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Research Article

Solvability of Some Boundary Value Problems for Fractional p-Laplacian Equation

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This paper considers the existence of solutions for two boundary value problems for fractional p-Laplacian equation. Under certain nonlinear growth conditions of the nonlinearity, two new existence results are obtained by using Schaefer's fixed point theorem. As an application, an example to illustrate our results is given.

1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. This subject, as old as the problem of ordinary differential calculus, can go back to the times when Leibniz and Newton invented differential calculus. As is known to all, the problem for fractional derivative was originally raised by Leibniz in a letter, dated September 30, 1695. A fractional derivative arises from many physical processes, such as a non-Markovian diffusion process with memory [1], charge transport in amorphous semiconductors [2], and propagations of mechanical waves in viscoelastic media [3], and so forth. Moreover, phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry, and material science are also described by differential equations of fractional order [4–8]. For instance, Pereira et al. [9] considered the following fractional Van der Pol equation:

$$D^{\lambda}x(t)+\alpha\left(x^{2}(t)-1\right)x'(t)+x(t)=0,\quad 1<\lambda<2, \tag{1}$$

where D^{λ} is the fractional derivative of order λ and α is a control parameter that reflects the degree of nonlinearity of the system. Equation (1) is obtained by substituting the capacitance by a fractance in the nonlinear RLC circuit model.

Recently, fractional differential equations have been of great interest due to the intensive development of the theory of fractional calculus itself and its applications. For example, for fractional initial value problems, the existence and multiplicity of solutions (or positive solutions) were discussed in [10–13]. On the other hand, for fractional boundary value problems (FBVPs), Agarwal et al. [14] considered a two-point boundary value problem at nonresonance, and Bai [15] considered a m-point boundary value problem at resonance. For more papers on FBVPs, see [16–21] and the references therein.

The turbulent flow in a porous medium is a fundamental mechanics problem. For studying this type of problems, Leibenson [22] introduced the p-Laplacian equation as follows:

$$\left(\phi_{p}\left(x'(t)\right)\right)'=f\left(t,x(t),x'(t)\right), \tag{2}$$

where $\phi_p(s)=|s|^{p-2}s$, $p>1$. Obviously, ϕ_p is invertible and its inverse operator is ϕ_q, where $q>1$ is a constant such that $1/p+1/q=1$.

In the past few decades, many important results relative to (2) with certain boundary value conditions have been obtained. We refer the reader to [23–27] and the references cited therein. For boundary value problems of fractional p-Laplacian equations, Chen and Liu [28] considered an antiperiodic boundary value problem with the following form:

$$\begin{gathered}D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}x(t)\right)=f\left(t,x(t)\right),\quad t\in[0,1],\\ x(0)=-x(1),\qquad D_{0^+}^{\alpha}x(0)=-D_{0^+}^{\alpha}x(1),\end{gathered} \tag{3}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, and $D_{0^+}^{\alpha}$ is Caputo fractional derivative. Under certain nonlinear growth conditions of the nonlinearity, an existence result was obtained by using degree theory. In addition, Yao et al. [29] studied a three-point boundary value problem given by

$$\begin{gathered} -D_t^{\beta}\left(\phi_p\left(D_t^{\alpha}x\right)\right)(t) = h(t) f(t, x(t)), \quad t \in (0,1), \\ x(0) = 0, \quad D_t^{\gamma} x(1) = a D_t^{\gamma} x(\xi), \quad D_t^{\alpha} x(0) = 0, \end{gathered} \tag{4}$$

where D_t^{α} is the standard Riemann-Liouville derivative with $1 < \alpha \leq 2$, $0 < \beta, \gamma \leq 1$, $0 \leq \alpha - \gamma - 1$, $\xi \in (0,1)$, and the constant a is a positive number satisfying $a\xi^{\alpha-\gamma-2} \leq 1-\gamma$. The monotone iterative technique was applied to establish the existence results on multiple positive solutions in [29].

