}
&\varphi(z_2,z_1) + \varphi(z_1,z_2) + e\langle Tz_1 - Tz_2, \eta(z_2,z_1)\rangle \\
&\quad + \frac{e}{r}\langle z_1 - z_2, z_2 - z_1\rangle \in C.
\end{aligned} \tag{34}$$

By the monotonicity of φ, we have

$$e\left\langle Tz_1 - Tz_2, \eta\left(z_2, z_1\right)\right\rangle + \frac{e}{r}\left\langle z_1 - z_2, z_2 - z_1\right\rangle \in C. \quad (35)$$

Thus

$$\frac{e}{r}\left\langle z_1 - z_2, z_2 - z_1\right\rangle - e\left\langle Tz_2 - Tz_1, \eta\left(z_2, z_1\right)\right\rangle \in C. \quad (36)$$

Since T is relaxed η-α-monotone and $r > 0$ and the property of C, one has

$$e\left\langle z_1 - z_2, z_2 - z_1\right\rangle - er\alpha\left(z_2 - z_1\right) \in C. \quad (37)$$

In (36) exchanging the position of z_1 and z_2, we get

$$\frac{e}{r}\left\langle z_2 - z_1, z_1 - z_2\right\rangle - e\alpha\left(z_1 - z_2\right) \in C, \quad (38)$$

that is,

$$e\left\langle z_2 - z_1, z_1 - z_2\right\rangle - er\alpha\left(z_1 - z_2\right) \in C. \quad (39)$$

Now, adding the inequalities (37) and (39),

$$\begin{aligned} &e\left\langle z_1 - z_2, z_2 - z_1\right\rangle - er\alpha\left(z_2 - z_1\right) \\ &\quad + e\left\langle z_2 - z_1, z_1 - z_2\right\rangle - er\alpha\left(z_1 - z_2\right) \in C. \end{aligned} \quad (40)$$

By using (iv), we have

$$2e\left\langle z_2 - z_1, z_1 - z_2\right\rangle \in C. \quad (41)$$

If $\left\langle z_2 - z_1, z_1 - z_2\right\rangle < 0$, then

$$-2\left\langle z_2 - z_1, z_1 - z_2\right\rangle > 0. \quad (42)$$

This implies that

$$-2e\left\langle z_2 - z_1, z_1 - z_2\right\rangle \in C. \quad (43)$$

From (41) and (43), we have $z_1 = z_2$ which is a contradiction. Thus

$$\left\langle z_2 - z_1, z_1 - z_2\right\rangle \geq 0, \quad (44)$$

so

$$-\left\|z_1 - z_2\right\|^2 = \left\langle z_1 - z_2, z_2 - z_1\right\rangle \geq 0. \quad (45)$$

Hence $z_1 = z_2$. Therefore B_r is single value.

(3) For any $x_1, x_2 \in H$, let $z_1 = B_r(x_1)$ and $z_2 = B_r(x_2)$. Then

$$\begin{aligned} &\varphi\left(z_1, y\right) + e\left\langle Tz_1, \eta\left(y, z_1\right)\right\rangle \\ &\quad + \frac{e}{r}\left\langle y - z_1, z_1 - x_1\right\rangle \in C, \quad \forall y \in X, \end{aligned} \quad (46)$$

$$\begin{aligned} &\varphi\left(z_2, y\right) + e\left\langle Tz_2, \eta\left(y, z_2\right)\right\rangle \\ &\quad + \frac{e}{r}\left\langle y - z_2, z_2 - x_2\right\rangle \in C, \quad \forall y \in X. \end{aligned} \quad (47)$$

Letting $y = z_2$ in (46) and $y = z_1$ in (47), adding (46) and (47), we have

$$\begin{aligned} &\varphi\left(z_1, z_2\right) + \varphi\left(z_2, z_1\right) + e\left\langle Tz_1, \eta\left(z_2, z_1\right)\right\rangle \\ &\quad + e\left\langle Tz_2, \eta\left(z_1, z_2\right)\right\rangle \\ &\quad + \frac{e}{r}\left\langle z_2 - z_1, z_1 - z_2 - \left(x_1 - x_2\right)\right\rangle \in C. \end{aligned} \quad (48)$$

