Research Article

Periodic Loop Solutions and Their Limit Forms for the Kudryashov-Sinelshchikov Equation

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The Kudryashov-Sinelshchikov equation is studied by using the bifurcation method of dynamical systems and the method of phase portraits analysis. We show that the limit forms of periodic loop solutions contain loop soliton solutions, smooth periodic wave solutions, and periodic cusp wave solutions. Also, some new exact travelling wave solutions are presented through some special phase orbits.

1. Introduction

A mixture of liquid and gas bubbles of the same size may be considered as an example of a classic nonlinear medium. In practice, analysis of propagation of the pressure waves in a liquid with gas bubbles is important problem. We know that there are solitary and periodic waves in a mixture of a liquid and gas bubbles and these waves can be described by nonlinear partial differential equations. As for examples of nonlinear differential equations to describe the pressure waves in bubbly liquids, we can point out the Burgers equation, the Korteweg-de Vries equation, the Burgers-Korteweg-de Vries equation, and so on [1].

In 2010, Kudryashov and Sinelshchikov [1] obtained a more common nonlinear partial differential equation for describing the pressure waves in a mixture liquid and gas bubbles taking into consideration the viscosity of liquid and the heat transfer, and the equation reads as follows:

$$u_t + \alpha u u_x + u_{xxx} - (u u_{xx})_x - \beta u_x u_{xx} = 0, \tag{1.1}$$

where u is a density and which model heat transfer and viscosity, α, β are real parameters. Equation (1.1) is called Kudryashov-Sinelshchikov equation, it is generalization of the

KdV and the BKdV equation and similar but not identical to the Camassa-Holm equation. Undistorted waves are governed by a corresponding ordinary differential equation which, for special values of some integration constant, is solved analytically in [1]. Ryabov [2] obtained some exact solutions for $\beta = -3$ and $\beta = -4$ using a modification of the truncated expansion method. Solutions are derived in a more straightforward manner and cast into a simpler form, and some new types of solutions which contain solitary wave and periodic wave solutions are presented in [3].

In this paper, we focus on the case $\beta = 2$ of (1.1) using the bifurcation theory and the method of phase portraits analysis [4–6], we will investigate periodic loop solutions and their limit forms and give some new exact travelling wave solutions.

2. Preliminary

In this paper, we always consider the case $\beta = 2$, so from now on we assume $\beta = 2$ in (1.1) without mentioning it further.

Substituting $u(x,t) = 1 - \phi(x + \alpha t) = 1 - \phi(\xi)$ into (1.1) and integrating the resulting equation once with respect to ξ, we obtain

$$g - 2\alpha\phi + \frac{\alpha}{2}\phi^2 - \phi\phi'' - (\phi')^2 = 0, \tag{2.1}$$

where g is the integral constant.

Letting $y = d\phi/d\xi$, we get the following planar system:

$$\frac{d\phi}{d\xi} = y, \qquad \frac{dy}{d\xi} = \frac{g - 2\alpha\phi + (\alpha/2)\phi^2 - y^2}{\phi}. \tag{2.2}$$

Using the transformation $d\xi = \phi d\tau$, it carries (2.2) into the Hamiltonian system:

$$\frac{d\phi}{d\tau} = \phi y, \qquad \frac{dy}{d\tau} = g - 2\alpha\phi + \frac{\alpha}{2}\phi^2 - y^2. \tag{2.3}$$

Since both system (2.2) and (2.3) have the same first integral:

$$\phi^2\left(y^2 - g + \frac{4}{3}\alpha\phi - \frac{1}{4}\alpha\phi^2\right) = h, \tag{2.4}$$

then the two systems above have the same topological phase portraits except the line $\phi = 0$. Therefore, we can obtain the bifurcation phase portraits of system (2.2) from that of system (2.3).

Write $\Delta = 2\alpha(2\alpha - g)$. Clearly, when $\Delta > 0$, system (2.3) has two equilibrium points at $(\phi_{1,2}, 0)$ in ϕ-axis, where $\phi_{1,2} = (2\alpha \pm \sqrt{\Delta})/\alpha$. When $\Delta = 0$, system (2.3) has only one equilibrium point at $(2, 0)$ in ϕ-axis. When $\Delta < 0$, system (2.3) has no any equilibrium point in ϕ-axis. When $g > 0$, there exist two equilibrium points of system (2.3) in line $\phi = 0$ at $(0, \pm\sqrt{g})$.

Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system of (2.3) at equilibrium point (ϕ_e, y_e), $J = \det(M(\phi_e, y_e))$, and $T = \text{trace}(M(\phi_e, y_e))$. By the theory of planar

dynamical systems, we know that for an equilibrium point (ϕ_e, y_e) of a planar integrable system, (ϕ_e, y_e) is a saddle point if $J < 0$, a center point if $J > 0$ and $T = 0$, a cusp if $J = 0$ and the Poincaré index of (ϕ_e, y_e) is zero. By using the properties of equilibrium points and bifurcation method of dynamical systems, we can show that the bifurcation phase portraits of systems (2.2) and (2.3) is as drawn in Figure 1.

From Figures 1(b), 1(c), 1(d), 1(e), and 1(l), we have the following results.

3. Main Results

Proposition 3.1. (i) *When $\alpha < 0$, $g = 2\alpha$, for $h = (-4/3)\alpha$ defined by (2.4), (1.1) has a loop-soliton solution.*

(ii) *When $\alpha < 0$, $g = 2\alpha$, for $h \in (0, (-4/3)\alpha)$, there exists a family of uncountably infinite many periodic loop solutions of (1.1). Moreover, the periodic loop solutions converge to the loop-soliton solution as h approaches $(-4/3)\alpha$.*

(iii) *When $\alpha < 0$, $g = 2\alpha$, for $h \in ((-4/3)\alpha, +\infty)$, there exists a family of uncountably infinite many periodic loop solutions of (1.1). Moreover, the periodic loop solutions converge to the loop-soliton solution as h approaches $(-4/3)\alpha$.*

Proposition 3.2. *Denote that $h_1 = H(\phi_1, 0)$ and $h_2 = H(\phi_2, 0)$.*

(i) *When $\alpha < 0$, $2\alpha < g < 0$, for $h = h_1$ defined by (2.4), (1.1) has a loop-soliton solution and has a solitary wave solution.*

(ii) *When $\alpha < 0$, $2\alpha < g < 0$, for $h \in (h_2, h_1)$, there exist a family of uncountably infinite many periodic loop solutions and a family of uncountably infinite many smooth periodic wave solutions of (1.1). Moreover, the periodic loop solutions converge to the loop-soliton solution and the smooth periodic wave solutions converge to the solitary wave solution as h approaches h_1.*

(iii) *When $\alpha < 0$, $2\alpha < g < 0$, for $h \in (h_1, +\infty)$, there exists a family of uncountably infinite many periodic loop solutions of (1.1). Moreover, the periodic loop solutions converge to the loop-soliton solution as h approaches h_1.*

(iv) *When $\alpha < 0$, $2\alpha < g < 0$, for $h \in (0, h_2]$, there exists a family of uncountably infinite many periodic loop solutions of (1.1).*

Proposition 3.3. (i) *When $\alpha < 0$, $g = 0$, for $h = 0$ defined by (2.4), (1.1) has a smooth periodic wave solution.*