Motivated by the works mentioned previously, in this paper, we investigate the existence of solutions for fractional p-Laplacian equation of the form

$$D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}x(t)\right) = f\left(t, x(t), D_{0^+}^{\alpha}x(t)\right), \quad t \in [0,1] \tag{5}$$

subject to either boundary value conditions

$$x(0) = D_{0^+}^{\alpha}x(1) = 0 \tag{6}$$

or

$$x(0) = \int_0^1 D_{0^+}^{\alpha}x(t)\, dt = 0, \tag{7}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha + \beta \leq 2$, $D_{0^+}^{\alpha}$ is a Caputo fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Note that the nonlinear operator $D_{0^+}^{\beta}\phi_p(D_{0^+}^{\alpha})$ reduces to the linear operator $D_{0^+}^{\beta}D_{0^+}^{\alpha}$ when $p = 2$ and the additive index law

$$D_{0^+}^{\beta}D_{0^+}^{\alpha}u(t) = D_{0^+}^{\alpha+\beta}u(t) \tag{8}$$

holds under some reasonable constraints on the function $u(t)$ [30].

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions, and lemmas. In Section 3, based on Schaefer's fixed point theorem, we establish two theorems on existence of solutions for FBVP (5) and (6) (Theorem 7) and FBVP (5) and (7) (Theorem 8). Finally, in Section 4, an explicit example is given to illustrate the main results. Our results are different from those of bibliographies listed in the previous texts.

2. Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory, which can be found, for instance, in [31, 32].

Definition 1. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\, ds, \tag{9}$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2. The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$\begin{aligned} D_{0^+}^{\alpha}u(t) &= I_{0^+}^{n-\alpha}\frac{d^n u(t)}{dt^n} \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)\, ds, \end{aligned} \tag{10}$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 3 (see [33]). *Let $\alpha > 0$. Assume that $u, D_{0^+}^{\alpha}u \in L(0,1)$. Then the following equality holds:*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}, \tag{11}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$; here n is the smallest integer greater than or equal to α.

Lemma 4 (see [34]). *For fixed $l \in C[0,1]$, let one define*

$$G_l(a) = \int_0^1 \phi_q(l(t) + a)\, dt. \tag{12}$$

Then the equation $G_l(a) = 0$ has a unique solution $\widetilde{a}(l)$.

In this paper, we take $Y = C[0,1]$ with the norm $\|y\|_{\infty} = \max_{t\in[0,1]}|y(t)|$ and $X = \{x \mid x, D_{0^+}^{\alpha}x \in Y\}$ with the norm $\|x\|_X = \max\{\|x\|_{\infty}, \|D_{0^+}^{\alpha}x\|_{\infty}\}$. By means of the linear functional analysis theory, we can prove that X is a Banach space.

3. Existence Results

In this section, two theorems on existence of solutions for FBVP (5) and (6) and FBVP (5) and (7) will be given under nonlinear growth restriction of f.

As a consequence of Lemma 3, we have the following results that are useful in what follows.

Lemma 5. *Given $h \in Y$, the unique solution of*

$$\begin{gathered} D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}x(t)\right) = h(t), \quad t \in [0,1], \\ x(0) = D_{0^+}^{\alpha}x(1) = 0 \end{gathered} \tag{13}$$

is

$$\begin{aligned} x(t) &= I_{0^+}^{\alpha}\phi_q\left(I_{0^+}^{\beta}h(t) + Ah(t)\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)\, d\tau \right. \\ &\qquad \left. +Ah(s)\right) ds, \end{aligned} \tag{14}$$

where

$$\begin{aligned} Ah(t) &= -I_{0^+}^{\beta} h(t)\Big|_{t=1} \\ &= -\frac{1}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} h(s)\, ds. \end{aligned} \tag{15}$$

Proof. Assume that $x(t)$ satisfies the equation of FBVP (13); then Lemma 3 implies that

$$\phi_p\left(D_{0^+}^{\alpha} x(t)\right) = I_{0^+}^{\beta} h(t) + c_0, \quad c_0 \in \mathbb{R}. \tag{16}$$

From the boundary value condition $D_{0^+}^{\alpha} x(1) = 0$, one has

$$c_0 = -I_{0^+}^{\beta} h(t)\Big|_{t=1} = Ah(t). \tag{17}$$

Thus, we have

$$x(t) = I_{0^+}^{\alpha} \phi_q\left(I_{0^+}^{\beta} h(t) + Ah(t)\right) + c_1, \quad c_1 \in \mathbb{R}. \tag{18}$$

By condition $x(0) = 0$, we get $c_1 = 0$. The proof is complete. □

Define the operator $F : X \to X$ by

$$\begin{aligned} &Fx(t) \\ &= I_{0^+}^{\alpha} \phi_q\left(I_{0^+}^{\beta} Nx(t) + ANx(t)\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\ &\quad \times \phi_q\Bigg(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} \\ &\qquad \times f\left(\tau, x(\tau), D_{0^+}^{\alpha} x(\tau)\right) d\tau \\ &\qquad - \frac{1}{\Gamma(\beta)} \int_0^1 (1-\tau)^{\beta-1} \\ &\qquad \times f\left(\tau, x(\tau), D_{0^+}^{\alpha} x(\tau)\right) d\tau\Bigg) ds, \\ &\qquad \forall t \in [0,1], \end{aligned} \tag{19}$$

where $N : X \to Y$ is the Nemytskii operator defined by

$$Nx(t) = f\left(t, x(t), D_{0^+}^{\alpha} x(t)\right), \quad \forall t \in [0,1]. \tag{20}$$

Clearly, the fixed points of the operator F are solutions of FBVP (5) and (6).