Since φ is monotone and C is closed convex cone, we get

$$\begin{aligned} &\left\langle Tz_1 - Tz_2, \eta\left(z_2, z_1\right)\right\rangle \\ &\quad + \frac{1}{r}\left\langle z_2 - z_1, z_1 - z_2 - x_1 + x_2\right\rangle \geq 0, \end{aligned} \quad (49)$$

that is,

$$\begin{aligned} &\frac{1}{r}\left\langle z_2 - z_1, z_1 - z_2 - x_1 + x_2\right\rangle \\ &\quad \geq \left\langle Tz_2 - Tz_1, \eta\left(z_2, z_1\right)\right\rangle \\ &\quad \geq \alpha\left(z_2 - z_1\right). \end{aligned} \quad (50)$$

In (50) exchanging the position of z_1 and z_2, we get

$$\begin{aligned} &\frac{1}{r}\left\langle z_1 - z_2, z_2 - z_1 - x_2 + x_1\right\rangle \\ &\quad \geq \alpha\left(z_1 - z_2\right). \end{aligned} \quad (51)$$

Adding the inequalities (50) and (51), we have

$$\begin{aligned} &2\left\langle z_1 - z_2, z_2 - z_1 - x_2 + x_1\right\rangle \\ &\quad \geq r\left(\alpha\left(z_1 - z_2\right) + \alpha\left(z_2 - z_1\right)\right). \end{aligned} \quad (52)$$

It follows from (iv) that

$$\left\langle z_1 - z_2, z_2 - z_1 - x_2 + x_1\right\rangle \geq 0. \quad (53)$$

This implies that

$$\left\|B_r x_1 - B_r x_2\right\|^2 \leq \left\langle B_r x_1 - B_r x_2, x_1 - x_2\right\rangle. \quad (54)$$

This shows that B_r is firmly nonexpansive.

(4) We claim that $F(B_r) = \mathrm{ASGVEPR}(\varphi, T)$. Indeed, we have the following:

$$\begin{aligned} x \in F\left(B_r\right) &\Longleftrightarrow x = B_r x \\ &\Longleftrightarrow \varphi\left(x, y\right) + e\left\langle Tx, \eta\left(y, x\right)\right\rangle \\ &\qquad + \frac{e}{r}\left\langle y - x, x - x\right\rangle \in C, \quad \forall y \in X \\ &\Longleftrightarrow \varphi\left(x, y\right) + e\left\langle Tx, \eta\left(y, x\right)\right\rangle \in C, \quad \forall y \in X \\ &\Longleftrightarrow x \in \mathrm{ASGVEPR}\left(\varphi, T\right). \end{aligned} \quad (55)$$

(5) Since every firmly nonexpansive mapping is nonexpansive, we see that B_r is nonexpansive. Since the set of fixed point of every nonexpansive mapping is closed and convex, we have that ASGVEPR(φ, T) is closed and convex. This completes the proof. □

4. Convergence Analysis

In this section, we prove a strong convergence theorem which is one of our main results.

Theorem 7. *Let X be a nonempty, compact, convex subset of a real Hilbert space H. Let C be a closed, convex cone of a real Hausdorff topological vector space Y and $e \in C \setminus \{0\}$. Let $\varphi : X \times X \to Y$ satisfy (A_1)–(A_6). Let $T : X \to H$ be an η-hemicontinuous and relaxed η-α-monotone mapping. Let $A : X \to H$ be a λ-inverse-strongly monotone mapping, and let $\{S_n\}_{n=1}^{\infty}$ be a countable family of nonexpansive mappings from X onto itself such that*

$$F := \cap_{n=1}^{\infty} Fix(S_n) \cap SGVEPR(\varphi, T) \neq \emptyset. \tag{56}$$

