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Research Article

Nonoscillation of Second-Order Dynamic Equations with Several Delays

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Existence of nonoscillatory solutions for the second-order dynamic equation $(A_0x^{\Delta})^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)x(\alpha_i(t))=0$ for $t\in[t_0,\infty)_{\mathbb{T}}$ is investigated in this paper. The results involve nonoscillation criteria in terms of relevant dynamic and generalized characteristic inequalities, comparison theorems, and explicit nonoscillation and oscillation conditions. This allows to obtain most known nonoscillation results for second-order delay differential equations in the case $A_0(t)\equiv 1$ for $t\in[t_0,\infty)_{\mathbb{R}}$ and for second-order nondelay difference equations ($\alpha_i(t)=t+1$ for $t\in[t_0,\infty)_{\mathbb{N}}$). Moreover, the general results imply new nonoscillation tests for delay differential equations with arbitrary A_0 and for second-order delay difference equations. Known nonoscillation results for quantum scales can also be deduced.

1. Introduction

This paper deals with second-order linear delay dynamic equations on time scales. Differential equations of the second order have important applications and were extensively studied; see, for example, the monographs of Agarwal et al. [1], Erbe et al. [2], Győri and Ladas [3], Ladde et al. [4], Myškis [5], Norkin [6], Swanson [7], and references therein. Difference equations of the second order describe finite difference approximations of second-order differential equations, and they also have numerous applications.

We study nonoscillation properties of these two types of equations and some of their generalizations. The main result of the paper is that under some natural assumptions for a delay dynamic equation the following four assertions are equivalent: nonoscillation of solutions of the equation on time scales and of the corresponding dynamic inequality, positivity of the fundamental function, and the existence of a nonnegative solution for a generalized Riccati inequality. The equivalence of oscillation properties of the differential equation and the corresponding differential inequality can be applied to obtain new explicit nonoscillation and oscillation conditions and also to prove some well-known results in a different way. A generalized Riccati inequality is used to compare oscillation properties of two equations without comparing their solutions. These results can be regarded as a natural generalization of the well-known Sturm-Picone comparison theorem for a second-order ordinary differential equation; see [7, Section 1.1]. Applying positivity of the fundamental function, positive solutions of two equations can be compared. There are many results of this kind for delay differential equations of the first-order and only a few for second-order equations. Myškis [5] obtained one of the first comparison theorems for second-order differential equations. The results presented here are generalizations of known nonoscillation tests even for delay differential equations (when the time scale is the real line).

The paper also contains conditions on the initial function and initial values which imply that the corresponding solution is positive. Such conditions are well known for first-order delay differential equations; however, to the best of our knowledge, the only paper concerning second-order equations is [8].

From now on, we will without furthermore mentioning suppose that the time scale $\mathbb{T}$ is unbounded from above. The purpose of the present paper is to study nonoscillation of the delay dynamic equation

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where $n \in \mathbb{N}$, $t_0 \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the forcing term, $A_0 \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, and for all $i \in [1,n]_{\mathbb{N}}$, $A_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the coefficient corresponding to the function α_i, where $\alpha_i \le \sigma$ on $[t_0,\infty)_{\mathbb{T}}$.

In this paper, we follow the method employed in [8] for second-order delay differential equations. The method can also be regarded as an application of that used in [9] for first-order dynamic equations.

As a special case, the results of the present paper allow to deduce nonoscillation criteria for the delay differential equation

$$\left(A_0x'\right)'(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{R}}, \tag{1.2}$$

in the case $A_0(t) \equiv 1$ for $t \in [t_0,\infty)_{\mathbb{R}}$, they coincide with theorems in [8]. The case of a "quickly growing" function A_0 when the integral of its reciprocal can converge is treated separately.

Let us recall some known nonoscillation and oscillation results for the ordinary differential equations

$$\left(A_0x'\right)'(t) + A_1(t)x(t) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{R}}, \tag{1.3}$$

$$x''(t) + A_1(t)x(t) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{R}}, \tag{1.4}$$

where A_1 is nonnegative, which are particular cases of (1.2) with $n = 1$, $\alpha_1(t) = t$, and $A_0(t) \equiv 1$ for all $t \in [t_0,\infty)_{\mathbb{R}}$.

In [10], Leighton proved the following well-known oscillation test for (1.4); see [10, 11].

Theorem A (see [10]). *Assume that*

$$\int_{t_0}^{\infty} \frac{1}{A_0(\eta)} d\eta = \infty, \qquad \int_{t_0}^{\infty} A_1(\eta) d\eta = \infty, \tag{1.5}$$

then (1.3) *is oscillatory.*

This result for (1.4) was obtained by Wintner in [12] without imposing any sign condition on the coefficient A_1.

In [13], Kneser proved the following result.

Theorem B (see [13]). *Equation* (1.4) *is nonoscillatory if* $t^2 A_1(t) \leq 1/4$ *for all* $t \in [t_0, \infty)_{\mathbb{R}}$, *while oscillatory if* $t^2 A_1(t) > \lambda_0/4$ *for all* $t \in [t_0, \infty)_{\mathbb{R}}$ *and some* $\lambda_0 \in (1, \infty)_{\mathbb{T}}$.

In [14], Hille proved the following result, which improves the one due to Kneser; see also [14–16].

Theorem C (see [14]). *Equation* (1.4) *is nonoscillatory if*

$$t \int_{t}^{\infty} A_1(\eta) d\eta \leq \frac{1}{4} \quad \forall t \in [t_0, \infty)_{\mathbb{R}}, \tag{1.6}$$

while it is oscillatory if

$$t \int_{t}^{\infty} A_1(\eta) d\eta > \frac{\lambda_0}{4} \quad \forall t \in [t_0, \infty)_{\mathbb{R}} \text{ and some } \lambda_0 \in (1, \infty)_{\mathbb{R}}. \tag{1.7}$$

Another particular case of (1.1) is the second-order delay difference equation

$$\Delta(A_0 \Delta x)(k) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(k) x(\alpha_i(k)) = 0 \quad \text{for} \in [k_0, \infty)_{\mathbb{N}}, \tag{1.8}$$

to the best of our knowledge, there are very few nonoscillation results for this equation; see, for example, [17]. However, nonoscillation properties of the nondelay equations

$$\Delta(A_0 \Delta x)(k) + A_1(k) x(k+1) = 0 \quad \text{for } k \in [k_0, \infty)_{\mathbb{N}}, \tag{1.9}$$

$$\Delta^2 x(k) + A_1(k) x(k+1) = 0 \quad \text{for } k \in [k_0, \infty)_{\mathbb{N}} \tag{1.10}$$

have been extensively studied in [1, 18–22]; see also [23]. In particular, the following result is valid.

Theorem D. *Assume that*

$$\sum_{j=k_0}^{\infty} A_1(j) = \infty, \tag{1.11}$$

then (1.10) *is oscillatory.*

The following theorem can be regarded as the discrete analogue of the nonoscillation result due to Kneser.

Theorem E. *Assume that* $k(k+1)A_1(k) \le 1/4$ *for all* $k \in [k_0, \infty)_{\mathbb{N}}$, *then* (1.10) *is nonoscillatory.*

Hille's result in [14] also has a counterpart in the discrete case. In [22], Zhou and Zhang proved the nonoscillation part, and in [24], Zhang and Cheng justified the oscillation part which generalizes Theorem E.

Theorem F (see [22, 24]). *Equation* (1.10) *is nonoscillatory if*

$$k\sum_{j=k}^{\infty} A_1(j) \le \frac{1}{4} \quad \forall k \in [k_0, \infty)_{\mathbb{N}}, \tag{1.12}$$

while is oscillatory if

$$k\sum_{j=k}^{\infty} A_1(j) > \frac{\lambda_0}{4} \quad \forall k \in [k_0, \infty)_{\mathbb{N}} \text{ and some } \lambda_0 \in (1, \infty)_{\mathbb{R}}. \tag{1.13}$$

In [23], Tang et al. studied nonoscillation and oscillation of the equation

$$\Delta^2 x(k) + A_1(k)x(k) = 0 \quad \text{for } k \in [k_0, \infty)_{\mathbb{N}}, \tag{1.14}$$

where $\{A_1(k)\}$ is a sequence of nonnegative reals and obtained the following theorem.

Theorem G (see [23]). *Equation* (1.14) *is nonoscillatory if* (1.12) *holds, while is it oscillatory if* (1.13) *holds.*

These results together with some remarks on the q-difference equations will be discussed in Section 7. The readers can find some nonoscillation results for second-order nondelay dynamic equations in the papers [20, 25–29], some of which generalize some of those mentioned above.

The paper is organized as follows. In Section 2, some auxiliary results are presented. In Section 3, the equivalence of the four above-mentioned properties is established. Section 4 is dedicated to comparison results. Section 5 includes some explicit nonoscillation and oscillation conditions. A sufficient condition for existence of a positive solution is given in Section 6. Section 7 involves some discussion and states open problems. Section 7 as an appendix contains a short account on the fundamentals of the time scales theory.

2. Preliminary Results

Consider the following delay dynamic equation:

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}},$$
$$x(t_0) = x_1, \quad x^{\Delta}(t_0) = x_2, \quad x(t) = \varphi(t) \quad \text{for } t \in [t_{-1},t_0)_{\mathbb{T}}, \tag{2.1}$$

where $n \in \mathbb{N}$, $\mathbb{T}$ is a time scale unbounded above, $t_0 \in \mathbb{T}$, $x_1, x_2 \in \mathbb{R}$ are the initial values, $\varphi \in C_{\mathrm{rd}}([t_{-1},t_0)_{\mathbb{T}},\mathbb{R})$ is the initial function, such that φ has a finite left-sided limit at the initial point t_0 provided that it is left dense, $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the forcing term, $A_0 \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, and for all $i \in [1,n]_{\mathbb{N}}$, $A_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the coefficient corresponding to the function $\alpha_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, which satisfies $\alpha_i(t) \le \sigma(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ and $\lim_{t\to\infty}\alpha_i(t) = \infty$. Here, we denoted

$$\alpha_{\min}(t) := \min_{i\in[1,n]_{\mathbb{N}}}\{\alpha_i(t)\} \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \quad t_{-1} := \inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha_{\min}(t)\}, \tag{2.2}$$

then t_{-1} is finite, since $\alpha_{\min}$ asymptotically tends to infinity.

Definition 2.1. A function $x : [t_{-1},\infty)_{\mathbb{T}} \to \mathbb{R}$ with $x \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and a derivative satisfying $A_0x^{\Delta} \in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is called a *solution* of (2.1) if it satisfies the equation in the first line of (2.1) identically on $[t_0,\infty)_{\mathbb{T}}$ and also the initial conditions in the second line of (2.1).

For a given function $\varphi \in C_{\mathrm{rd}}([t_{-1},t_0)_{\mathbb{T}},\mathbb{R})$ with a finite left-sided limit at the initial point t_0 provided that it is left-dense and $x_1, x_2 \in \mathbb{R}$, problem (2.1) admits a unique solution satisfying $x = \varphi$ on $[t_{-1},t_0)_{\mathbb{T}}$ with $x(t_0) = x_1$ and $x^{\Delta}(t_0) = x_2$ (see [30] and [31, Theorem 3.1]).

Definition 2.2. A solution of (2.1) is called *eventually positive* if there exists $s \in [t_0,\infty)_{\mathbb{T}}$ such that $x > 0$ on $[s,\infty)_{\mathbb{T}}$, and if $(-x)$ is eventually positive, then x is called *eventually negative*. If (2.1) has a solution which is either eventually positive or eventually negative, then it is called *nonoscillatory*. A solution, which is neither eventually positive nor eventually negative, is called *oscillatory*, and (2.1) is said to be *oscillatory* provided that every solution of (2.1) is oscillatory.