Lemma 6. *Given $h \in Y$, the unique solution of*

$$\begin{gathered} D_{0^+}^{\beta} \phi_p\left(D_{0^+}^{\alpha} x(t)\right) = h(t), \quad t \in [0,1], \\ x(0) = \int_0^1 D_{0^+}^{\alpha} x(t)\, dt = 0 \end{gathered} \tag{21}$$

is

$$\begin{aligned} x(t) &= I_{0^+}^{\alpha} \phi_q\left(I_{0^+}^{\beta} h(t) + Bh(t)\right) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q\Bigg(\frac{1}{\Gamma(\beta)} \int_0^s (s-\tau)^{\beta-1} h(\tau)\, d\tau \\ &\qquad + Bh(s)\Bigg) ds, \end{aligned} \tag{22}$$

where

$$\begin{aligned} Bh(t) &= -I_{0^+}^{\beta} h(t)\Big|_{t=\eta(h)} \\ &= -\frac{1}{\Gamma(\beta)} \int_0^{\eta(h)} (\eta(h) - s)^{\beta-1} h(s)\, ds, \end{aligned} \tag{23}$$

here $\eta(h) \in (0,1)$ is a constant dependent on $h(t)$.

Proof. Assume that $x(t)$ satisfies the equation of FBVP (21); then Lemma 3 implies that

$$D_{0^+}^{\alpha} x(t) = \phi_q\left(I_{0^+}^{\beta} h(t) + c_0\right), \quad c_0 \in \mathbb{R}. \tag{24}$$

From condition $\int_0^1 D_{0^+}^{\alpha} x(t) dt = 0$, one has

$$\int_0^1 \phi_q\left(I_{0^+}^{\beta} h(t) + c_0\right) dt = 0. \tag{25}$$

Based on Lemma 4, we know that (25) has a unique solution $\widetilde{c_0}(h)$. Moreover, by the integral mean value theorem, there exists a constant $\eta(h) \in (0,1)$ such that $\phi_q(I_{0^+}^{\beta} h(t) + \widetilde{c_0}(h))|_{t=\eta(h)} = 0$, which implies that $(I_{0^+}^{\beta} h(t) + \widetilde{c_0}(h))|_{t=\eta(h)} = 0$. Thus, we have

$$\widetilde{c_0}(h) = -I_{0^+}^{\beta} h(t)\Big|_{t=\eta(h)} = Bh(t). \tag{26}$$

Hence

$$x(t) = I_{0^+}^{\alpha} \phi_q\left(I_{0^+}^{\beta} h(t) + Bh(t)\right) + c_1, \quad c_1 \in \mathbb{R}. \tag{27}$$

From condition $x(0) = 0$, we get $c_1 = 0$. The proof is complete. □

Define the operator $P : X \to X$ by

$$\begin{aligned} Px(t) &= I_{0^+}^{\alpha}\phi_q\left(I_{0^+}^{\beta}Nx(t) + BNx(t)\right) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ &\quad \times \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} \right. \\ &\qquad \times f\left(\tau, x(\tau), D_{0^+}^{\alpha}x(\tau)\right)d\tau \\ &\qquad - \frac{1}{\Gamma(\beta)}\int_0^{\eta(x)} (\eta(x)-\tau)^{\beta-1} \\ &\qquad \left. \times f\left(\tau, x(\tau), D_{0^+}^{\alpha}x(\tau)\right)d\tau\right) ds, \\ &\qquad \forall t \in [0,1], \end{aligned} \tag{28}$$

where $\eta(x) \in (0,1)$ and N is the Nemytskii operator defined by (20). Clearly, the fixed points of the operator P are solutions of FBVP (5) and (7).

Our first result, based on Schaefer's fixed point theorem and Lemma 5, is stated as follows.