Assume that the conditions (i)-(ii) of Theorem 6 are satisfied. Put $\alpha_0 = 1$ and assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$ is a strictly decreasing sequence. Assume that $\{\beta_n\}_{n=1}^{\infty} \subset (c,d)$ with some $c, d \in (0,1)$ and $\{\lambda_n\}_{n=1}^{\infty} \subset [a,b]$ with some $a, b \in (0, 2\lambda)$. Then, for any $x_1 \in X$, the sequence $\{x_n\}$, generated by

$$\begin{aligned}
&\varphi(u_n, y) + e\langle Tu_n, \eta(y, u_n)\rangle + e\langle Ax_n, y - u_n\rangle \\
&\quad + \frac{e}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \in C, \quad \forall y \in X, \\
&y_n = \alpha_n x_n + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n S_i x_n \\
&\quad + (1-\alpha_n)(1-\beta_n) u_n, \\
&C_n = \{z \in X : \|y_n - z\| \leq \|x_n - z\|\}, \\
&D_n = \cap_{j=1}^{n} C_j, \\
&x_{n+1} = P_{D_n} x_1, \quad n \geq 1,
\end{aligned} \tag{57}$$

converges strongly to $x^* \in P_F x_1$. In particular, if X contains the origin 0 and taking $x_1 = 0$, then the sequence $\{x_n\}$ generated by (57) converges strongly to the minimum norm element in F, that is $x^* = P_F 0$.

Proof. We divide the proof into several steps.

Step 1. We will show that F is closed and convex, the sequence $\{x_n\}$ generated by (57) is well defined, and $F \subset D_n$, for all $n \geq 1$.

First, we prove that F is closed and convex. It suffices to prove that SGVEPR(φ, T) is closed and convex. Indeed, it is easy to prove the conclusion by the following fact:

$$\begin{aligned}
\forall p \in \mathrm{SGVEPR}(\varphi, T) &\Longleftrightarrow \varphi(p, y) + e\langle Tp, \eta(y, p)\rangle \\
&\qquad + \frac{e}{\lambda_n}\langle y - p, \lambda_n Ap\rangle \in C \\
&\qquad \forall y \in X \\
&\Longleftrightarrow \varphi(p, y) + e\langle Tp, \eta(y, p)\rangle + \frac{e}{\lambda_n} \\
&\qquad \times \langle y - p, p - (p - \lambda_n Ap)\rangle \in C \\
&\qquad \forall y \in X \\
&\Longleftrightarrow p = B_{\lambda_n}(I - \lambda_n A)p.
\end{aligned} \tag{58}$$

This implies that

$$\mathrm{SGVEPR}(\varphi, T) = \mathrm{Fix}\left[B_{\lambda_n}(I - \lambda_n A)\right]. \tag{59}$$

Since $B_{\lambda_n}(I - \lambda_n A)$ is a nonexpansive mapping for $\lambda_n < 2\lambda$ and the set of fixed points of a nonexpansive mapping is closed and convex, we have that SGVEPR(φ, T) is closed and convex.

Next, we prove that the sequence $\{x_n\}$ generated by (57) is well defined and $F \subset D_n$ for all $n \geq 1$. By Definition of C_n, for all $z \in X$, the inequality

$$\|y_n - z\| \leq \|x_n - z\| \tag{60}$$

is equivalent to

$$\langle y_n - x_n, y_n + x_n\rangle - 2\langle y_n - x_n, z\rangle \leq 0. \tag{61}$$

It is easy to see that C_n is closed and convex for all $n \in \mathbb{N}$. Hence D_n is closed and convex for all $n \in \mathbb{N}$. For any $p \in F$, and since $u_n = B_{\lambda_n}(x_n - \lambda_n Ax_n)$ and $I - \lambda_n A$ is nonexpansive, we have

$$\begin{aligned}
\|y_n - p\| &= \Bigg\|\alpha_n(x_n - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n(S_i x_n - p) \\
&\qquad + (1-\alpha_n)(1-\beta_n)(u_n - p)\Bigg\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|S_i x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|u_n - p\| \\
&\leq \alpha_n\|x_n - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n) \\
&\quad \times \left\|B_{\lambda_n}(x_n - \lambda_n Ax_n) - B_{\lambda_n}(p - \lambda_n Ap)\right\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n) \\
&\quad \times \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\
&\leq \alpha_n\|x_n - p\| + (1-\alpha_n)\beta_n\|x_n - p\| \\
&\quad + (1-\alpha_n)(1-\beta_n)\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned} \tag{62}$$

This implies that $F \subset C_n$ for all $n \in \mathbb{N}$. Hence $F \subset \cap_{j=1}^{n} C_j$. That is

$$F \subset D_n, \quad \forall n \in \mathbb{N}. \tag{63}$$

Since D_n is nonempty closed convex, we get that the sequence $\{x_n\}$ is well defined. This completes the proof of Step 1.