For convenience in the notation and simplicity in the proofs, we suppose that functions vanish out of their specified domains, that is, let $f : D \to \mathbb{R}$ be defined for some $D \subset \mathbb{R}$, then it is always understood that $f(t) = \chi_D(t)f(t)$ for $t \in \mathbb{R}$, where χ_D is the characteristic function of the set $D \subset \mathbb{R}$ defined by $\chi_D(t) \equiv 1$ for $t \in D$ and $\chi_D(t) \equiv 0$ for $t \notin D$.

Definition 2.3. Let $s \in \mathbb{T}$ and $s_{-1} := \inf_{t\in[s,\infty)_{\mathbb{T}}}\{\alpha_{\min}(t)\}$. The solutions $\mathcal{X}_1 = \mathcal{X}_1(\cdot, s)$ and $\mathcal{X}_2 = \mathcal{X}_2(\cdot, s)$ of the problems

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [s,\infty)_{\mathbb{T}}, \\ x^{\Delta}(s) = \frac{1}{A_0(s)}, \quad x(t) \equiv 0 \quad \text{for } t \in [s_{-1}, s]_{\mathbb{T}},\end{gathered} \tag{2.3}$$

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [s,\infty)_{\mathbb{T}}, \\ x^{\Delta}(s) = 0, \quad x(t) = \chi_{\{s\}}(t) \quad \text{for } t \in [s_{-1}, s]_{\mathbb{T}},\end{gathered} \tag{2.4}$$

which satisfy $\mathcal{X}_1(\cdot, s), \mathcal{X}_2(\cdot, s) \in C^1_{\mathrm{rd}}([s,\infty)_{\mathbb{T}}, \mathbb{R})$, are called the *first fundamental solution* and the *second fundamental solution* of (2.1), respectively.

The following lemma plays the major role in this paper; it presents a representation formula to solutions of (2.1) by the means of the fundamental solutions $\mathcal{X}_1$ and $\mathcal{X}_2$.

Lemma 2.4. *Let x be a solution of* (2.1)*, then x can be written in the following form:*

$$x(t) = x_2\mathcal{X}_1(t,t_0) + x_1\mathcal{X}_2(t,t_0) + \int_{t_0}^{t} \mathcal{X}_1(t,\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta \tag{2.5}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. For $t \in [t_{-1},\infty)_{\mathbb{T}}$, let

$$y(t) := \begin{cases} \displaystyle\int_{t_0}^{t} \mathcal{X}_1(t,\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta & \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ \varphi(t) & \text{for } t \in [t_1, t_0)_{\mathbb{T}}. \end{cases} \tag{2.6}$$

We recall that $\mathcal{X}_1(\cdot, t_0)$ and $\mathcal{X}_2(\cdot, t_0)$ solve (2.3) and (2.4), respectively. To complete the proof, let us demonstrate that y solves

$$\begin{gathered}\left(A_0 y^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)y(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ y(t_0) = 0, \quad y^{\Delta}(t_0) = 0, \quad y(t) = \varphi(t) \quad \text{for } t \in [t_{-1}, t_0)_{\mathbb{T}}.\end{gathered} \tag{2.7}$$

This will imply that the function z defined by $z := x_2\mathcal{X}_1(\cdot,t_0) + x_1\mathcal{X}_2(\cdot,t_0) + y$ on $[t_0,\infty)_{\mathbb{T}}$ is a solution of (2.1). Combining this with the uniqueness result in [31, Theorem 3.1] will complete the proof. For all $t \in [t_0,\infty)_{\mathbb{T}}$, we can compute that

$$\begin{aligned} y^{\Delta}(t) &= \int_{t_0}^{t} \mathcal{X}_1^{\Delta}(t,\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta \\ &\quad + \mathcal{X}_1(\sigma(t),\sigma(t))\left[f(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t))\right] \\ &= \int_{t_0}^{t} \mathcal{X}_1^{\Delta}(t,\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta. \end{aligned} \tag{2.8}$$

Therefore, $y(t_0) = 0$, $y^{\Delta}(t_0) = 0$, and $y = \varphi$ on $[t_{-1},t_0)_{\mathbb{T}}$, that is, y satisfies the initial conditions in (2.7). Differentiating y^{Δ} after multiplying by A_0 and using the properties of the first fundamental solution $\mathcal{X}_1$, we get

$$\begin{aligned} \left(A_0 y^{\Delta}\right)^{\Delta}(t) &= \int_{t_0}^{t} \left(A_0\mathcal{X}_1^{\Delta}(\cdot,\sigma(\eta))\right)^{\Delta}(t)\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta \\ &\quad + A_0^{\sigma}(t)\mathcal{X}_1^{\Delta}(\sigma(t),\sigma(t))\left[f(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t))\right] \\ &= -\sum_{j\in[1,n]_{\mathbb{N}}} A_j(t)\int_{t_0}^{\alpha_j(t)} \mathcal{X}_1(\alpha_j(t),\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta \\ &\quad - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) + f(t) \end{aligned} \tag{2.9}$$

for all $t \in [t_0,\infty)_{\mathbb{T}}$. For $t \in [t_0,\infty)_{\mathbb{T}}$, set $I(t) = \{i \in [1,n]_{\mathbb{N}} : \chi_{[t_0,\infty)_{\mathbb{T}}}(\alpha_i(t)) = 1\}$ and $J(t) := \{i \in [1,n]_{\mathbb{N}} : \chi_{[t_{-1},t_0)_{\mathbb{T}}}(\alpha_i(t)) = 1\}$. Making some arrangements, for all $t \in [t_0,\infty)_{\mathbb{T}}$, we find

$$\begin{aligned} \left(A_0 y^{\Delta}\right)^{\Delta}(t) &= -\sum_{j\in I(t)} A_j(t)\int_{t_0}^{\alpha_j(t)} \mathcal{X}_1(\alpha_j(t),\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta \\ &\quad - \sum_{j\in J(t)} A_j(t)\int_{t_0}^{\alpha_j(t)} \mathcal{X}_1(\alpha_j(t),\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta \\ &\quad - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) + f(t), \end{aligned} \tag{2.10}$$

and thus

$$\begin{aligned}\left(A_0y^{\Delta}\right)^{\Delta}(t)&=-\sum_{j\in I(t)}A_j(t)\int_{t_0}^{\alpha_j(t)}\mathcal{X}_1(\alpha_j(t),\sigma(\eta))f(\eta)\Delta\eta-\sum_{j\in J(t)}A_i(t)\varphi(\alpha_i(t))+f(t)\\&=-\sum_{j\in I(t)}A_j(t)y(\alpha_j(t))-\sum_{j\in J(t)}A_j(t)y(\alpha_j(t))+f(t),\end{aligned}\tag{2.11}$$

which proves that y satisfies (2.7) on $[t_0,\infty)_{\mathbb{T}}$ since $I(t)\cap J(t)=\emptyset$ and $I(t)\cup J(t)=[1,n]_{\mathbb{N}}$ for each $t\in[t_0,\infty)_{\mathbb{T}}$. The proof is therefore completed. □

Next, we present a result from [9] which will be used in the proof of the main result.

Lemma 2.5 (see [9, Lemma 2.5]). *Let $t_0\in\mathbb{T}$ and assume that K is a nonnegative Δ-integrable function defined on $\{(t,s)\in\mathbb{T}\times\mathbb{T}:t\in[t_0,\infty)_{\mathbb{T}},s\in[t_0,t]_{\mathbb{T}}\}$. If $f,g\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ satisfy*

$$f(t)=\int_{t_0}^{t}K(t,\eta)f(\eta)\Delta\eta+g(t)\quad\forall t\in[t_0,\infty)_{\mathbb{T}},\tag{2.12}$$

then $g(t)\geq 0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ implies $f(t)\geq 0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$.

3. Nonoscillation Criteria

Consider the delay dynamic equation

$$\left(A_0x^{\Delta}\right)^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)x(\alpha_i(t))=0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}}\tag{3.1}$$

and its corresponding inequalities

$$\left(A_0x^{\Delta}\right)^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)x(\alpha_i(t))\leq 0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}},\tag{3.2}$$

$$\left(A_0x^{\Delta}\right)^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)x(\alpha_i(t))\geq 0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}}.\tag{3.3}$$

We now prove the following result, which plays a major role throughout the paper.

Theorem 3.1. *Suppose that the following conditions hold:*

($A1$) $A_0\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$,

($A2$) *for* $i\in[1,n]_{\mathbb{N}}$, $A_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$,

($A3$) *for* $i\in[1,n]_{\mathbb{N}}$, $\alpha_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ *satisfies* $\alpha_i(t)\leq\sigma(t)$ *for all* $t\in[t_0,\infty)_{\mathbb{T}}$ *and* $\lim_{t\to\infty}\alpha_i(t)=\infty$, *then the following conditions are equivalent:*

(i) *the second-order dynamic equation* (3.1) *has a nonoscillatory solution,*

(ii) *the second-order dynamic inequality* (3.2) *has an eventually positive solution and/or* (3.3) *has an eventually negative solution,*

(iii) *there exist a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and a function* $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ *with* $\Lambda/A_0 \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ *satisfying the first-order dynamic Riccati inequality*

$$\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(t, \alpha_i(t)) \leq 0 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.4}$$

(iv) *the first fundamental solution* $\mathcal{X}_1$ *of* (3.1) *is eventually positive, that is, there exists a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that* $\mathcal{X}_1(t, s) > 0$ *for all* $t \in (s, \infty)_{\mathbb{T}}$ *and all* $s \in [t_1, \infty)_{\mathbb{T}}$.

Proof. The proof follows the scheme: (i)⇒(ii)⇒(iii)⇒(iv)⇒(i).

(i)⇒(ii) This part is trivial, since any eventually positive solution of (3.1) satisfies (3.2) too, which indicates that its negative satisfies (3.3).

(ii)⇒(iii) Let x be an eventually positive solution of (3.2), then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t_1) = 1$ (otherwise, we may proceed with the function $x/x(t_1)$, which is also a solution since (3.2) is linear). Let

$$\Lambda(t) := A_0(t)\frac{x^{\Delta}(t)}{x(t)} \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.5}$$

then evidently $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ and

$$1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} = 1 + \mu(t)\frac{x^{\Delta}(t)}{x(t)} = \frac{x^{\sigma}(t)}{x(t)} > 0 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.6}$$

which proves that $\Lambda/A_0 \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$. This implies that the exponential function $\mathrm{e}_{\Lambda/A_0}(\cdot, t_1)$ is well defined and is positive on the entire time scale $[t_1, \infty)_{\mathbb{T}}$; see [32, Theorem 2.48]. From (3.5), we see that Λ satisfies the ordinary dynamic equation

$$\begin{gathered} x^{\Delta}(t) = \frac{\Lambda(t)}{A_0(t)}x(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \\ x(t_1) = 1, \end{gathered} \tag{3.7}$$

whose unique solution is

$$x(t) = \mathrm{e}_{\Lambda/A_0}(t, t_1) \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.8}$$

see [32, Theorem 2.77]. Hence, using (3.8), for all $t \in [t_1, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned} x^{\Delta}(t) &= \frac{\Lambda(t)}{A_0(t)} \mathrm{e}_{\Lambda/A_0}(t, t_1), \\ \left(A_0 x^{\Delta}\right)^{\Delta}(t) &= \left(\Lambda \mathrm{e}_{\Lambda/A_0}(\cdot, t_1)\right)^{\Delta}(t) = \Lambda^{\Delta}(t) \mathrm{e}_{\Lambda/A_0}(t, t_1) + \Lambda^{\sigma}(t) \mathrm{e}^{\Delta}_{\Lambda/A_0}(t, t_1) \\ &= \Lambda^{\Delta}(t) \mathrm{e}_{\Lambda/A_0}(t, t_1) + \frac{1}{A_0(t)} \Lambda^{\sigma}(t) \Lambda(t) \mathrm{e}_{\Lambda/A_0}(t, t_1), \end{aligned} \tag{3.9}$$

which gives by substituting into (3.2) and using [32, Theorem 2.36] that

$$\begin{aligned} 0 &\geq \Lambda^{\Delta}(t) \mathrm{e}_{\Lambda/A_0}(t, t_1) + \frac{1}{A_0(t)} \Lambda^{\sigma}(t) \Lambda(t) \mathrm{e}_{\Lambda/A_0}(t, t_1) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\Lambda/A_0}(\alpha_i(t), t_1) \\ &= \mathrm{e}_{\Lambda/A_0}(t, t_1) \left[\Lambda^{\Delta}(t) + \frac{1}{A_0(t)} \Lambda^{\sigma}(t) \Lambda(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \frac{\mathrm{e}_{\Lambda/A_0}(\alpha_i(t), t_1)}{\mathrm{e}_{\Lambda/A_0}(t, t_1)} \right] \\ &= \mathrm{e}_{\Lambda/A_0}(t, t_1) \left[\Lambda^{\Delta}(t) + \frac{1}{A_0(t)} \Lambda^{\sigma}(t) \Lambda(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(t, \alpha_i(t)) \right] \end{aligned} \tag{3.10}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Since the expression in the brackets is the same as the left-hand side of (3.4) and $\mathrm{e}_{\Lambda/A_0}(\cdot, t_1) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, the function Λ is a solution of (3.4).