(ii) *When $\alpha < 0$, $g = 0$, for $h \in (0, +\infty)$, there exists a family of uncountably infinite many periodic loop solutions of (1.1). Moreover, the periodic loop solutions converge to the smooth periodic wave solution as h approaches 0.*
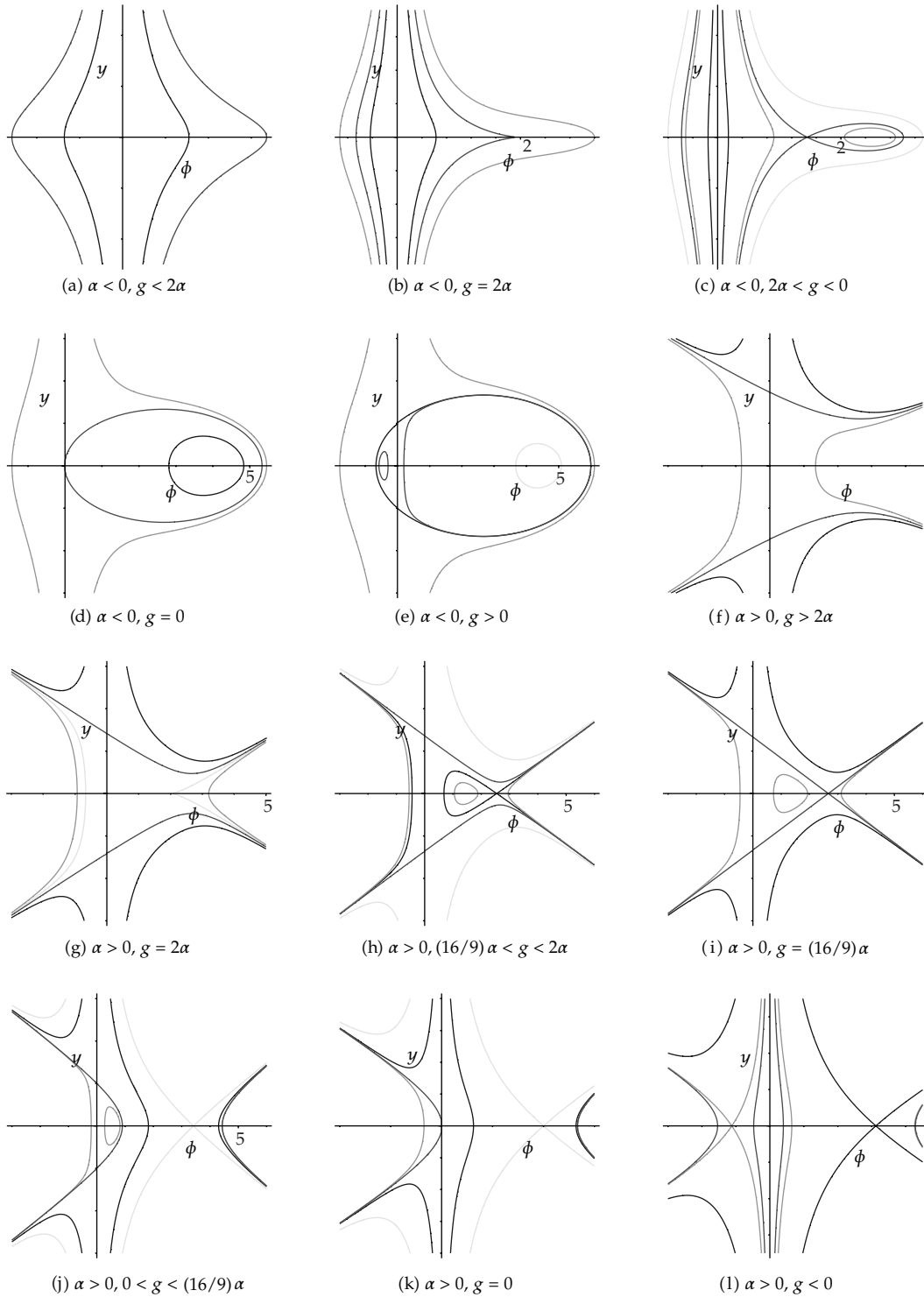
Proposition 3.4. (i) *When $\alpha < 0$, $g > 0$, for $h = 0$ defined by (2.4), (1.1) has two cusp periodic wave solutions.*

(ii) *When $\alpha < 0$, $g > 0$, for $h \in (0, +\infty)$, there exists a family of uncountably infinite many periodic loop solutions of (1.1). Moreover, the periodic loop solutions converge to the cusp periodic wave solutions as h approaches 0.*

Proposition 3.5. *Denote that $h_2 = H(\phi_2, 0)$.*

(i) *When $\alpha > 0$, $g < 0$, for $h = h_2$ defined by (2.4), (1.1) has a loop-soliton solution.*

(ii) *When $\alpha > 0$, $g < 0$, for $h \in (0, h_2)$, there exists a family of uncountably infinite many periodic loop solutions of (1.1). Moreover, the periodic loop solutions converge to the loop-soliton solution as h approaches h_2.*

(a) $\alpha < 0$, $g < 2\alpha$

(b) $\alpha < 0$, $g = 2\alpha$

(c) $\alpha < 0$, $2\alpha < g < 0$

(d) $\alpha < 0$, $g = 0$

(e) $\alpha < 0$, $g > 0$

(f) $\alpha > 0$, $g > 2\alpha$

(g) $\alpha > 0$, $g = 2\alpha$

(h) $\alpha > 0$, $(16/9)\alpha < g < 2\alpha$

(i) $\alpha > 0$, $g = (16/9)\alpha$

(j) $\alpha > 0$, $0 < g < (16/9)\alpha$

(k) $\alpha > 0$, $g = 0$

(l) $\alpha > 0$, $g < 0$

Figure 1: The bifurcation phase portraits of systems (2.2) and (2.3).