Step 2. We shall show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$ and there is $x^* \in C$ such that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

It easy to see that $D_{n+1} \subset D_n$ for all $n \in \mathbb{N}$ from the construction of D_n. Hence

$$x_{n+2} = P_{D_{n+1}} x_1 \in D_{n+1} \subset D_n. \tag{64}$$

Since $x_{n+1} = P_{D_n} x_1$, we have

$$\|x_{n+1} - x_1\| \leq \|x_{n+2} - x_1\|, \tag{65}$$

for all $n \geq 1$. This implies that $\{\|x_n - x_1\|\}$ is increasing. Note that C is bounded, we get that $\{\|x_n - x_1\|\}$ is bounded. This shows that $\lim_{n\to\infty}\|x_n - x_1\|$ exists.

Since $x_{n+1} = P_{D_n} x_1$ and $x_{m+1} = P_{D_m} x_1 \in D_m \subset D_n$ for all $m \geq n$, we have

$$\langle x_{n+1} - x_1, x_{m+1} - x_{n+1} \rangle \geq 0. \tag{66}$$

It follows from (66) that

$$\begin{aligned}
&\|x_{m+1} - x_{n+1}\|^2 \\
&\quad = \|x_{m+1} - x_1 - (x_{n+1} - x_1)\|^2 \\
&\quad = \|x_{m+1} - x_1\|^2 + \|x_{n+1} - x_1\|^2 \\
&\qquad - 2\langle x_{m+1} - x_1, x_{n+1} - x_1 \rangle \\
&\quad = \|x_{m+1} - x_1\|^2 + \|x_{n+1} - x_1\|^2 \\
&\qquad - 2\langle x_{n+1} - x_1, x_{m+1} - x_{n+1} + x_{n+1} - x_1 \rangle \\
&\quad = \|x_{m+1} - x_1\|^2 + \|x_{n+1} - x_1\|^2 \\
&\qquad - 2\langle x_{n+1} - x_1, x_{m+1} - x_{n+1} \rangle \\
&\quad \leq \|x_{m+1} - x_1\|^2 - \|x_{n+1} - x_1\|^2.
\end{aligned} \tag{67}$$

By taking $m = n + 1$ in (67), we have

$$\|x_{n+2} - x_{n+1}\|^2 \leq \|x_{n+2} - x_1\|^2 - \|x_{n+1} - x_1\|^2. \tag{68}$$

Since the limits of $\| x_n - x_1 \|$ exist, we get that

$$\|x_{n+2} - x_{n+1}\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{69}$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{70}$$

Moreover, from (67), we also have

$$\lim_{m,n\to\infty} \|x_{m+1} - x_{n+1}\| = 0. \tag{71}$$

This shows that the sequence $\{x_n\}$ is a Cauchy sequence. Hence there is $x^* \in C$ such that

$$x_n \longrightarrow x^* \in C, \quad \text{as } n \longrightarrow \infty. \tag{72}$$

Step 3. We shall show that $\|x_n - u_n\| \to 0$ as $n \to \infty$.