(iii)⇒(iv) Consider the initial value problem

$$\begin{gathered} \left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \\ x^{\Delta}(t_1) = 0, \quad x(t) \equiv 0 \quad \text{for } t \in [t_{-1}, t_1]_{\mathbb{T}}. \end{gathered} \tag{3.11}$$

Denote

$$g(t) := A_0(t) x^{\Delta}(t) - \Lambda(t) x(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.12}$$

where x is any solution of (3.11) and Λ is a solution of (3.4). From (3.12), we have

$$\begin{gathered} x^{\Delta}(t) = \frac{\Lambda(t)}{A_0(t)} x(t) + \frac{g(t)}{A_0(t)} \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \\ x(t_1) = 0, \end{gathered} \tag{3.13}$$

whose unique solution is

$$x(t) = \int_{t_1}^{t} \mathrm{e}_{\Lambda/A_0}(t, \sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.14}$$

see [32, Theorem 2.77]. Now, for all $t \in [t_1, \infty)_{\mathbb{T}}$, we compute that

$$\begin{aligned} x(t) &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), t)\left[\int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A(\eta)}\Delta\eta - \mu(t)\mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(t))\frac{g(t)}{A_0(t)}\right] \\ &= \frac{A_0(t)}{A_0(t) + \mu(t)\Lambda(t)}\left[x^{\sigma}(t) - \mu(t)\frac{g(t)}{A_0(t)}\right] \\ &= \frac{1}{A_0(t) + \mu(t)\Lambda(t)}\left[A_0(t)x^{\sigma}(t) - \mu(t)g(t)\right], \end{aligned} \tag{3.15}$$

and similarly

$$\begin{aligned} x(\alpha_i(t)) &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t)) \\ &\quad \times \left[\int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A(\eta)}\Delta\eta - \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A_0(\eta)}\Delta\eta\right] \\ &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t))\left[x^{\sigma}(t) - \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A_0(\eta)}\Delta\eta\right] \\ &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t))x^{\sigma}(t) - \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t), \sigma(\eta))\frac{g(\eta)}{A_0(\eta)}\Delta\eta \end{aligned} \tag{3.16}$$

for $i \in [1, n]_{\mathbb{N}}$. From (3.12) and (3.15), we have

$$\begin{aligned} \left(A_0 x^{\Delta}\right)^{\Delta}(t) &= (\Lambda x + g)^{\Delta}(t) = \Lambda^{\Delta}(t)x^{\sigma}(t) + \Lambda(t)x^{\Delta}(t) + g^{\Delta}(t) \\ &= \Lambda^{\Delta}(t)x^{\sigma}(t) + \frac{\Lambda^2(t)}{A_0(t)}x(t) + \frac{\Lambda(t)}{A_0(t)}g(t) + g^{\Delta}(t) \end{aligned} \tag{3.17}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. We substitute (3.14), (3.15), (3.16), and (3.17) into (3.11) and find that

$$\begin{aligned} f(t) &= \left[\Lambda^{\Delta}(t)x^{\sigma}(t) + \frac{\Lambda^2(t)}{A_0(t)}x(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t))\right] + \frac{\Lambda(t)}{A_0(t)}g(t) + g^{\Delta}(t) \\ &= \left[\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t))\right]x^{\sigma}(t) \end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{\mu(t)\Lambda^2(t)}{A_0(t)\left(A_0(t)+\mu(t)\Lambda(t)\right)} g(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t),\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \right] \\
&+ \frac{\Lambda(t)}{A_0(t)} g(t) + g^{\Delta}(t) \\
= &\left[\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t)+\mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t),\alpha_i(t)) \right] \\
&\times \left[1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} \right] \int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(t,\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \\
&- \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t),\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \\
&+ \frac{\Lambda(t)}{A_0(t)+\mu(t)\Lambda(t)} g(t) + g^{\Delta}(t)
\end{aligned}
\tag{3.18}
$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Then, (3.18) can be rewritten as

$$
\begin{aligned}
g^{\Delta}(t) = &-\frac{\Lambda(t)}{A_0(t)+\mu(t)\Lambda(t)} g(t) + \Upsilon(t) \int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(t,\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \\
&+ \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t),\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta + f(t)
\end{aligned}
\tag{3.19}
$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$, where

$$
\Upsilon(t) := -\left[1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} \right] \left[\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t)+\mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t),\alpha_i(t)) \right]
\tag{3.20}
$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. We now show that $\Upsilon \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$. Indeed, by using (3.4) and the simple useful formula (A.2), we get

$$
\begin{aligned}
\Upsilon(t) &= -\left[\left(1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} \right) \Lambda^{\Delta}(t) + \frac{1}{A_0(t)} \Lambda^2(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(t,\alpha_i(t)) \right] \\
&= -\left[\Lambda^{\Delta}(t) + \frac{1}{A_0(t)} \Lambda^{\sigma}(t)\Lambda(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(t,\alpha_i(t)) \right] \geq 0
\end{aligned}
\tag{3.21}
$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. On the other hand, from (3.11) and (3.12), we see that $g(t_1) = 0$. Then, by [32, Theorem 2.77], we can write (3.19) in the equivalent form

$$g = \mathscr{A} g + h \quad on \; [t_1, \infty)_{\mathbb{T}}, \tag{3.22}$$

where, for $t \in [t_1, \infty)_{\mathbb{T}}$, we have defined

$$\begin{aligned}(\mathscr{A} g)(t) := \int_{t_1}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Bigg[\Upsilon(\eta) \int_{t_1}^{\sigma(\eta)} \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\zeta \\ + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(\eta) \int_{\alpha_i(\eta)}^{\sigma(\eta)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\zeta \Bigg] \Delta\eta,\end{aligned} \tag{3.23}$$

$$h(t) := \int_{t_1}^{t} \mathrm{e}_{\Lambda/A_0}(t, \sigma(\zeta)) f(\eta) \Delta\eta. \tag{3.24}$$

Note that $\Lambda/A_0 \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ implies $-\Lambda/(A_0 + \mu\Lambda) \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ (indeed, we have $1 - \mu\Lambda/(A_0 + \mu\Lambda) = A_0/(A_0 + \mu\Lambda) > 0$ on $[t_1, \infty)_{\mathbb{T}}$), and thus the exponential function $\mathrm{e}_{\ominus(\Lambda/A_0)}(\cdot, t_1)$ is also well defined and positive on the entire time scale $[t_1, \infty)_{\mathbb{T}}$, see [32, Exercise 2.28]. Thus, $f \geq 0$ on $[t_1, \infty)_{\mathbb{T}}$ implies $h \geq 0$ on $[t_1, \infty)_{\mathbb{T}}$. For simplicity of notation, for $s, t \in [t_1, \infty)_{\mathbb{T}}$, we let

$$\begin{aligned}K_1(t, s) &:= \frac{1}{A_0(s)} \int_{s}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Upsilon(\eta) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(s)) \Delta\eta, \\ K_2(t, s) &:= \frac{1}{A_0(s)} \int_{s}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \sum_{i \in [1,n]_{\mathbb{N}}} A_i(\eta) \chi_{[\alpha_i(\eta), \infty)_{\mathbb{T}}}(s) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(s)) \Delta\eta.\end{aligned} \tag{3.25}$$

Using the change of integration order formula in [33, Lemma 1], for all $t \in [t_1, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}&\int_{t_1}^{t} \int_{t_1}^{\sigma(\eta)} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Upsilon(\eta) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\zeta \Delta\eta \\ &\quad = \int_{t_1}^{t} \int_{\zeta}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Upsilon(\eta) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\eta \Delta\zeta \\ &\quad = \int_{t_1}^{t} K_1(t, \zeta) g(\zeta) \Delta\zeta,\end{aligned} \tag{3.26}$$

and similarly

$$\begin{aligned}
&\int_{t_1}^{t}\int_{t_1}^{\sigma(\eta)} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\chi_{[\alpha_i(\eta),\infty)_{\mathbb{T}}}(\zeta)\mathrm{e}_{\Lambda/A_0}(\sigma(\eta),\sigma(\zeta))\frac{g(\zeta)}{A_0(\zeta)}\Delta\zeta\Delta\eta \\
&\quad = \int_{t_1}^{t} K_2(t,\zeta)g(\zeta)\Delta\zeta.
\end{aligned} \tag{3.27}$$

Therefore, we can rewrite (3.23) in the equivalent form of the integral operator

$$(\mathscr{H}g)(t) = \int_{t_1}^{t}\left[K_1(t,\eta)+K_2(t,\eta)\right]g(\eta)\Delta\eta \quad \text{for } t\in[t_1,\infty)_{\mathbb{T}}, \tag{3.28}$$

whose kernel is nonnegative. Consequently, using (3.22), (3.24), and (3.28), we obtain that $f \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$ implies $h \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$; this and Lemma 2.5 yield that $g \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore, from (3.14), we infer that if $f \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$, then $x \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$ too. On the other hand, by Lemma 2.4, x has the following representation:

$$x(t) = \int_{t_1}^{t} \mathcal{X}_1(t,\sigma(\eta))f(\eta)\Delta\eta \quad \text{for } t\in[t_1,\infty)_{\mathbb{T}}. \tag{3.29}$$

Since x is eventually nonnegative for any eventually nonnegative function f, we infer that the kernel $\mathcal{X}_1$ of the integral on the right-hand side of (3.29) is eventually nonnegative. Indeed, assume to the contrary that $x \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$ but $\mathcal{X}_1$ is not nonnegative, then we may pick $t_2 \in [t_1,\infty)_{\mathbb{T}}$ and find $s \in [t_1,t_2)_{\mathbb{T}}$ such that $\mathcal{X}_1(t_2,\sigma(s)) < 0$. Then, letting $f(t) := -\min\{\mathcal{X}_1(t_2,\sigma(t)),0\} \geq 0$ for $t\in[t_1,\infty)_{\mathbb{T}}$, we are led to the contradiction $x(t_2) < 0$, where x is defined by (3.29). To prove that $\mathcal{X}_1$ is eventually positive, set $x(t) := \mathcal{X}_1(t,s)$ for $t \in [t_0,\infty)_{\mathbb{T}}$, where $s \in [t_1,\infty)_{\mathbb{T}}$, to see that $x \geq 0$ and $(A_0x^{\Delta})^{\Delta} \leq 0$ on $[s,\infty)_{\mathbb{T}}$, which implies A_0x^{Δ} is nonincreasing on $[s,\infty)_{\mathbb{T}}$. So that, we may let $t_1 \in [t_0,\infty)_{\mathbb{T}}$ so large that x^{Δ} (i.e., A_0x^{Δ}) is of fixed sign on $[s,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$. The initial condition and (A1) together with $x^{\Delta}(s) = 1/A_0(s) > 0$ imply that $x^{\Delta} > 0$ on $[s,\infty)_{\mathbb{T}}$. Consequently, we have $x(t) = \mathcal{X}_1(t,s) > \mathcal{X}_1(s,s) = 0$ for all $t\in(s,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$.