4. Exact Traveling Wave Solutions of the Kudryashov-Sinelshchikov Equation

Corresponding to Figure 1(b), the graph defined by $H(\phi, y) = (-4/3)\alpha$ consist of two hyperbolic sectors of the cusp $(2,0)$ and an open-end curve Γ_0 passing through the point $(-2/3, 0)$. It follows from (2.4) that

$$y = \pm \frac{(2 - \phi)\sqrt{-\alpha(2 - \phi)(\phi + 2/3)}}{2\phi}, \quad -\frac{2}{3} \leq \phi < 2, \ \phi \neq 0. \tag{4.1}$$

Substituting (4.1) into the $d\phi/d\xi = y$ and integrating along the curve Γ_0 and noting that $u(x, t) = 1 - \phi(x + \alpha t) = 1 - \phi(\xi)$, we obtain the following representation of loop-soliton solution:

$$u(\chi) = 1 - 2\sin^2(\chi) + \frac{2}{3}\cos^2(\chi),$$

$$\xi(\chi) = \frac{4}{\sqrt{-\alpha}}\left(\frac{3}{4}\tan(\chi) - \chi\right), \tag{4.2}$$

where χ is a new parametric variable.

Corresponding to Figure 1(c), the graph defined by $H(\phi, y) = H(\phi_1, 0)$ consists of an open-end curve Γ_1 passing through the point $(\phi_m, 0)$ and a homoclinic orbit connecting with saddle point $(\phi_1, 0)$ and passing point $(\phi_M, 0)$, where $\phi_m = (3\sqrt{\Delta} - 2\alpha - 2\sqrt{2\alpha(2\alpha - 3\sqrt{\Delta})})/-3\alpha$, $\phi_M = (3\sqrt{\Delta} - 2\alpha + 2\sqrt{2\alpha(2\alpha - 3\sqrt{\Delta})})/-3\alpha$. It follows from (2.4) that

$$y = \pm \frac{(\phi_1 - \phi)\sqrt{-\alpha(\phi_M - \phi)(\phi - \phi_m)}}{2\phi}, \quad \phi_m \leq \phi < \phi_1, \ \phi \neq 0, \tag{4.3}$$

$$y = \pm \frac{(\phi - \phi_1)\sqrt{-\alpha(\phi_M - \phi)(\phi - \phi_m)}}{2\phi}, \quad \phi_1 \leq \phi \leq \phi_M. \tag{4.4}$$

Substituting (4.3) into the $d\phi/d\xi = y$ and integrating along the curve Γ_1, we can obtain the following representation of loop-soliton solution:

$$u(\chi) = 1 - \frac{\phi_1\left[\phi_M\sinh^2(\omega\chi) - \phi_m\cosh^2(\omega\chi)\right] + \phi_m\phi_M}{\left[\phi_M\cosh^2(\omega\chi) - \phi_m\sinh^2(\omega\chi)\right] - \phi_1},$$

$$\xi(\chi) = \frac{2}{\sqrt{-\alpha}}\left(\chi - 2\arctan\left(\sqrt{\frac{\phi_1 - \phi_m}{\phi_M - \phi_1}}\tanh(\omega\chi)\right)\right), \tag{4.5}$$

where $\omega = \sqrt{(\phi_1 - \phi_m)(\phi_M - \phi_1)}/2\phi_1$.

Substituting (4.4) into the $d\phi/d\xi = y$ and integrating along the homoclinic orbit, we can obtain the following representation of solitary wave solution:

$$u(\chi) = 1 - \frac{\phi_1\left[\phi_M\cosh^2(\omega\chi) - \phi_m\sinh^2(\omega\chi)\right] - \phi_m\phi_M}{\left[\phi_M\sinh^2(\omega\chi) - \phi_m\cosh^2(\omega\chi)\right] + \phi_1},$$

$$\xi(\chi) = \frac{2}{\sqrt{-\alpha}}\left(\chi - 2\arctan\left(\sqrt{\frac{\phi_M - \phi_1}{\phi_1 - \phi_m}}\tanh(\omega\chi)\right)\right), \tag{4.6}$$

where $\omega = \sqrt{(\phi_1 - \phi_m)(\phi_M - \phi_1)}/2\phi_1$.