Since $x_{n+1} \in C_n$ and $\lim_{n\to\infty} \| x_n - x_{n+1} \| = 0$, we have

$$\|y - x_{n+1}\| \leq \|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{73}$$

and hence

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{74}$$

Note that u_n can be rewritten as $u_n = B_{\lambda_n}(x_n - \lambda_n A x_n)$ for all $n \geq 1$. We take $p \in F$ thus we have $p = B_{\lambda_n}(p - \lambda_n A p)$. Since A is λ-inverse-strongly monotone, and $0 < \lambda_n < 2\lambda$, we know that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\|u_n - p\|^2 \\
&\quad = \|B_{\lambda_n}(x_n - \lambda_n A x_n) - B_{\lambda_n}(p - \lambda_n A p)\|^2 \\
&\quad \leq \|x_n - \lambda_n A x_n - p + \lambda_n A p\|^2 \\
&\quad = \|(x_n - p) - \lambda_n (A x_n - A p)\|^2 \\
&\quad = \|x_n - p\|^2 - 2\lambda_n \langle x_n - p, A x_n - A p \rangle \\
&\qquad + \lambda_n^2 \|A x_n - A p\|^2 \\
&\quad \leq \|x_n - p\|^2 - 2\lambda_n \lambda \|A x_n - A p\|^2 \\
&\qquad + \lambda_n^2 \|A x_n - A p\|^2 \\
&\quad = \|x_n - p\|^2 + \lambda_n (\lambda_n - 2\lambda) \|A x_n - A p\|^2 \\
&\quad \leq \|x_n - p\|^2.
\end{aligned} \tag{75}$$

Using (57) and (75), we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \Bigg\| \alpha_n (x_n - p) \\
&\qquad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \beta_n (S_i x_n - p) \\
&\qquad + (1 - \alpha_n)(1 - \beta_n)(u_n - p) \Bigg\|^2 \\
&\quad \leq \alpha_n \|x_n - p\|^2 \\
&\qquad + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \beta_n \|S_i x_n - p\|^2 \\
&\qquad + (1 - \alpha_n)(1 - \beta_n) \|u_n - p\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|x_n-p\|^2+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|x_n-p\|^2\\
&\quad+(1-\alpha_n)(1-\beta_n)\\
&\quad\times\left(\|x_n-p\|^2+\lambda_n(\lambda_n-2\lambda)\|Ax_n-Ap\|^2\right)\\
&=\|x_n-p\|^2+(1-\alpha_n)\\
&\quad\times(1-\beta_n)\lambda_n(\lambda_n-2\lambda)\|Ax_n-Ap\|^2,
\end{aligned}\tag{76}$$

and hence

$$\begin{aligned}
&(1-\alpha_n)(1-d)a(2\lambda-b)\|Ax_n-Ap\|^2\\
&\quad\leq(1-\alpha_n)(1-\beta_n)\lambda_n(2\lambda-\lambda_n)\|Ax_n-Ap\|^2\\
&\quad\leq\|x_n-p\|^2-\|y_n-p\|^2\\
&\quad\leq\|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|).
\end{aligned}\tag{77}$$

Note that $\{x_n\}$ and $\{y_n\}$ are bounded, $\alpha_n \to 0$, and x_n-y_n converges to 0, we get that

$$\lim_{n\to\infty}\|Ax_n-Ap\|\longrightarrow 0.\tag{78}$$

Using Theorem 6, we have

$$\begin{aligned}
&\|u_n-p\|^2\\
&\quad=\left\|B_{\lambda_n}(x_n-\lambda_nAx_n)-B_{\lambda_n}(p-\lambda_nAp)\right\|^2\\
&\quad\leq\langle x_n-\lambda_nAx_n-(p-\lambda_nAp),u_n-p\rangle\\
&\quad=\frac{1}{2}\left(\|x_n-\lambda_nAx_n-(p-\lambda_nAp)\|^2+\|u_n-p\|^2\right.\\
&\qquad\left.-\|x_n-\lambda_nAx_n-(p-\lambda_nAp)-(u_n-p)\|^2\right)\\
&\quad\leq\frac{1}{2}\left(\|x_n-p\|^2+\|u_n-p\|^2\right.\\
&\qquad\left.-\|x_n-u_n-\lambda_n(Ax_n-Ap)\|^2\right)\\
&\quad=\frac{1}{2}\left(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2\right.\\
&\qquad\left.+2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2\right).
\end{aligned}\tag{79}$$