(iv)⇒(i) Clearly, $\mathcal{X}_1(\cdot,t_0)$ is an eventually positive solution of (3.1).

The proof is completed. □

Let us introduce the following condition:

$(A4)$ $A_0 \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ with

$$\int_{t_0}^{\infty}\frac{1}{A_0(\eta)}\Delta\eta = \infty. \tag{3.30}$$

Remark 3.2. It is well known that (A4) ensures existence of $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)x^{\Delta}(t) \geq 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, for any nonoscillatory solution x of (3.1). This fact follows from the formula

$$x(t) = x(s) + A_0(s)x^{\Delta}(s)\int_s^t \frac{1}{A_0(\eta)}\Delta\eta - \int_s^t \frac{1}{A_0(\eta)}\left[\int_s^{\eta} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\zeta)x(\alpha_i(\zeta))\Delta\zeta\right]\Delta\eta \tag{3.31}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$, obtained by integrating (3.1) twice, where $s \in [t_0, \infty)_{\mathbb{T}}$. In the case when (A4) holds, (iii) of Theorem 3.1 can be assumed to hold with $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}^+_0)$, which means that any positive (negative) solution is nondecreasing (nonincreasing).

Remark 3.3. Let (A4) hold and exist $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and the function $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}^+_0)$ satisfying inequality (3.4), then the assertions (i), (iii), and (iv) of Theorem 3.1 are also valid on $[t_1, \infty)_{\mathbb{T}}$.

Remark 3.4. It should be noted that (3.4) is also equivalent to the inequality

$$\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t)) \leq 0 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.32}$$

see (3.20) and compare with [26, 28, 29, 34].

Example 3.5. For $\mathbb{T} = \mathbb{R}$, (3.4) has the form

$$\Lambda'(t) + \frac{1}{A_0(t)}\Lambda^2(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\exp\left\{-\int_{\alpha_i(t)}^t \frac{\Lambda(\eta)}{A_0(\eta)}\mathrm{d}\eta\right\} \leq 0 \quad \forall t \in [t_1, \infty)_{\mathbb{R}}, \tag{3.33}$$

see [8] for the case $A_0(t) \equiv 1,\ t \in [t_0, \infty)_{\mathbb{R}}$, and [35] for $n = 1,\ \alpha_1(t) = t,\ t \in [t_0, \infty)_{\mathbb{R}}$.

Example 3.6. For $\mathbb{T} = \mathbb{N}$, (3.4) becomes

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{A_0(k) + \Lambda(k)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(k)\prod_{j=\alpha_i(k)}^{k} \frac{A_0(j)}{A_0(j) + \Lambda(j)} \leq 0 \quad \forall k \in [k_1, \infty)_{\mathbb{N}}, \tag{3.34}$$

where the product over the empty set is assumed to be equal to one; see [1, 18] (or (1.10)) for $n = 1$, $\alpha_1(k) = k + 1$, $k \in [k_0, \infty)_{\mathbb{N}}$, and [20] for $n = 1$, $A_0(k) \equiv 1$, $\alpha_1(k) = k + 1$, $k \in [k_0, \infty)_{\mathbb{N}}$. It should be mentioned that in the literature all the results relating difference equations with discrete Riccati equations consider only the nondelay case. This result in the discrete case is therefore new.

Example 3.7. For $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ with $q \in (1,\infty)_{\mathbb{R}}$, under the same assumption on the product as in the previous example, condition (3.4) reduces to the inequality

$$D_q\Lambda(t) + \frac{\Lambda^2(t)}{A_0(t) + (q-1)t\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \prod_{\eta=\log_q(\alpha_i(t))}^{\log_q(t)} \frac{A_0(q^\eta)}{A_0(q^\eta) + (q-1)q^\eta\Lambda(q^\eta)} \le 0 \quad (3.35)$$

for all $t \in [t_1,\infty)_{\overline{q^{\mathbb{Z}}}}$.

4. Comparison Theorems

Theorem 3.1 can be employed to obtain comparison nonoscillation results. To this end, together with (3.1), we consider the second-order dynamic equation

$$\left(A_0x^\Delta\right)^\Delta(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \quad (4.1)$$

where $B_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ for $i \in [1,n]_{\mathbb{N}}$.

The following theorem establishes the relation between the first fundamental solution of the model equation with positive coefficients and comparison (4.1) with coefficients of arbitrary signs.

Theorem 4.1. *Suppose that (A2), (A3), (A4), and the following condition hold:*

(A5) *for $i \in [1,n]_{\mathbb{N}}$, $B_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ with $A_i(t) \ge B_i(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.*

Assume further that (3.4) *admits a solution $\Lambda \in C^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$, then the first fundamental solution $\mathcal{Y}_1$ of* (4.1) *satisfies $\mathcal{Y}_1(t,s) \ge \mathcal{X}_1(t,s) > 0$ for all $t \in (s,\infty)_{\mathbb{T}}$ and all $s \in [t_1,\infty)_{\mathbb{T}}$, where $\mathcal{X}_1$ denotes the first fundamental solution of* (3.1).

Proof. We consider the initial value problem

$$\begin{gathered}\left(A_0x^\Delta\right)^\Delta(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ x^\Delta(t_0) = 0, \quad x(t) \equiv 0 \quad \text{for } t \in [t_{-1},t_0]_{\mathbb{T}},\end{gathered} \quad (4.2)$$

where $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. Let $g \in C_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R})$, and define the function x as

$$x(t) = \int_{t_1}^{t} \mathcal{X}_1(t,\sigma(\eta))g(\eta)\Delta\eta \quad \forall t \in [t_1,\infty)_{\mathbb{T}}. \quad (4.3)$$

By the Leibnitz rule (see [32, Theorem 1.117]), for all $t \in [t_1, \infty)_{\mathbb{T}}$, we have

$$x^{\Delta}(t) = \int_{t_1}^{t} \mathcal{X}_1^{\Delta}(t, \sigma(\eta)) g(\eta) \Delta\eta, \tag{4.4}$$

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) = \int_{t_1}^{t} \left(A_0 \mathcal{X}_1^{\Delta}(\cdot, \sigma(\eta))\right)^{\Delta}(t) g(\eta) \Delta\eta + g(t). \tag{4.5}$$

Substituting (4.3) and (4.5) into (4.2), we get

$$\begin{aligned} f(t) &= \int_{t_1}^{t} \left(A_0 \mathcal{X}_1^{\Delta}(\cdot, \sigma(\eta))\right)^{\Delta}(t) g(\eta) \Delta\eta + \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t) \int_{t_1}^{\alpha_i(t)} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) g(\eta) \Delta\eta + g(t) \\ &= \sum_{i \in [1,n]_{\mathbb{N}}} [B_i(t) - A_i(t)] \int_{t_1}^{\alpha_i(t)} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) g(\eta) \Delta\eta + g(t) \\ &= \sum_{i \in [1,n]_{\mathbb{N}}} [B_i(t) - A_i(t)] \int_{t_1}^{t} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) g(\eta) \Delta\eta + g(t), \end{aligned} \tag{4.6}$$

where in the last step, we have used the fact that $\mathcal{X}_1(t, \sigma(s)) \equiv 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$ and all $s \in [t, \infty)_{\mathbb{T}}$. Therefore, we obtain the operator equation

$$g = \mathscr{H} g + f \quad \text{on } [t_1, \infty)_{\mathbb{T}}, \tag{4.7}$$

where

$$(\mathscr{H} g)(t) := \int_{t_1}^{t} \sum_{i \in [1,n]_{\mathbb{N}}} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) [A_i(t) - B_i(t)] g(\eta) \Delta\eta \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{4.8}$$

whose kernel is nonnegative. An application of Lemma 2.5 shows that nonnegativity of f implies the same for g, and thus x is nonnegative by (4.3). On the other hand, by Lemma 2.4, x has the representation

$$x(t) = \int_{t_0}^{t} \mathcal{Y}_1(t, \sigma(\eta)) f(\eta) \Delta\eta \quad \forall t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.9}$$

Proceeding as in the proof of the part (iii)⇒(iv) of Theorem 3.1, we conclude that the first fundamental solution $\mathcal{Y}_1$ of (4.1) satisfies $\mathcal{Y}_1(t, s) \geq 0$ for all $t \in (s, \infty)_{\mathbb{T}}$ and all $s \in [t_1, \infty)_{\mathbb{T}}$. To complete the proof, we have to show that $\mathcal{Y}_1(t, s) \geq \mathcal{X}_1(t, s) > 0$ for all $t \in (s, \infty)_{\mathbb{T}}$ and all $s \in [t_1, \infty)_{\mathbb{T}}$. Clearly, for any fixed $s \in [t_1, \infty)_{\mathbb{T}}$ and all $t \in [s, \infty)_{\mathbb{T}}$, we have

$$\left(A_0 \mathcal{Y}_1^{\Delta}(\cdot, s)\right)^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \mathcal{Y}_1(\alpha_i(t), s) = \sum_{i \in [1,n]_{\mathbb{N}}} [A_i(t) - B_i(t)] \mathcal{Y}_1(\alpha_i(t), s), \tag{4.10}$$

which by the solution representation formula yields that

$$\mathcal{Y}_1(t,s) = \mathcal{X}_1(t,s) + \int_s^t \mathcal{X}_1(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} [A_i(\eta) - B_i(\eta)]\mathcal{Y}_1(\alpha_i(\eta),s)\Delta\eta \geq \mathcal{X}_1(t,s) \tag{4.11}$$

for all $t \in [s,\infty)_{\mathbb{T}}$. This completes the proof since the first fundamental solution $\mathcal{X}_1$ satisfies $\mathcal{X}_1(t,s) > 0$ for all $t \in (s,\infty)_{\mathbb{T}}$ and all $s \in [t_1,\infty)_{\mathbb{T}}$ by Remark 3.3. □

Corollary 4.2. *Suppose that (A1), (A2), (A3), and (A5) hold, and* (3.1) *has a nonoscillatory solution on* $[t_1,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$*, then* (4.1) *admits a nonoscillatory solution on* $[t_2,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$.

Corollary 4.3. *Assume that (A2) and (A3) hold.*

(i) *If (A1) holds and the dynamic inequality*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t)x(\alpha_i(t)) \leq 0 \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{4.12}$$

where $A_i^{+}(t) := \max\{A_i(t),0\}$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$ *and* $i \in [1,n]_{\mathbb{N}}$*, has a positive solution on* $[t_0,\infty)_{\mathbb{T}}$*, then* (3.1) *also admits a positive solution on* $[t_1,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$.