Theorem 7. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

(H) *there exist nonnegative functions $a, b, c \in Y$ such that*

$$\begin{aligned} |f(t,u,v)| &\le a(t) + b(t)|u|^{p-1} \\ &\quad + c(t)|v|^{p-1}, \quad \forall t \in [0,1], (u,v) \in \mathbb{R}^2. \end{aligned} \tag{29}$$

Then FBVP (5) *and* (6) *has at least one solution, provided that*

$$\frac{2}{\Gamma(\beta+1)}\left(\frac{\|b\|_{\infty}}{(\Gamma(\alpha+1))^{p-1}} + \|c\|_{\infty}\right) < 1. \tag{30}$$

Proof. We will use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given in the following two steps.

Step 1. $F : X \to X$ is completely continuous.

Let $\Omega \subset X$ be an open bounded subset. By the continuity of f, we can get that F is continuous and $F(\overline{\Omega})$ is bounded. Moreover, there exists a constant $T > 0$ such that $|I_{0^+}^{\beta}Nx + ANx| \le T$, for all $x \in \overline{\Omega}, t \in [0,1]$. Thus, in view of the Arzelà-Ascoli theorem, we need only to prove that $F(\overline{\Omega}) \subset X$ is equicontinuous.

For $0 \le t_1 < t_2 \le 1$ and $x \in \overline{\Omega}$, we have

$$\begin{aligned} &\left|Fx(t_2) - Fx(t_1)\right| \\ &= \frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}\phi_q\left(I_{0^+}^{\beta}Nx(s) + ANx(s)\right)ds \right. \\ &\qquad \left. - \int_0^{t_1}(t_1-s)^{\alpha-1}\phi_q\left(I_{0^+}^{\beta}Nx(s) + ANx(s)\right)ds\right| \\ &\le \frac{T^{q-1}}{\Gamma(\alpha)}\left\{\int_0^{t_1}\left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right]ds \right. \\ &\qquad \left. + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right\} \\ &= \frac{T^{q-1}}{\Gamma(\alpha+1)}\left[t_1^{\alpha} - t_2^{\alpha} + 2(t_2-t_1)^{\alpha}\right] \\ &\le \frac{T^{q-1}}{\Gamma(\alpha+1)}\left[t_2^{\alpha} - t_1^{\alpha} + 2(t_2-t_1)^{\alpha}\right]. \end{aligned} \tag{31}$$

Since t^{α} is uniformly continuous on $[0,1]$, we can obtain that $F(\overline{\Omega}) \subset Y$ is equicontinuous. A similar proof can show that $(I_{0^+}^{\beta}N + AN)(\overline{\Omega}) \subset Y$ is equicontinuous. This, together with the uniform continuity of $\phi_q(s)$ on $[-T,T]$, yields that $D_{0^+}^{\alpha}F(\overline{\Omega})(= \phi_q(I_{0^+}^{\beta}N + AN)(\overline{\Omega})) \subset Y$ is also equicontinuous.

Step 2 (priori bounds). Set

$$\Omega = \left\{x \in X \mid x = \lambda^{q-1}Fx, \lambda \in (0,1)\right\}. \tag{32}$$

Now it remains to show that the set Ω is bounded.

From Lemma 3 and boundary value condition $x(0) = 0$, one has

$$\begin{aligned} x(t) &= I_{0+}^{\alpha}D_{0^+}^{\alpha}x(t) \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}D_{0^+}^{\alpha}x(s)\,ds. \end{aligned} \tag{33}$$

Thus, we get

$$\begin{aligned} |x(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|D_{0^+}^{\alpha}x(s)\right|ds \\ &\le \frac{1}{\Gamma(\alpha)}\left\|D_{0^+}^{\alpha}x\right\|_{\infty} \cdot \frac{1}{\alpha}t^{\alpha} \\ &\le \frac{1}{\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}x\right\|_{\infty}, \quad \forall t \in [0,1]. \end{aligned} \tag{34}$$

That is,

$$\|x\|_{\infty} \le \frac{1}{\Gamma(\alpha+1)}\left\|D_{0^+}^{\alpha}x\right\|_{\infty}. \tag{35}$$

For $x \in \Omega$, we have

$$\begin{aligned}\phi_p\left(D_{0^+}^{\alpha} x(t)\right) &= \lambda\left(I_{0^+}^{\beta} N x(t) + A N x(t)\right) \\ &= \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\left(s, x(s), D_{0^+}^{\alpha} x(s)\right) ds \\ &\quad - \frac{\lambda}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f\left(s, x(s), D_{0^+}^{\alpha} x(s)\right) ds.\end{aligned} \tag{36}$$