This implies that

$$\begin{aligned}
&\|u_n-p\|^2\\
&\quad\leq\|x_n-p\|^2-\|x_n-u_n\|^2\\
&\qquad+2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2.
\end{aligned}\tag{80}$$

From (80), we have

$$\begin{aligned}
&\|y_n-p\|^2\\
&\quad=\left\|\alpha_n(x_n-p)+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n(S_ix_n-p)\right.\\
&\qquad\left.+(1-\alpha_n)(1-\beta_n)(u_n-p)\right\|^2\\
&\quad\leq\alpha_n\|x_n-p\|^2+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n\|S_ix_n-p\|^2\\
&\qquad+(1-\alpha_n)(1-\beta_n)\|u_n-p\|^2\\
&\quad\leq\alpha_n\|x_n-p\|^2+(1-\alpha_n)\beta_n\|x_n-p\|^2\\
&\qquad+(1-\alpha_n)(1-\beta_n)\\
&\qquad\times\left(\|x_n-p\|^2-\|x_n-u_n\|^2\right.\\
&\qquad\quad\left.+2\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle-\lambda_n^2\|Ax_n-Ap\|^2\right)\\
&\quad\leq\|x_n-p\|^2-(1-\alpha_n)(1-\beta_n)\|x_n-u_n\|^2\\
&\qquad+2(1-\alpha_n)(1-\beta_n)\lambda_n\langle x_n-u_n,Ax_n-Ap\rangle,
\end{aligned}\tag{81}$$

and hence

$$\begin{aligned}
&(1-d)(1-\alpha_n)\|x_n-u_n\|^2\\
&\quad\leq(1-\beta_n)(1-\alpha_n)\|x_n-u_n\|^2\\
&\quad\leq\|x_n-y_n\|(\|x_n-p\|+\|y_n-p\|)\\
&\qquad+2(1-\alpha_n)(1-\beta_n)\lambda_n\\
&\qquad\times\|x_n-u_n\|\left\|Ax_n-A_p\right\|.
\end{aligned}\tag{82}$$

From (78) and $\lim_{n\to\infty}\|x_n-y_n\|=0$, we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{83}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n-S_ix_n\|=0$, for all $i=0,1,\ldots$.

It follows from definition of scheme (57) that

$$\begin{aligned}
&y_n+\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n(x_n-S_ix_n)-(1-\alpha_n)\beta_nx_n\\
&\quad=\alpha_nx_n+(1-\alpha_n)(1-\beta_n)u_n,
\end{aligned}\tag{84}$$

that is,

$$\begin{aligned}
&\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\beta_n(x_n-S_ix_n)\\
&\quad=x_n-y_n-x_n+\alpha_nx_n+(1-\alpha_n)\beta_nx_n\\
&\qquad+(1-\alpha_n)(1-\beta_n)u_n
\end{aligned}$$

$$\begin{aligned}&= x_n - y_n + (1-\alpha_n)(\beta_n - 1)x_n\\&\quad + (1-\alpha_n)(1-\beta_n)u_n\\&= x_n - y_n + (1-\alpha_n)(1-\beta_n)(u_n - x_n).\end{aligned} \tag{85}$$

Hence, for any $p \in F$, one has

$$\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)\beta_n \langle x_n - S_i x_n, x_n - p\rangle \tag{86}$$

$$\begin{aligned}&= (1-\alpha_n)(1-\beta_n)\langle u_n - x_n, x_n - p\rangle\\&\quad + \langle x_n - y_n, x_n - p\rangle.\end{aligned} \tag{87}$$

Since each S_i is nonexpansive and by (36) we get that

$$\|S_i x_n - x_n\|^2 \le \langle x_n - S_i x_n, x_n - p\rangle. \tag{88}$$

Hence, combining this inequality with (86), we have

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)\beta_n \|S_i x_n - x_n\|^2\\&\quad \le (1-\alpha_n)(1-\beta_n)\langle u_n - x_n, x_n - p\rangle\\&\qquad + \langle x_n - y_n, x_n - p\rangle,\end{aligned} \tag{89}$$