(ii) *If (A4) holds and there exist a sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *and a function* $\Lambda \in C^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^{+})$ *satisfying the inequality*

$$\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(t,\alpha_i(t)) \leq 0 \quad \forall t \in [t_1,\infty)_{\mathbb{T}}, \tag{4.13}$$

then the first fundamental solution $\mathcal{X}_1$ *of* (3.1) *satisfies* $\mathcal{X}_1(t,s) > 0$ *for all* $t \in (s,\infty)_{\mathbb{T}}$ *and all* $s \in [t_1,\infty)_{\mathbb{T}}$.

Proof. Consider the dynamic equation

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.14}$$

Theorem 3.1 implies that for this equation the assertions (i) and (ii) hold. Since for all $i \in [1,n]_{\mathbb{N}}$, we have $A_i(t) \leq A_i^{+}(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, the application of Corollary 4.2 and Theorem 4.1 completes the proof. □

Now, let us compare the solutions of problem (2.1) and the following initial value problem:

$$\begin{gathered}
\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = g(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\
x(t_0) = y_1, \quad x^{\Delta}(t_0) = y_2, \quad x(t) = \varphi(t) \quad \text{for } t \in [t_{-1},t_0)_{\mathbb{T}},
\end{gathered} \tag{4.15}$$

where $y_1, y_2 \in \mathbb{R}$ are the initial values, $\psi \in C_{\mathrm{rd}}([t_{-1}, t_0)_{\mathbb{T}}, \mathbb{R})$ is the initial function such that ψ has a finite left-sided limit at the initial point t_0 provided that it is left dense, $g \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ is the forcing term.

Theorem 4.4. *Suppose that (A2), (A3), (A4), (A5), and the following condition hold:*

($A6$) *$f, g \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $\varphi, \psi \in C_{\mathrm{rd}}([t_{-1}, t_0)_{\mathbb{T}}, \mathbb{R})$ satisfy*

$$f(t) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t)\varphi(\alpha_i(t)) \le g(t) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t)\psi(\alpha_i(t)) \quad \forall t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.16}$$

Moreover, let (2.1) *have a positive solution x on $[t_0, \infty)_{\mathbb{T}}$, $y_1 = x_1$, and $y_2 \ge x_2$, then the solution y of* (4.15) *satisfies $y(t) \ge x(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.*

Proof. By Theorem 3.1 and Remark 3.3, we can assume that $\Lambda \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ is a solution of the dynamic Riccati inequality (3.4), then by ($A5$), the function Λ is also a solution of the dynamic Riccati inequality

$$\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t) e_{\ominus(\Lambda/A_0)}(t, \alpha_i(t)) \le 0 \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.17}$$

which is associated with (4.15). Hence, by Theorem 3.1 and Remark 3.3, the first fundamental solution $\mathcal{Y}_1$ of (4.15) satisfies $\mathcal{Y}_1(t, s) > 0$ for all $t \in (s, \infty)_{\mathbb{T}}$ and all $s \in [t_0, \infty)_{\mathbb{T}}$. Rewriting (2.1) in the form

$$\begin{gathered} \left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t) x(\alpha_i(t)) = f(t) - \sum_{i \in [1,n]_{\mathbb{N}}} [A_i(t) - B_i(t)] x(\alpha_i(t)), \quad t \in [t_0, \infty)_{\mathbb{T}} \\ x(t_0) = x_1, \quad x^{\Delta}(t_0) = x_2, \quad x(t) = \varphi(t), \quad t \in [t_{-1}, t_0)_{\mathbb{T}}, \end{gathered} \tag{4.18}$$

applying Lemma 2.4, and using ($A6$), we have

$$\begin{aligned} x(t) &= x_2 \mathcal{Y}_1(t, t_0) + x_1 \mathcal{Y}_2(t, t_0) + \int_{t_0}^{t} \mathcal{Y}_1(t, \sigma(\eta)) \\ &\quad \times \left[f(\eta) - \sum_{i \in [1,n]_{\mathbb{N}}} [A_i(\eta) - B_i(\eta)] \chi_{[t_0,\infty)_{\mathbb{T}}}(\alpha_i(\eta)) x(\alpha_i(\eta)) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta \\ &\le y_2 \mathcal{Y}_1(t, t_0) + y_1 \mathcal{Y}_2(t, t_0) + \int_{t_0}^{t} \mathcal{Y}_1(t, \sigma(\eta)) \left[g(\eta) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(\eta)\psi(\alpha_i(\eta)) \right] \Delta\eta \\ &= y(t) \end{aligned} \tag{4.19}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. This completes the proof. $\square$

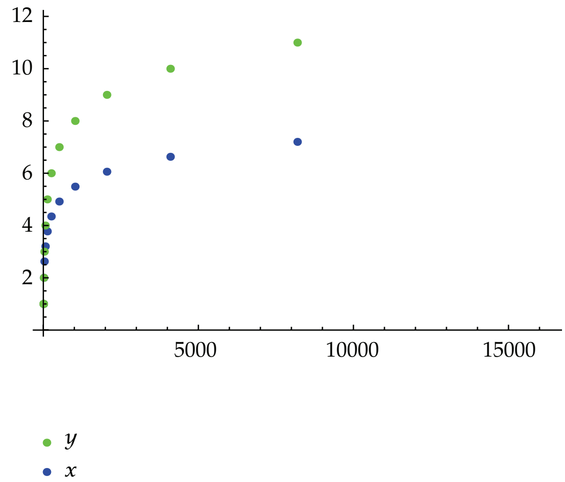

Figure 1: The graph of 10 iterates for the solutions of (4.20) and (4.22) illustrates the result of Theorem 4.4, here $y(t) > x(t)$ for all $t \in (1, \infty)_{\overline{2^{\mathbb{Z}}}}$.

Remark 4.5. If $B_i \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ for $i \in [1, n]_{\mathbb{N}}$, $f(t) \leq g(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $\varphi(t) \geq \psi(t)$ for all $t \in [t_{-1}, t_0)_{\mathbb{T}}$, then $(A6)$ holds.

The following example illustrates Theorem 4.4 for the quantum time scale $\mathbb{T} = \overline{2^{\mathbb{Z}}}$.

Example 4.6. Let $\overline{2^{\mathbb{Z}}} := \{2^k : k \in \mathbb{Z}\} \cup \{0\}$, and consider the following initial value problems:

$$\begin{gathered} D_2\left(\mathrm{Id}_{\overline{2^{\mathbb{Z}}}} D_2 x\right)(t) + \frac{2}{t^4} x\left(\frac{t}{4}\right) = -\frac{1}{t^4} \quad \text{for } t \in [1, \infty)_{\overline{2^{\mathbb{Z}}}}, \\ D_2 x(1) = 1, \quad x(t) \equiv 1 \quad \text{for } t \in \left[\frac{1}{4}, 1\right]_{\overline{2^{\mathbb{Z}}}}, \end{gathered} \tag{4.20}$$

where $\mathrm{Id}_{\overline{2^{\mathbb{Z}}}}$ is the identity function on $\overline{2^{\mathbb{Z}}}$, that is, $\mathrm{Id}_{\overline{2^{\mathbb{Z}}}}(t) = t$ for $t \in \overline{2^{\mathbb{Z}}}$, and

$$D_2 x(t) = \frac{1}{t}(x(2t) - x(t)) \quad \text{for } t \in \overline{2^{\mathbb{Z}}}, \tag{4.21}$$

$$\begin{gathered} D_2\left(\mathrm{Id}_{\overline{2^{\mathbb{Z}}}} D_2 x\right)(t) + \frac{1}{t^4} x\left(\frac{t}{4}\right) = \frac{1}{t^4} \quad \text{for } t \in [1, \infty)_{\overline{2^{\mathbb{Z}}}}, \\ D_2 x(1) = 1, \quad x(t) \equiv 1 \quad \text{for } t \in \left[\frac{1}{4}, 1\right]_{\overline{2^{\mathbb{Z}}}}. \end{gathered} \tag{4.22}$$

Denoting by x and y the solutions of (4.20) and (4.22), respectively, we obtain $y(t) \geq x(t)$ for all $t \in [1, \infty)_{\overline{2^{\mathbb{Z}}}}$ by Theorem 4.4. For the graph of the first 10 iterates, see Figure 1.

As an immediate consequence of Theorem 4.4, we obtain the following corollary.

Corollary 4.7. *Suppose that (A1), (A2), and (A3) hold and that* (3.1) *is nonoscillatory, then, for* $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$*, the dynamic equation*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \textit{for } t\in[t_0,\infty)_{\mathbb{T}} \tag{4.23}$$

is also nonoscillatory.

We now consider the following dynamic equation:

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = g(t) \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \\ x(t_0)=y_1, \quad x^{\Delta}(t_0)=y_2, \quad x(t)=\psi(t) \quad \text{for } t\in[t_{-1},t_0)_{\mathbb{T}},\end{gathered} \tag{4.24}$$

where the parameters are the same as in (4.15).

We obtain the most complete result if we compare solutions of (2.1) and (4.24) by omitting the condition ($A2$) and assuming that the solution of (2.1) is positive.

Corollary 4.8. *Suppose that (A3), (A4), and the following condition hold:*

($A7$) $f,g \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *and* $\varphi,\psi \in C_{\mathrm{rd}}([t_{-1},t_0)_{\mathbb{T}},\mathbb{R})$ *satisfy*

$$f(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) \le g(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\psi(\alpha_i(t)) \quad \forall t\in[t_0,\infty)_{\mathbb{T}}. \tag{4.25}$$

If x is a positive solution of (2.1) *on* $[t_0,\infty)_{\mathbb{T}}$ *with* $x_1 = y_1$ *and* $y_2 \ge x_2$*, then for the solution y of* (4.24)*, one has* $y(t)\ge x(t)$ *for all* $t\in[t_0,\infty)_{\mathbb{T}}$.

Proof. Corollary 4.3 and Remark 3.3 imply that the first fundamental solution $\mathcal{X}_1$ associated with (2.1) (and (4.24)) satisfies $\mathcal{X}_1(t,s)>0$ for all $t\in(s,\infty)_{\mathbb{T}}$ and all $s\in[t_0,\infty)_{\mathbb{T}}$. Hence, the claim follows from the solution representation formula. □

Remark 4.9. If at least one of the inequalities in the statements of Theorem 4.4 and Corollary 4.8 is strict, then the conclusions hold with the strict inequality too.

Let us compare equations with different coefficients and delays. Now, we consider

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\beta_i(t)) = 0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}. \tag{4.26}$$

Theorem 4.10. *Suppose that (A2), (A4), (A5), and the following condition hold:*

($A8$) *for* $i\in[1,n]_{\mathbb{N}}$, $\beta_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ *satisfies* $\beta_i(t)\le\alpha_i(t)$ *for all* $t\in[t_0,\infty)_{\mathbb{T}}$ *and* $\lim_{t\to\infty}\beta_i(t)=\infty$.

Assume further that the first-order dynamic Riccati inequality (3.4) *has a solution* $\Lambda\in C^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ *for some* $t_1\in[t_0,\infty)_{\mathbb{T}}$*, then the first fundamental solution* $\mathcal{Y}_1$ *of* (4.26) *satisfies* $\mathcal{Y}_1(t,s)>0$ *for all* $t\in(s,\infty)_{\mathbb{T}}$ *and all* $s\in[t_1,\infty)_{\mathbb{T}}$.

Proof. Note that $(A5)$ implies $A_i(t) \geq B_i^+(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $i \in [1, n]_{\mathbb{N}}$, then we have

$$\begin{aligned} 0 &\geq \Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) e_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t)) \\ &\geq \Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i \in [1,n]_{\mathbb{N}}} B_i^+(t) e_{\ominus(\Lambda/A_0)}(\sigma(t), \beta_i(t)) \end{aligned} \tag{4.27}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. The reference to Corollary 4.3 (ii) concludes the proof. □

Remark 4.11. If the condition $(A4)$ in Theorem 4.1, Theorem 4.4, Corollary 4.8, and Theorem 4.10 is replaced with $(A1)$, then the claims of the theorems are valid eventually.