Moreover, the graph defined by $H(\phi, y) = h$, $h \in (H(\phi_2, 0), H(\phi_1, 0))$, consists of two open-end curves Γ_2, Γ_3, and a periodic orbit, say Ψ, enclosing the center point $(\phi_2, 0)$. The curve Ψ passes through the points $(\gamma_1, 0)$ and $(\gamma_2, 0)$, while Γ_2, Γ_3 pass through the points $(\gamma_3, 0)$ and $(\gamma_4, 0)$, respectively, where $\gamma_1, \gamma_2, \gamma_3, \gamma_4(\gamma_4 < 0 < \gamma_3 < \gamma_2 < \gamma_1)$ are four real roots of $\psi^4 - (16/3)\psi^3 + (4g/\alpha)\psi^2 + (4h/\alpha) = 0$. It follows from (2.4) that

$$y = \pm\frac{\sqrt{-\alpha(\gamma_1 - \phi)(\gamma_2 - \phi)(\gamma_3 - \phi)(\phi - \gamma_4)}}{2\phi}, \quad \gamma_4 \le \phi \le \gamma_3, \ \phi \ne 0, \tag{4.7}$$

$$y = \pm\frac{\sqrt{-\alpha(\gamma_1 - \phi)(\phi - \gamma_2)(\phi - \gamma_3)(\phi - \gamma_4)}}{2\phi}, \quad \gamma_2 \le \phi \le \gamma_1. \tag{4.8}$$

Let us denote by $F(\cdot, k)$ and $\Pi(\cdot, \cdot, k)$ the Legendre's incomplete elliptic integrals of the first and third kinds, respectively, with the modulus k (see [7]).

Substituting (4.7) into the $d\phi/d\xi = y$ and integrating along the curve Γ_2, we can obtain the implicit representation of periodic loop solution for $u \in [1 - \gamma_3, 1 - \gamma_4]$:

$$\frac{4\gamma_3}{\alpha_1^2\sqrt{-\alpha(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}}$$

$$\times\left[(\alpha_1^2 - \alpha_2^2)\Pi\left(\arcsin\left(\sqrt{\frac{(\gamma_2 - \gamma_4)(\gamma_3 + u - 1)}{(\gamma_3 - \gamma_4)(\gamma_2 + u - 1)}}\right), \alpha_1^2, k\right)\right.$$

$$\left.+\alpha_2^2 F\left(\arcsin\left(\sqrt{\frac{(\gamma_2 - \gamma_4)(\gamma_3 + u - 1)}{(\gamma_3 - \gamma_4)(\gamma_2 + u - 1)}}\right), k\right)\right] = \pm\xi, \tag{4.9}$$

where $\alpha_1^2 = (\gamma_3 - \gamma_4)/(\gamma_2 - \gamma_4)$, $\alpha_2^2 = \gamma_2(\gamma_3 - \gamma_4)/\gamma_3(\gamma_2 - \gamma_4)$, $k = \sqrt{(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)/(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}$.

Substituting (4.8) into the $d\phi/d\xi = y$ and integrating along the periodic orbit, we can obtain the implicit representation of smooth periodic wave solution for $u \in [1 - \gamma_1, 1 - \gamma_2]$:

$$
\frac{4\gamma_2}{\alpha_1^2 \sqrt{-\alpha(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}}
$$

$$
\times \left[(\alpha_1^2 - \alpha_2^2) \Pi \left(\arcsin \left(\sqrt{\frac{(\gamma_1 - \gamma_3)(1 - u - \gamma_2)}{(\gamma_1 - \gamma_2)(1 - u - \gamma_3)}} \right), \alpha_1^2, k \right) \right.
$$

$$
\left. + \alpha_2^2 F \left(\arcsin \left(\sqrt{\frac{(\gamma_1 - \gamma_3)(1 - u - \gamma_2)}{(\gamma_1 - \gamma_2)(1 - u - \gamma_3)}} \right), k \right) \right] = \pm \xi, \tag{4.10}
$$

where $\alpha_1^2 = (\gamma_1 - \gamma_2)/(\gamma_1 - \gamma_3)$, $\alpha_2^2 = \gamma_3(\gamma_1 - \gamma_2)/\gamma_2(\gamma_1 - \gamma_3)$, $k = \sqrt{(\gamma_1 - \gamma_2)(\gamma_3 - \gamma_4)/(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}$.