that is,

$$\begin{aligned}\|S_i x_n - x_n\|^2 &\le \frac{2(1-\alpha_n)(1-\beta_n)}{(\alpha_{i-1}-\alpha_i)\beta_n}\langle u_n - x_n, x_n - p\rangle\\&\quad + \frac{2}{(\alpha_{i-1}-\alpha_i)\beta_n}\langle x_n - y_n, x_n - p\rangle\\&\le \frac{2(1-\alpha_n)(1-\beta_n)}{(\alpha_{i-1}-\alpha_i)\beta_n}\|u_n - x_n\|\|x_n - p\|\\&\quad + \frac{2}{(\alpha_{i-1}-\alpha_i)\beta_n}\|x_n - y_n\|\|x_n - p\|.\end{aligned} \tag{90}$$

Since $\|u_n - x_n\| \to 0$ and $\|x_n - y_n\| \to 0$, we have

$$\lim_{n\to\infty}\|S_i x_n - x_n\| = 0, \quad \forall i = 1, 2, \ldots. \tag{91}$$

Step 5. We show that $x_n \to x^* = P_F x_1$.

First, we show that $x^* \in \cap_{i=1}^{\infty}\mathrm{Fix}(S_i)$. Since

$$\lim_{n\to\infty} x_n = x^*, \qquad \lim_{n\to\infty}\|S_i x_n - x_n\| = 0, \tag{92}$$

we have

$$x^* \in \mathrm{Fix}(S_i) \quad \text{for each } i = 1, 2, \ldots. \tag{93}$$

Hence $x^* \in \cap_{i=1}^{\infty}\mathrm{Fix}(S_i)$. Next, we show that $x^* \in \mathrm{SGVEPR}(\varphi, T)$. Noting that $u_n = B_{\lambda_n}(x_n - \lambda_n A x_n)$, one obtains

$$\begin{aligned}&\varphi(u_n, y) + e\langle T u_n, \eta(y, u_n)\rangle\\&\quad + e\langle A x_n, y - u_n\rangle + \frac{e}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \in C,\\&\qquad \forall y \in X,\end{aligned} \tag{94}$$

which implies that

$$\begin{aligned}0 \in \varphi(y, u_n) - \Big\{&e\langle T u_n, \eta(y, u_n)\rangle\\&+ e\langle A x_n, y - u_n\rangle\\&+ \frac{e}{\lambda_n}\langle y - u_n, u_n - x_n\rangle\Big\} + C,\\&\qquad \forall y \in X.\end{aligned} \tag{95}$$

Put $v_t = ty + (1-t)x^*$, for all $t \in (0,1)$ and $y \in X$. Then, we have $v_t \in X$. So, from (95), we have

$$\begin{aligned}&e\langle v_t - u_n, A v_t\rangle\\&\quad \in e\langle v_t - u_n, A v_t\rangle\\&\qquad - e\langle v_t - u_n, A x_n\rangle - e\left\langle v_t - u_n, \frac{u_n - x_n}{\lambda_n}\right\rangle\\&\qquad + \varphi(v_t, u_n) + e\langle T u_n, \eta(u_n, v_t)\rangle + C.\\&\quad = e\langle v_t - u_n, A v_t - A u_n\rangle\\&\qquad + e\langle v_t - u_n, A u_n - A x_n\rangle\\&\qquad - e\left\langle v_t - u_n, \frac{u_n - x_n}{\lambda_n}\right\rangle + \varphi(v_t, u_n)\\&\qquad + e\langle T u_n, \eta(u_n, v_t)\rangle + C.\end{aligned} \tag{96}$$

Since $\|x_n - u_n\| \to 0$ and the properties of T, we have

$$\begin{gathered}\|A u_n - A x_n\| \longrightarrow 0,\\ \frac{u_n - x_n}{\lambda_n} \longrightarrow 0,\\ \langle v_t - u_n, A u_n - A x_n\rangle \longrightarrow 0.\end{gathered} \tag{97}$$

From the monotonicity of A, we have

$$\langle v_t - u_n, A v_t - A u_n\rangle \ge 0. \tag{98}$$

Thus

$$e\langle v_t - u_n, A v_t - A u_n\rangle \in C. \tag{99}$$

So, from (96)–(99) and η-hemicontinuity of T, we have

$$e\langle v_t - x^*, A v_t\rangle \in \varphi(v_t, x^*) + e\langle T x^*, \eta(x^*, v_t)\rangle + C. \tag{100}$$