Let us introduce the function

$$\alpha_{\max}(t) := \max_{i \in [1,n]_{\mathbb{N}}} \{\alpha_i(t)\} \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.28}$$

Corollary 4.12. *Suppose that (A1), (A2), (A3), and (A5) hold. If*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \left(\sum_{i \in [1,n]_{\mathbb{N}}} A_i(t)\right) x(\alpha_{\max}(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{4.29}$$

is nonoscillatory, then (4.1) *is also nonoscillatory.*

Remark 4.13. The claim of Corollary 4.12 is also true when $\alpha_{\max}$ is replaced by σ.

5. Explicit Nonoscillation and Oscillation Results

Theorem 5.1. *Suppose that (A1), (A2), and (A3) hold and that*

$$\frac{\sigma(t)}{2tA_0(t) + \mu(t)} + 2t\sigma(t) \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) e_{\ominus(1/(2\mathrm{Id}_{\mathbb{T}} A_0))}(\sigma(t), \alpha_i(t)) \leq 1 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{5.1}$$

where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and* $\mathrm{Id}_{\mathbb{T}}$ *is the identity function on* $\mathbb{T}$, *then* (3.1) *is nonoscillatory.*

Proof. The statement of the theorem yields that $\Lambda(t) = 1/(2t)$ for $t \in [t_0, \infty)_{\mathbb{T}^+}$ is a positive solution of the Riccati inequality (3.32). □

Next, let us apply Theorem 5.1 to delay differential equations.

Corollary 5.2. *Let* $A_0 \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R}^+)$*, for* $i \in [1,n]_{\mathbb{N}}$*,* $A_i \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R}_0^+)$*, and* $\alpha_i \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R})$ *such that* $\alpha_i(t) \leq t$ *for all* $t \in [t_0,\infty)_{\mathbb{R}}$ *and* $\lim_{t\to\infty}\alpha_i(t) = \infty$*. If*

$$\frac{1}{2A_0(t)} + 2t^2 \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\exp\left\{-\int_{\alpha_i(t)}^{t} \frac{1}{2\eta A_0(\eta)}d\eta\right\} \leq 1 \quad \forall t \in [t_1,\infty)_{\mathbb{R}} \tag{5.2}$$

for some $t_1 \in [t_0,\infty)_{\mathbb{R}}$*, then* (1.2) *is nonoscillatory.*

Now, let us proceed with the discrete case.

Corollary 5.3. *Let* $\{A_0(k)\}$ *be a positive sequence, for* $i \in [1,n]_{\mathbb{N}}$*, let* $\{A_i(k)\}$ *be a nonnegative sequence, and let* $\{\alpha_i(k)\}$ *be a divergent sequence such that* $\alpha_i(k) \leq k+1$ *for all* $k \in [k_0,\infty)_{\mathbb{N}}$*. If*

$$\frac{k+1}{2kA_0(k)+1} + 2k(k+1)\sum_{i\in[1,n]_{\mathbb{N}}} A_i(k)\prod_{j=\alpha_i(k)}^{k}\frac{2jA_0(j)}{2jA_0(j)+1} \leq 1 \quad \forall k \in [k_1,\infty)_{\mathbb{N}} \tag{5.3}$$

for some $k_1 \in [k_0,\infty)_{\mathbb{N}}$*, then* (1.8) *is nonoscillatory.*

Let us introduce the function

$$A(t,s) := \int_s^t \frac{1}{A_0(\eta)}\Delta\eta \quad \text{for } s,t \in [t_0,\infty)_{\mathbb{T}}. \tag{5.4}$$

Theorem 5.4. *Suppose that (A1), (A2), and (A3) hold, and for every* $t_1 \in [t_0,\infty)_{\mathbb{T}}$*, the dynamic equation*

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + \frac{1}{A(\alpha_{\max}(t),t_1)}\left(\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)A(\alpha_i(t),t_1)\right)x(\alpha_{\max}(t)) = 0, \quad t \in [t_2,\infty)_{\mathbb{T}} \tag{5.5}$$

is oscillatory, where $t_2 \in [t_1,\infty)_{\mathbb{T}}$ *satisfies* $\alpha_{\min}(t) > t_1$ *for all* $t \in [t_2,\infty)_{\mathbb{T}}$*, then* (3.1) *is also oscillatory.*

Proof. Assume to the contrary that (3.1) is nonoscillatory, then there exists a solution x of (3.1) such that $x > 0$, $(A_0x^{\Delta})^{\Delta} \leq 0$ on $[t_1,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$. This implies that A_0x^{Δ} is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$, then it follows that

$$x(t) \geq x(t) - x(t_1) = \int_{t_1}^{t}\frac{1}{A_0(\eta)}A_0(\eta)x^{\Delta}(\eta)\Delta\eta \geq A(t,t_1)A_0(t)x^{\Delta}(t) \quad \forall t \in [t_1,\infty)_{\mathbb{T}}, \tag{5.6}$$

or simply by using (5.4),

$$x(t) - A(t,t_1)A_0(t)x^{\Delta}(t) \geq 0 \quad \forall t \in [t_1,\infty)_{\mathbb{T}}. \tag{5.7}$$

Now, let

$$\varphi(t) := \frac{x(t)}{A(t,t_1)} \quad \text{for } t \in (t_1,\infty)_{\mathbb{T}}. \tag{5.8}$$

By the quotient rule, (5.4) and (5.7), we have

$$\varphi^{\Delta}(t) = \frac{A(t,t_1)A_0(t)x^{\Delta}(t) - x(t)}{A(\sigma(t),t_1)A(t,t_1)A_0(t)} \leq 0 \quad \forall t \in (t_1,\infty)_{\mathbb{T}}, \tag{5.9}$$

proving that φ is nonincreasing on $(t_1,\infty)_{\mathbb{T}}$. Therefore, for all $i \in [1,n]_{\mathbb{N}}$, we obtain

$$\frac{x(\alpha_{\max}(t))}{A(\alpha_{\max}(t),t_1)} = \varphi(\alpha_{\max}(t)) \leq \varphi(\alpha_i(t)) = \frac{x(\alpha_i(t))}{A(\alpha_i(t),t_1)} \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{5.10}$$

where $t_2 \in [t_1,\infty)_{\mathbb{T}}$ satisfies $\alpha_{\min}(t) > t_1$ for all $t \in [t_2,\infty)_{\mathbb{T}}$. Using (5.10) in (3.1), we see that x solves

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + \frac{1}{A(\alpha_{\max}(t),t_1)}\left(\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)A(\alpha_i(t),t_1)\right)x(\alpha_{\max}(t)) \leq 0 \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{5.11}$$

which shows that (5.5) is also nonoscillatory by Theorem 3.1. This is a contradiction, and the proof is completed. □

The following theorem can be regarded as the dynamic generalization of Leighton's result (Theorem A).

Theorem 5.5. *Suppose that (A2), (A3), and (A4) hold and that*

$$\int_{t_2}^{\infty} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\mathrm{e}_{\ominus(1/(A_0A(\cdot,t_1)))}(\sigma(\eta),\alpha_i(\eta))\Delta\eta = \infty, \tag{5.12}$$

where $t_2 \in (t_1,\infty) \subset [t_0,\infty)_{\mathbb{T}}$, then every solution of (3.1) is oscillatory.

Proof. Assume to the contrary that (3.1) is nonoscillatory. It follows from Theorem 3.1 and Remark 3.2 that (3.4) has a solution $\Lambda \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$. Using (3.5) and (5.7), we see that

$$\Lambda(t) \leq \frac{1}{A(t,t_1)} \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{5.13}$$

which together with (3.4) implies that

$$\Lambda^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(1/(A_0A(\cdot,t_1)))}(\sigma(t),\alpha_i(t)) \leq 0 \quad \forall t \in [t_2,\infty)_{\mathbb{T}}. \tag{5.14}$$

Integrating the last inequality, we get

$$\Lambda(t) - \Lambda(t_2) + \int_{t_2}^{t} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta) e_{\ominus(1/(A_0 A(\cdot,t_1)))}(\sigma(\eta), \alpha_i(\eta))\Delta\eta \le 0 \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{5.15}$$

which is in a contradiction with (5.12). This completes the proof. □

We conclude this section with applications of Theorem 5.5 to delay differential equations and difference equations.

Corollary 5.6. *Let $A_0 \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R}^+)$, for $i \in [1,n]_{\mathbb{N}}$, $A_i \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R}_0^+)$, and $\alpha_i \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R})$ such that $\alpha_i(t) \le t$ for all $t \in [t_0,\infty)_{\mathbb{R}}$ and $\lim_{t\to\infty}\alpha_i(t) = \infty$. If*

$$\lim_{t\to\infty} A(t,t_0) = \infty, \qquad \int_{t_0}^{\infty} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\frac{A(\alpha_i(\eta),t_0)}{A(\eta,t_0)}d\eta = \infty, \tag{5.16}$$

where

$$A(t,s) := \int_s^t \frac{1}{A_0(\eta)}d\eta \quad \text{for } s,t \in [t_0,\infty)_{\mathbb{R}}, \tag{5.17}$$

then (1.2) *is oscillatory.*

Corollary 5.7. *Let $\{A_0(k)\}$ be a positive sequence, for $i \in [1,n]_{\mathbb{N}}$, let $\{A_i(k)\}$ be a nonnegative sequence and let $\{\alpha_i(k)\}$ be a divergent sequence such that $\alpha_i(k) \le k+1$ for all $k \in [k_0,\infty)_{\mathbb{N}}$. If*

$$\lim_{k\to\infty} A(k,k_0) = \infty, \qquad \sum_{j=k_0}^{\infty}\sum_{i\in[1,n]_{\mathbb{N}}} A_i(j)\prod_{\ell=\alpha_i(j)}^{j}\frac{A_0(\ell)A(\ell,k_0)}{A_0(\ell)A(\ell,k_0)+1} = \infty, \tag{5.18}$$

where

$$A(k,l) := \sum_{j=l}^{k-1}\frac{1}{A_0(j)} \quad \text{for } l,k \in [k_0,\infty)_{\mathbb{N}}, \tag{5.19}$$

then (1.8) *is oscillatory.*

6. Existence of a Positive Solution

Theorem 6.1. *Suppose that (A2), (A3), and (A4) hold, $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$, and the first-order dynamic Riccati inequality* (3.4) *has a solution $\Lambda \in C_{\mathrm{rd}}^1([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$. Moreover, suppose that there exist $x_1,x_2 \in \mathbb{R}^+$ such that $\varphi(t) \le x_1$ for all $t \in [t_{-1},t_0)_{\mathbb{T}}$ and $x_2 \ge \Lambda(t_0)x_1/A_0(t_0)$, then* (2.1) *admits a positive solution x such that $x(t) \ge x_1$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.*

Proof. First assume that y is the solution of the following initial value problem:

$$\left(A_0 y^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) y(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$
$$y^{\Delta}(t_0) = \frac{\Lambda(t_0)}{A_0(t_0)} x_1, \quad y(t) \equiv x_1 \quad \text{for } t \in [t_{-1}, t_0]_{\mathbb{T}}. \tag{6.1}$$

Denote

$$z(t) := \begin{cases} x_1 \mathrm{e}_{\Lambda/A_0}(t, t_0) & \text{for } t \in [t_0, \infty)_{\mathbb{T}}, \\ x_1 & \text{for } t \in [t_{-1}, t_0)_{\mathbb{T}}, \end{cases} \tag{6.2}$$

then, by following similar arguments to those in the proof of the part (ii)⇒(iii) of Theorem 3.1, we obtain

$$g(t) := \left(A_0 z^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) z(\alpha_i(t))$$
$$= x_1 \mathrm{e}_{\Lambda/A_0}(t, t_0) \left[\Lambda^{\Delta}(t) + \frac{1}{A_0(t)} \Lambda^{\sigma}(t) \Lambda(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(t, \alpha_i(t))\right] \le 0 \tag{6.3}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. So z is a solution to

$$\left(A_0 z^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) z(\alpha_i(t)) = g(t) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$
$$z^{\Delta}(t_0) = \frac{\Lambda(t_0)}{A_0(t_0)} x_1, \quad z(t) \equiv x_1 \quad \text{for } t \in [t_{-1}, t_0]_{\mathbb{T}}. \tag{6.4}$$

Theorem 4.4 implies that $y(t) \ge z(t) \ge x_1 > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. By the hypothesis of the theorem, Theorem 4.4, and Corollary 4.8, we have $x(t) \ge y(t) \ge x_1 > 0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. This completes the proof for the case $f \equiv 0$ and $g \equiv 0$ on $[t_0, \infty)_{\mathbb{T}}$.