So, from (H), we obtain that

$$\begin{aligned}&\left|\phi_p\left(D_{0^+}^{\alpha} x(t)\right)\right| \\ &\leq \frac{2}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}\left|f\left(s, x(s), D_{0^+}^{\alpha} x(s)\right)\right| ds \\ &\leq \frac{2}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} \\ &\qquad \times\left(a(s) + b(s)|x(s)|^{p-1} + c(s)\left|D_{0^+}^{\alpha} x(s)\right|^{p-1}\right) ds \\ &\leq \frac{2}{\Gamma(\beta)}\left(\|a\|_{\infty} + \|b\|_{\infty}\|x\|_{\infty}^{p-1} + \|c\|_{\infty}\left\|D_{0^+}^{\alpha} x\right\|_{\infty}^{p-1}\right) \cdot \frac{1}{\beta} \\ &= \frac{2}{\Gamma(\beta+1)}\left(\|a\|_{\infty} + \|b\|_{\infty}\|x\|_{\infty}^{p-1} + \|c\|_{\infty}\left\|D_{0^+}^{\alpha} x\right\|_{\infty}^{p-1}\right), \\ &\qquad \forall t \in [0,1],\end{aligned} \tag{37}$$

which, together with $|\phi_p(D_{0^+}^{\alpha} x(t))| = |D_{0^+}^{\alpha} x(t)|^{p-1}$ and (35), yields that

$$\begin{aligned}\left\|D_{0^+}^{\alpha} x\right\|_{\infty}^{p-1} &\leq \frac{2}{\Gamma(\beta+1)} \\ &\quad \times\left[\|a\|_{\infty} + \frac{\|b\|_{\infty}}{(\Gamma(\alpha+1))^{p-1}}\left\|D_{0^+}^{\alpha} x\right\|_{\infty}^{p-1}\right. \\ &\quad \left. + \|c\|_{\infty}\left\|D_{0^+}^{\alpha} x\right\|_{\infty}^{p-1}\right].\end{aligned} \tag{38}$$

In view of (30), from (38), we can see that there exists a constant $M_1 > 0$ such that

$$\left\|D_{0^+}^{\alpha} x\right\|_{\infty} \leq M_1. \tag{39}$$

Thus, from (35), we get

$$\|x\|_{\infty} \leq \frac{M_1}{\Gamma(\alpha+1)} := M_2. \tag{40}$$

Combining (39) with (40), we have

$$\|x\|_X = \max\left\{\|x\|_{\infty}, \left\|D_{0^+}^{\alpha} x\right\|_{\infty}\right\} \leq \max\left\{M_1, M_2\right\}. \tag{41}$$

As a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is the solution of FBVP (5) and (6). The proof is complete. □

Our second result, based on Schaefer's fixed point theorem and Lemma 6, is stated as follows.

Theorem 8. *Let $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Suppose that (H) holds; then FBVP (5) and (7) has at least one solution, provided that (30) is satisfied.*

Proof. The proof work is similar to the proof of Theorem 7, so we omit the details. □

4. An Example

In this section, we will give an example to illustrate our main results.

Example 1. Consider the following fractional p-Laplacian equation:

$$\begin{aligned}&D_{0^+}^{3/4} \phi_3\left(D_{0^+}^{1/2} x(t)\right) \\ &\quad = -\frac{49}{6} + \frac{1}{6} x^2(t) + t e^{-|D_{0^+}^{1/2} x(t)|}, \quad t \in [0,1].\end{aligned} \tag{42}$$

Corresponding to (5), we get that $p = 3$, $\alpha = 1/2$, $\beta = 3/4$, and

$$f(t, u, v) = -\frac{49}{6} + \frac{1}{6} u^2 + t e^{-|v|}. \tag{43}$$

Choose $a(t) = 10$, $b(t) = 1/6$, and $c(t) = 0$. By a simple calculation, we can obtain that $\|b\|_{\infty} = 1/6$, $\|c\|_{\infty} = 0$ and

$$\frac{2}{\Gamma(3/4+1)}\left(\frac{1/6}{(\Gamma(1/2+1))^2} + 0\right) < 1. \tag{44}$$

Obviously, (42) subject to boundary value conditions (6) (or (7)) satisfies all assumptions of Theorem 7 (or Theorem 8). Hence, FBVP (42) and (6) (or FBVP (42) and (7)) has at least one solution.

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