Since φ is C-convex, we have

$$t\varphi(v_t, y) + (1-t)\varphi(v_t, x^*) \in \varphi(v_t, v_t) + C. \tag{101}$$

Since for any $u, v \in X$, and the mapping $x \mapsto \langle T v, \eta(x, u)\rangle$ is convex, we have

$$\begin{aligned}\langle T x^*, \eta(v_t, v_t)\rangle &\le t\langle T x^*, \eta(y, v_t)\rangle\\&\quad + (1-t)\langle T x^*, \eta(x^*, v_t)\rangle.\end{aligned} \tag{102}$$

This implies that

$$\begin{aligned} & et\langle Tx^*, \eta(y, v_t)\rangle + e(1-t)\langle Tx^*, \eta(x^*, v_t)\rangle \\ & \qquad \in e\langle Tx^*, \eta(v_t, v_t)\rangle + C. \end{aligned} \tag{103}$$

From (101) and (103), we get that

$$\begin{aligned} & t\varphi(v_t, y) + (1-t)\varphi(v_t, x^*) \\ & \qquad + et\langle Tx^*, \eta(y, v_t)\rangle + e(1-t)\langle Tx^*, \eta(x^*, v_t)\rangle \\ & \quad \in e\langle Tx^*, \eta(v_t, v_t)\rangle + \varphi(v_t, v_t) + C = C, \end{aligned} \tag{104}$$

which implies that

$$\begin{aligned} & -t(\varphi(v_t, y) + e\langle Tx^*, \eta(y, v_t)\rangle) \\ & \qquad - (1-t)(\varphi(v_t, x^*) + e\langle Tx^*, \eta(x^*, v_t)\rangle) \in -C. \end{aligned} \tag{105}$$

From (100) and (105), we have

$$\begin{aligned} & -t(\varphi(v_t, y) + e\langle Tx^*, \eta(y, v_t)\rangle) \\ & \quad \in (1-t)(\varphi(v_t, x^*) + e\langle Tx^*, \eta(x^*, v_t)\rangle) - C \\ & \quad \in (1-t)e\langle v_t - x^*, Av_t\rangle - C. \end{aligned} \tag{106}$$

This implies that

$$\begin{aligned} & -t(\varphi(v_t, y) + e\langle Tx^*, \eta(y, v_t)\rangle) \\ & \qquad - e(1-t)t\langle y - x^*, Av_t\rangle \in -C. \end{aligned} \tag{107}$$

It follows that

$$\begin{aligned} & (\varphi(v_t, y) + e\langle Tx^*, \eta(y, v_t)\rangle) \\ & \qquad + e(1-t)\langle y - x^*, Av_t\rangle \in C. \end{aligned} \tag{108}$$

As $t \to 0$, we obtain that for each $y \in X$,

$$\begin{aligned} & (\varphi(x^*, y) + e\langle Tx^*, \eta(y, x^*)\rangle) \\ & \qquad + e(1-t)\langle y - x^*, Ax^*\rangle \in C. \end{aligned} \tag{109}$$

Hence $x^* \in \mathrm{SGVEPR}(\varphi, T)$. Finally, we prove that $x^* = P_F x$. From $x_{n+1} = P_{D_n} x$ and $F \subset D_n$, we have

$$\langle x - x_{n+1}, x_{n+1} - v\rangle \geq 0, \quad \forall v \in F. \tag{110}$$

Note that $\lim_{n\to\infty} x_n = x^*$; we take the limit in (110), and then we have

$$\langle x - x^*, x^* - v\rangle \geq 0, \quad \forall v \in F. \tag{111}$$

We see that $x^* = P_F x$ by (33). This completes the proof. □

Remark 8. If $Y = \mathbb{R}$, $C = \mathbb{R}^+$, and $e = 1$, then Theorem 7 extends and improves Theorem 3.1 of Wang et al. [9].

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