The general case where $f \not\equiv 0$ on $[t_0, \infty)_{\mathbb{T}}$ is also a consequence of Theorem 4.4. □

Let us illustrate the result of Theorem 6.1 with the following example.

Example 6.2. Let $\sqrt{\mathbb{N}_0} := \{\sqrt{k} : k \in \mathbb{N}_0\}$, and consider the following delay dynamic equation:

$$\left(\mathrm{Id}_{\sqrt{\mathbb{N}_0}} x^{\Delta}\right)^{\Delta}(t) + \frac{1}{8t\sqrt{t^2+1}} \left(x(t) + \frac{1}{2} x\left(\sqrt{t^2-1}\right)\right) = \frac{1}{t\sqrt{t^2+1}}, \quad t \in [1, \infty)_{\sqrt{\mathbb{N}_0}},$$
$$x^{\Delta}(1) = 2, \quad x(t) \equiv 2 \quad \text{for } t \in [0, 1]_{\sqrt{\mathbb{N}_0}}, \tag{6.5}$$

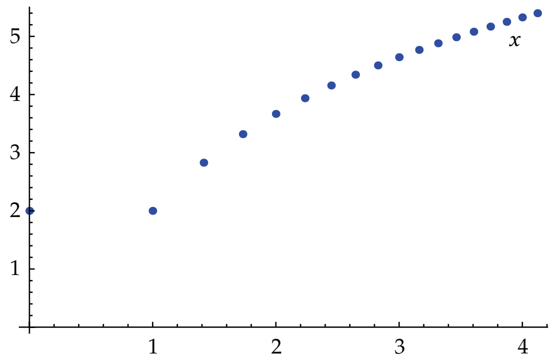

Figure 2: The graph of 15 iterates for the solution of (6.5) illustrates the result of Theorem 6.1.

then (5.1) takes the form $\Phi(t) \leq 1$ for all $t \in [1,\infty)_{\sqrt{\mathbb{N}_0}}$, where the function Φ is defined by

$$\Phi(t) := \frac{1}{2t^2 + \left(\sqrt{t^2+1} - t\right)} \left(\sqrt{t^2+1} + \frac{t^2}{2} \left(1 + \frac{t^2 - 1}{2(t^2-1) + \left(t - \sqrt{t^2-1}\right)} \right) \right) \quad \text{for } t \in [1,\infty)_{\mathbb{R}} \tag{6.6}$$

and is decreasing on $[1,\infty)_{\mathbb{R}}$ and thus is not greater than $\Phi(1) \approx 0.79$, that is, Theorem 5.1 holds. Theorem 6.1 therefore ensures that the solution is positive on $[1,\infty)_{\sqrt{\mathbb{N}_0}}$. For the graph of 15 iterates, see Figure 2.

7. Discussion and Open Problems

We start this section with discussion of explicit nonoscillation conditions for delay differential and difference equations. Let us first consider the continuous case. Corollary 5.6 with $n = 1$ and $\alpha_1(t) = t$ for $t \in [t_0,\infty)_{\mathbb{R}}$ reduces to Theorem A. Nonoscillation part of Kneser's result for (1.4) follows from Corollary 5.2 by letting $n = 1$, $A_0(t) \equiv 1$, and $\alpha_1(t) = t$ for $t \in [t_0,\infty)_{\mathbb{R}}$. Theorem E is obtained by applying Corollary 5.3 to (1.10).

Known nonoscillation tests for difference equations can also be deduced from the results of the present paper. In [18, Lemma 1.2], Chen and Erbe proved that (1.9) is nonoscillatory if and only if there exists a sequence $\{\Lambda(k)\}$ with $A_0(k) + \Lambda(k) > 0$ for all $k \in [k_1,\infty)_{\mathbb{N}}$ and some $k_1 \in [k_0,\infty)_{\mathbb{N}}$ satisfying

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{A_0(k) + \Lambda(k)} + A_1(k) \leq 0 \quad \forall k \in [k_1,\infty)_{\mathbb{N}}. \tag{7.1}$$

Since this result is a necessary and sufficient condition, the conclusion of Theorem F could be deduced from

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{1 + \Lambda(k)} + A_1(k) \leq 0 \quad \forall k \in [k_1,\infty)_{\mathbb{N}}, \tag{7.2}$$

which is a particular case of (7.1) with $A_0(k) \equiv 1$ for $k \in [k_0, \infty)_{\mathbb{N}}$. We present below a short proof for the nonoscillation part only. Assuming (1.12) and letting

$$\Lambda(k) := \frac{1}{4(k-1)} + \sum_{j=k}^{\infty} A_1(j) \quad \text{for } k \in [k_1, \infty)_{\mathbb{N}} \subset [2, \infty)_{\mathbb{N}}, \tag{7.3}$$

we get

$$\frac{1}{4(k-1)} + \frac{1}{4k} \geq \Lambda(k) \geq \frac{1}{4(k-1)} \quad \forall k \in [k_1, \infty)_{\mathbb{N}}, \tag{7.4}$$

and this yields

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{1+\Lambda(k)} + A_1(k) \leq -\frac{1}{4k^2(4k-3)} < 0 \quad \forall\, k \in [k_1, \infty)_{\mathbb{N}}. \tag{7.5}$$

That is, the discrete Riccati inequality (7.2) has a positive solution implying that (1.10) is nonoscillatory. It is not hard to prove that (1.13) implies nonexistence of a sequence $\{\Lambda(k)\}$ satisfying the discrete Riccati inequality (7.2) (see the proof of [23, Lemma 3]). Thus, oscillation/nonoscillation results for (1.10) in [21] can be deduced from nonexistence/existence of a solution for the discrete Riccati inequality (7.2); see also [20].

An application of Theorem 3.1 with $\Lambda(t) := \lambda/t$ for $t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}^+}$ and $\lambda \in \mathbb{R}^+$ implies the following result for quantum scales.

Example 7.1. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$ with $q \in (1, \infty)_{\mathbb{R}}$. If there exist $\lambda \in \mathbb{R}_0^+$ and $t_1 \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}^+}$ such that

$$\frac{\lambda^2}{A_0(t) + (q-1)\lambda} + t^2 \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \prod_{\eta = \log_q(\alpha_i(t))}^{\log_q(t)} \frac{A_0(q^{\eta})}{A_0(q^{\eta}) + (q-1)\lambda} \leq \frac{\lambda}{q}, \quad t \in [t_1, \infty)_{\overline{q^{\mathbb{Z}}}}, \tag{7.6}$$

then the delay q-difference equation

$$D_q\big(A_0 D_q x\big)(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}} \tag{7.7}$$

is nonoscillatory.

In [36], Bohner and Ünal studied nonoscillation and oscillation of the q-difference equation

$$D_q^2 x(t) + \frac{a}{qt^2} x(qt) = 0 \quad \text{for } t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}}, \tag{7.8}$$

where $a \in \mathbb{R}_0^+$, and proved that (7.7) is nonoscillatory if and only if

$$a \leq \frac{1}{\left(\sqrt{q}+1\right)^2}. \tag{7.9}$$

For the above q-difference equation, (7.6) reduces to the algebraic inequality

$$\frac{\lambda^2}{1+(q-1)\lambda} + \frac{a}{q} \leq \frac{\lambda}{q} \quad \text{or} \quad \lambda^2 - (1-(q-1)a)\lambda + a \leq 0, \tag{7.10}$$

whose discriminant is $(1-(q-1)a)^2 - 4a = (q-1)^2a^2 - (q+1)a + 1$. The discriminant is nonnegative if and only if

$$a \geq \frac{q+2\sqrt{q}+1}{q^2-2q+1} = \frac{1}{\left(\sqrt{q}-1\right)^2} \quad \text{or} \quad a \leq \frac{q-2\sqrt{q}+1}{q^2-2q+1} = \frac{1}{\left(\sqrt{q}+1\right)^2}. \tag{7.11}$$

If the latter one holds, then the inequality (7.6) holds with an equality for the value

$$\lambda := \frac{1}{2}\left(1-(q-1)a+\sqrt{(1-(q-1)a)^2-4a}\right). \tag{7.12}$$

It is easy to check that this value is not less than $2/(\sqrt{q}+1)^2$, that is, the solution is nonnegative. This gives us the nonoscillation part of [36, Theorem 3].

Let us also outline connections to some known results in the theory of second-order ordinary differential equations. For example, the Sturm-Picone comparison theorem is an immediate corollary of Theorem 4.10 if we remark that a solution $\Lambda \in C_{\mathrm{rd}}^1([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ of the inequality (3.32) satisfying $\Lambda/A_0 \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ is also a solution of (3.32) with B_i instead of A_i for $i=0,1$.

Proposition 7.2 (see [28, 32, 36]). *Suppose that $B_0(t) \geq A_0(t) > 0$, $A_1(t) \geq 0$, and $A_1(t) \geq B_1(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, then nonoscillation of*

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + A_1(t)x^{\sigma}(t) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}} \tag{7.13}$$

implies nonoscillation of

$$\left(B_0x^{\Delta}\right)^{\Delta}(t) + B_1(t)x^{\sigma}(t) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}. \tag{7.14}$$

The following result can also be regarded as another generalization of the Sturm-Picone comparison theorem. It is easily deduced that there is a solution $\Lambda \in C_{\mathrm{rd}}^1([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ of the inequality (3.4).

Proposition 7.3. *Suppose that (A4) and the conditions of Proposition 7.2 are fulfilled, then nonoscillation of*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + A_1(t)x(t) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{7.15}$$

implies the same for

$$\left(B_0 x^{\Delta}\right)^{\Delta}(t) + B_1(t)x(t) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{7.16}$$

Finally, let us present some open problems. To this end, we will need the following definition.

Definition 7.4. A solution x of (3.1) is said to be *slowly oscillating* if for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$ there exist $t_2 \in (t_1, \infty)_{\mathbb{T}}$ with $\alpha_{\min}(t) \geq t_1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$ and $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $x(t_1)x^{\sigma}(t_1) \leq 0$, $x(t_2)x^{\sigma}(t_2) \leq 0$, $x(t) > 0$ for all $t \in (t_1, t_2)_{\mathbb{T}}$.

Following the method of [8, Theorem 10], we can demonstrate that if (A1), (A2) with positive coefficients and (A3) hold, then the existence of a slowly oscillating solution of (3.1) which has infinitely many zeros implies oscillation of all solutions.