Corresponding to Figure 1(d), the graph defined by $H(\phi, y) = 0$ is a periodic orbit enclosing the center point $(2(\alpha - \sqrt{\alpha(\alpha - 1)})/\alpha, 0)$ and passing through the points $(0, 0)$, $(16/3, 0)$. It follows from (2.4) that

$$
y = \pm \frac{1}{2} \sqrt{-\alpha\phi \left(\frac{16}{3} - \phi \right)}, \quad 0 \le \phi \le \frac{16}{3}. \tag{4.11}
$$

Substituting (4.11) into the $d\phi/d\xi = y$ and integrating along the periodic orbit, we can obtain the following representation of smooth periodic wave solution:

$$
u(x, t) = 1 - \frac{16}{3} \cos^2(\omega(x + \alpha t)), \tag{4.12}
$$

where $\omega = (1/4)\sqrt{-\alpha}$.

Corresponding to Figure 1(e), the graph defined by $H(\phi, y) = 0$ consists of four heteroclinic orbits: two of them connecting the saddle points $(0, \pm\sqrt{g})$ with $(\phi_m, 0)$, and the others connecting saddle points $(0, \pm\sqrt{g})$ with $(\phi_M, 0)$, where $\phi_m = 2(4\alpha + \sqrt{\alpha(16\alpha - 9g)})/3\alpha$, $\phi_M = 2(4\alpha - \sqrt{\alpha(16\alpha - 9g)})/3\alpha$. It follows from (2.4) that

$$
y = \pm \frac{1}{2} \sqrt{-\alpha(\phi - \phi_m)(\phi_M - \phi)}, \quad \phi_m \le \phi \le 0, \tag{4.13}
$$

$$
y = \pm \frac{1}{2} \sqrt{-\alpha(\phi - \phi_m)(\phi_M - \phi)}, \quad 0 \le \phi \le \phi_M. \tag{4.14}
$$

Substituting (4.13) into the $d\phi/d\xi = y$ and integrating along the heteroclinic orbit, we can obtain the following representation of cusp periodic wave solution:

$$
u(x, t) = 1 - \phi_M \sin^2(\Omega - \omega|x + \alpha t - 2nT|) - \phi_m \cos^2(\Omega - \omega|x + \alpha t - 2nT|), \tag{4.15}
$$

where $\omega = (1/4)\sqrt{-\alpha}$, $\Omega = \arctan(\sqrt{-\phi_m/\phi_M})$, $T = 2|\Omega|$, $n = 0, \pm 1, \pm 2, \ldots$, $(2n - 1)T \le x + \alpha t \le (2n + 1)T$.

Substituting (4.14) into the $d\phi/d\xi = y$ and integrating along the heteroclinic orbit, we can obtain the following representation of cusp periodic wave solution:

$$u(x,t) = 1 - \phi_M \cos^2(\Omega - \omega|x + \alpha t - 2nT|) - \phi_m \sin^2(\Omega - \omega|x + \alpha t - 2nT|), \qquad (4.16)$$

where $\omega = (1/4)\sqrt{-\alpha}$, $\Omega = \arctan(\sqrt{-\phi_M/\phi_m})$, $T = 2|\Omega|$, $n = 0, \pm1, \pm2, \ldots$, $(2n-1)T \leq x + \alpha t \leq (2n+1)T$.

Moreover, the graph defined by $H(\phi, y) = h$, $h \in (0, +\infty)$ consists of two open-end curves Γ_4, Γ_5 passing through the points $(\phi_m, 0)$ and $(\phi_M, 0)$, respectively, where $(\phi_M - \phi)(\phi - \phi_m)[(\phi - b_1)^2 + a_1^2] = -\phi^4 + (16/3)\phi^3 - (4g/\alpha)\phi^2 - 4h/\alpha$. It follows from (2.4) that

$$y = \pm\frac{\sqrt{-\alpha(\phi_M - \phi)(\phi - \phi_m)\left[(\phi - b_1)^2 + a_1^2\right]}}{2\phi}, \quad \phi_m \leq \phi \leq \phi_M, \ \phi \neq 0. \qquad (4.17)$$

Substituting (4.17) into the $d\phi/d\xi = y$ and integrating along the curve Γ_4, we can obtain the implicit representation of periodic loop solution for $u \in [1 - \phi_M, 1 - \phi_m]$:

$$\frac{2(\phi_m A + \phi_M B)}{(A - B)\sqrt{AB}}\left\{\alpha_2 F(\varphi, k) + \frac{\alpha_1 - \alpha_2}{1 - \alpha_1^2}\left[\Pi\left(\varphi, \frac{\alpha_1^2}{\alpha_1^2 - 1}, k\right) - \alpha_1 f_1\right] - \eta_0\right\} = \pm\xi, \qquad (4.18)$$

where $A = \sqrt{(\phi_M - b_1)^2 + a_1^2}$, $B = \sqrt{(\phi_m - b_1)^2 + a_1^2}$, $\alpha_1 = (A - B)/(A + B)$, $\alpha_2 = (\phi_m A - \phi_M B)/(\phi_m A + \phi_M B)$, $k = \sqrt{((\phi_M - \phi_m)^2 - (A - B)^2)/4AB}$, $k_1 = \sqrt{1 - k^2}$, $\eta_0 = [\alpha_2^2 F(\varphi, k) + ((\alpha_1 - \alpha_2)/(1 - \alpha_1^2))(\Pi(\varphi, \alpha_1^2/(\alpha_1^2 - 1), k) - \alpha_1 f_1)]|_{u=1-\phi_M}$, $\varphi = \arccos(((\phi_M + u - 1)B + (\phi_m + u - 1)A)/((\phi_M + u - 1)B - (\phi_m + u - 1)A))$, $f_1 = \sqrt{(1 - \alpha_1^2)/(k^2 + k_1^2\alpha_1^2)}\arctan(\sqrt{(\sin^2\varphi(k^2 + k_1^2\alpha_1^2))/(1 - k^2\sin^2\varphi)(1 - \alpha_1^2)})$.

Corresponding to Figure 1(l), the graph defined by $H(\phi, y) = H(\phi_2, 0)$ consists of two hyperbolic sectors of the saddle point $(\phi_2, 0)$ and two open-end curves Γ_6, Γ_7 passing through the points $(\phi_m, 0)$, $(\phi_M, 0)$, respectively, where $\phi_m = (2\alpha + 3\sqrt{\Delta} - 2\sqrt{2\alpha(2\alpha + 3\sqrt{\Delta})})/3\alpha$, $\phi_M = (2\alpha + 3\sqrt{\Delta} + 2\sqrt{2\alpha(2\alpha + 3\sqrt{\Delta})})/3\alpha$. It follows from (2.4) that

$$y = \pm\frac{(\phi - \phi_2)\sqrt{\alpha(\phi_m - \phi)(\phi_M - \phi)}}{2\phi}, \quad \phi_2 < \phi \leq \phi_m, \ \phi \neq 0. \qquad (4.19)$$

Substituting (4.19) into the $d\phi/d\xi = y$ and integrating along the curve Γ_6, we can obtain the following representation of loop-soliton solution:

$$u(\chi) = 1 - \frac{\phi_2\left[\phi_M \sinh^2(\omega\chi) - \phi_m \cosh^2(\omega\chi)\right] + \phi_m\phi_M}{\left[\phi_M \cosh^2(\omega\chi) - \phi_m \sinh^2(\omega\chi)\right] - \phi_2},$$

$$\xi(\chi) = \frac{2}{\sqrt{\alpha}}\left(\chi - 2\tanh^{-1}\left(\sqrt{\frac{\phi_m - \phi_2}{\phi_M - \phi_2}}\tanh(\omega\chi)\right)\right), \qquad (4.20)$$

where $\omega = \sqrt{(\phi_m - \phi_2)(\phi_M - \phi_2)}/2\phi_2$, $\tanh^{-1}(\cdot)$ is the inverse function of the hyperbolic function $\tanh(\cdot)$, see [7].

Moreover, the graph defined by $H(\phi, y) = h$, $h \in (0, H(\phi_2, 0))$ consist of four open-end curves Γ_8, Γ_9, Γ_{10} and Γ_{11} passing through the points $(\gamma_4, 0), (\gamma_3, 0), (\gamma_2, 0), (\gamma_1, 0)$ respectively, where $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ ($\gamma_4 < \gamma_3 < 0 < \gamma_2 < \gamma_1$) are four real roots of $\psi^4 - (16/3)\psi^3 + (4g/\alpha)\psi^2 + (4h/\alpha) = 0$. It follows from (2.4) that

$$y = \pm \frac{\sqrt{\alpha(\gamma_1 - \phi)(\gamma_2 - \phi)(\phi - \gamma_3)(\phi - \gamma_4)}}{2\phi}, \quad \gamma_3 \le \phi \le \gamma_2, \ \phi \ne 0. \tag{4.21}$$

Substituting (4.21) into $d\phi/d\xi = y$ and integrating along the curve Γ_{10}, we can obtain the implicit representation of periodic loop solution for $u \in [1 - \gamma_2, 1 - \gamma_3]$:

$$\frac{4\gamma_2}{\alpha_1^2 \sqrt{\alpha(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}}$$
$$\times \left[(\alpha_1^2 - \alpha_2^2)\Pi\left(\arcsin\left(\sqrt{\frac{(\gamma_1 - \gamma_3)(\gamma_2 + u - 1)}{(\gamma_2 - \gamma_3)(\gamma_1 + u - 1)}} \right), \alpha_1^2, k \right) \right. \tag{4.22}$$
$$\left. + \alpha_2^2 F\left(\arcsin\left(\sqrt{\frac{(\gamma_1 - \gamma_3)(\gamma_2 + u - 1)}{(\gamma_2 - \gamma_3)(\gamma_1 + u - 1)}} \right), k \right) \right] = \pm\xi,$$

where $\alpha_1^2 = (\gamma_2 - \gamma_3)/(\gamma_1 - \gamma_3)$, $\alpha_2^2 = \gamma_1(\gamma_2 - \gamma_3)/\gamma_2(\gamma_1 - \gamma_3)$, $k = \sqrt{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_4)/(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)}$.

Remark 4.1. Denote that (i) $\alpha < 0$, $g = 2\alpha$, $h \in (0, (-4/3)\alpha)$, (ii) $\alpha < 0$, $g = 2\alpha$, $h \in ((-4/3)\alpha, +\infty)$, (iii) $\alpha < 0$, $2\alpha < g < 0$, $h \in (0, H(\phi_2, 0)]$, (iv) $\alpha < 0$, $2\alpha < g < 0$, $h \in (H(\phi_1, 0), +\infty)$, (v) $\alpha < 0$, $g = 0$, $h \in (0, +\infty)$, we can obtain the implicit representation of periodic loop solution similar to (4.18) when β, α, g, and h satisfy one and only one of above conditions, we omit it for brevity.

Example 4.2. Taking $\alpha = -1, g = 1$ and $h = 1$, we get the approximations of $A, B, \phi_m, \phi_M, a_1, b_1, \alpha_1, \alpha_2, k, k_1$ in the formula (4.18), where $A \doteq 5.846662930$, $B \doteq 1.525667184$, $\phi_m \doteq -1.117067993$, $\phi_M \doteq 6.016534182$, $a_1 \doteq 0.7403378831$, $b_1 \doteq 0.2169335722$, $\alpha_1 \doteq 0.586109911$, $\alpha_2 \doteq -5.932667554$, $k \doteq 0.9502338139$, $k_1 \doteq 0.3115376364$.
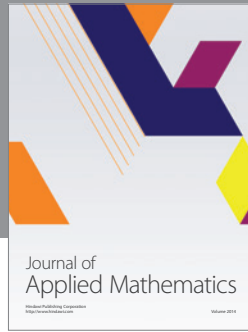
5. Conclusion

In this paper, using the bifurcation theory and the method of phase portraits analysis, we investigated periodic loop solutions and their limit forms of the Kudryashov-Sinelshchikov equation and show that the limit forms contain loop soliton solutions, smooth periodic wave solutions, and periodic cusp wave solutions. We also obtain the exact parametric representations above travelling wave solutions. The results of this paper have enriched results of [1–3]. We would like to study the Kudryashov-Sinelshchikov equation further.

Acknowledgments

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