(P1) Generally, will existence of a slowly oscillating solution imply oscillation of all solutions? To the best of our knowledge, slowly oscillating solutions have not been studied for difference equations yet, the only known result is [9, Proposition 5.2].

All the results of the present paper are obtained under the assumptions that all coefficients of (3.1) are nonnegative, and if some of them are negative, it is supposed that the equation with the negative terms omitted has a positive solution.

(P2) Obtain sufficient nonoscillation conditions for (3.1) with coefficients of an arbitrary sign, not assuming that all solutions of the equation with negative terms omitted are nonoscillatory. In particular, consider the equation with one oscillatory coefficient.

(P3) Describe the asymptotic and the global properties of nonoscillatory solutions.

(P4) Deduce nonoscillation conditions for linear second-order impulsive equations on time scales, where both the solution and its derivative are subject to the change at impulse points (and these changes can be matched or not). The results of this type for second-order delay differential equations were obtained in [37].

(P5) Consider the same equation on different time scales. In particular, under which conditions will nonoscillation of (1.8) imply nonoscillation of (1.2)?

(P6) Obtain nonoscillation conditions for neutral delay second-order equations. In particular, for difference equations some results of this type (a necessary oscillation conditions) can be found in [17].

(P7) In the present paper, all parameters of the equation are rd-continuous which corresponds to continuous delays and coefficients for differential equations. However, in [8], nonoscillation of second-order equations is studied under a more general assumption that delays and coefficients are Lebesgue measurable functions. Can the restrictions of rd-continuity of the parameters be relaxed to involve, for example, discontinuous coefficients which arise in the theory of impulsive equations?

Appendix

Time Scales Essentials

A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf(t, \infty)_{\mathbb{T}}$ while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty, t)_{\mathbb{T}}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined to be $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right dense* if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise, it is called *right scattered*, and similarly *left dense* and *left scattered* points are defined with respect to the backward jump operator. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, the Δ-derivative $f^{\Delta}(t)$ of f at the point t is defined to be the number, provided it exists, with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\Big|\big[f^{\sigma}(t) - f(s)\big] - f^{\Delta}(t)\big[\sigma(t) - s\big]\Big| \le \varepsilon|\sigma(t) - s| \quad \forall s \in U, \tag{A.1}$$

where $f^{\sigma} := f \circ \sigma$ on $\mathbb{T}$. We mean the Δ-derivative of a function when we only say derivative unless otherwise is specified. A function f is called *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$ and has a finite limit at left-dense points, and the *set of rd-continuous functions* is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. The set of functions $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ includes the functions whose derivative is in $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ too. For a function $f \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, the so-called *simple useful formula* holds

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t) \quad \forall t \in \mathbb{T}^{\kappa}, \tag{A.2}$$

where $\mathbb{T}^{\kappa} := \mathbb{T} \setminus \{\sup \mathbb{T}\}$ if $\sup \mathbb{T} = \max \mathbb{T}$ and satisfies $\rho(\max \mathbb{T}) \ne \max \mathbb{T}$; otherwise, $\mathbb{T}^{\kappa} := \mathbb{T}$. For $s, t \in \mathbb{T}$ and a function $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\eta)\Delta\eta = F(t) - F(s) \quad \text{for } s, t \in \mathbb{T}, \tag{A.3}$$

where $F \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ is an antiderivative of f, that is, $F^{\Delta} = f$ on $\mathbb{T}^{\kappa}$. Table 1 gives the explicit forms of the forward jump, graininess, Δ-derivative, and Δ-integral on the well-known time scales of reals, integers, and the quantum set, respectively.

A function $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ is called *regressive* if $1 + \mu f \ne 0$ on $\mathbb{T}^{\kappa}$, and *positively regressive* if $1 + \mu f > 0$ on $\mathbb{T}^{\kappa}$. The *set of regressive functions* and the *set of positively regressive functions* are denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R})$ and $\mathcal{R}^{+}(\mathbb{T}, \mathbb{R})$, respectively, and $\mathcal{R}^{-}(\mathbb{T}, \mathbb{R})$ is defined similarly.

Table 1: Forward jump, Δ-derivative, and Δ-integral.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$, $(q>1)$
$\sigma(t)$	t	$t+1$	qt
$f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$	$D_q f(t) := (f(qt)-f(t))/((q-1)t)$
$\int_s^t f(\eta)\Delta\eta$	$\int_s^t f(\eta)d\eta$	$\sum_{\eta=s}^{t-1} f(\eta)$	$\int_s^t f(\eta)d_q\eta := (q-1)\sum_{\eta=\log_q(s)}^{\log_q(t/q)} f(q^{\eta})q^{\eta}$

Table 2: The exponential function.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$, $(q>1)$
$\mathrm{e}_f(t,s)$	$\exp\{\int_s^t f(\eta)d\eta\}$	$\prod_{\eta=s}^{t-1}(1+f(\eta))$	$\prod_{\eta=\log_q(s)}^{\log_q(t/q)}(1+(q-1)q^{\eta}f(q^{\eta}))$

Let $f \in \mathcal{R}(\mathbb{T},\mathbb{R})$, then the *exponential function* $\mathrm{e}_f(\cdot,s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the initial value problem

$$x^{\Delta}(t) = f(t)x(t) \quad \text{for } t \in \mathbb{T}^{\kappa}, \qquad x(s) = 1 \tag{A.4}$$

for some fixed $s \in \mathbb{T}$. For $h \in \mathbb{R}^{+}$, set $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -1/h\}$, $\mathbb{Z}_h := \{z \in \mathbb{C} : -\pi/h < \mathrm{Im}(z) \leq \pi/h\}$, and $\mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C}$. For $h \in \mathbb{R}_0^{+}$, we define the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) := \begin{cases} z, & h = 0, \\ \dfrac{1}{h}\mathrm{Log}(1+hz), & h > 0 \end{cases} \tag{A.5}$$

for $z \in \mathbb{C}_h$, then the exponential function can also be written in the form

$$\mathrm{e}_f(t,s) := \exp\left\{\int_s^t \xi_{\mu(\eta)}(f(\eta))\Delta\eta\right\} \quad \text{for } s,t \in \mathbb{T}. \tag{A.6}$$

Table 2 illustrates the explicit forms of the exponential function on some well-known time scales.

The exponential function $\mathrm{e}_f(\cdot,s)$ is strictly positive on $[s,\infty)_{\mathbb{T}}$ if $f \in \mathcal{R}^{+}([s,\infty)_{\mathbb{T}},\mathbb{R})$, while $\mathrm{e}_f(\cdot,s)$ alternates in sign at right-scattered points of the interval $[s,\infty)_{\mathbb{T}}$ provided that $f \in \mathcal{R}^{-}([s,\infty)_{\mathbb{T}},\mathbb{R})$. For $h \in \mathbb{R}_0^{+}$, let $z,w \in \mathbb{C}_h$, the *circle plus* $\oplus_h$ and the *circle minus* $\ominus_h$ are defined by $z \oplus_h w := z + w + hzw$ and $z \ominus_h w := (z-w)/(1+hw)$, respectively. Further throughout the paper, we will abbreviate the operations $\oplus_{\mu}$ and $\ominus_{\mu}$ simply by $\oplus$ and $\ominus$, respectively. It is also known that $\mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ is a subgroup of $\mathcal{R}(\mathbb{T},\mathbb{R})$, that is, $0 \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$, $f,g \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ implies $f \oplus_{\mu} g \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$ and $\ominus_{\mu} f \in \mathcal{R}^{+}(\mathbb{T},\mathbb{R})$, where $\ominus_{\mu} f := 0 \ominus_{\mu} f$ on $\mathbb{T}$.

The readers are referred to [32] for further interesting details in the time scale theory.

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References

[1] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, *Discrete Oscillation Theory*, Hindawi Publishing Corporation, New York, NY, USA, 2005.

[2] L. H. Erbe, Q. K. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, vol. 190 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1995.

[3] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, NY, USA, 1991.

[4] G. S. Ladde, V. K. Lakshmikantham, and B. G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, vol. 110 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1987.

[5] A. D. Myškis, *Linear Differential Equations with Retarded Argument*, Izdat. "Nauka", Moscow, Russia, 1972.

[6] S. B. Norkin, *Differential Equations of the Second Order with Retarded Argument. Some Problems of the Theory of Vibrations of Systems with Retardation*, vol. 3 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1972.

[7] C. A. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, vol. 4 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1968.

[8] L. Berezansky and E. Braverman, "Some oscillation problems for a second order linear delay differential equation," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 2, pp. 719–740, 1998.

[9] E. Braverman and B. Karpuz, "Nonoscillation of first-order dynamic equations with several delays," *Advances in Difference Equations*, vol. 2010, Article ID 873459, 22 pages, 2010.

[10] W. Leighton, "On self-adjoint differential equations of second order," *Journal of the London Mathematical Society*, vol. 27, pp. 37–47, 1952.

[11] W. J. Coles, "A simple proof of a well-known oscillation theorem," *Proceedings of the American Mathematical Society*, vol. 19, p. 507, 1968.

[12] A. Wintner, "A criterion of oscillatory stability," *Quarterly of Applied Mathematics*, vol. 7, pp. 115–117, 1949.

[13] A. Kneser, "Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen," *Mathematische Annalen*, vol. 42, no. 3, pp. 409–435, 1893.

[14] E. Hille, "Non-oscillation theorems," *Transactions of the American Mathematical Society*, vol. 64, pp. 234–252, 1948.

[15] J. Deng, "Oscillation criteria for second-order linear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 271, no. 1, pp. 283–287, 2002.

[16] C. Huang, "Oscillation and nonoscillation for second order linear differential equations," *Journal of Mathematical Analysis and Applications*, vol. 210, no. 2, pp. 712–723, 1997.

[17] R. N. Rath, J. G. Dix, B. L. S. Barik, and B. Dihudi, "Necessary conditions for the solutions of second order non-linear neutral delay difference equations to be oscillatory or tend to zero," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 60907, 16 pages, 2007.

[18] S. Z. Chen and L. H. Erbe, "Riccati techniques and discrete oscillations," *Journal of Mathematical Analysis and Applications*, vol. 142, no. 2, pp. 468–487, 1989.

[19] S. S. Cheng, T. C. Yan, and H. J. Li, "Oscillation criteria for second order difference equation," *Funkcialaj Ekvacioj*, vol. 34, no. 2, pp. 223–239, 1991.

[20] P. Řehák, "Oscillation and nonoscillation criteria for second order linear difference equations," *Fasciculi Mathematici*, no. 31, pp. 71–89, 2001.

[21] B. G. Zhang and Y. Zhou, "Oscillation and nonoscillation for second-order linear difference equations," *Computers & Mathematics with Applications*, vol. 39, no. 1-2, pp. 1–7, 2000.

[22] Y. Zhou and B. G. Zhang, "Oscillations of delay difference equations in a critical state," *Computers & Mathematics with Applications*, vol. 39, no. 7-8, pp. 71–80, 2000.
[23] X. H. Tang, J. S. Yu, and D. H. Peng, "Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients," *Computers & Mathematics with Applications*, vol. 39, no. 7-8, pp. 169–181, 2000.
[24] B. G. Zhang and S. S. Cheng, "Oscillation criteria and comparison theorems for delay difference equations," *Fasciculi Mathematici*, no. 25, pp. 13–32, 1995.
[25] J. Baoguo, L. Erbe, and A. Peterson, "A Wong-type oscillation theorem for second order linear dynamic equations on time scales," *Journal of Difference Equations and Applications*, vol. 16, no. 1, pp. 15–36, 2010.
[26] O. Došlý and S. Hilger, "A necessary and sufficient condition for oscillation of the Sturm-Liouville dynamic equation on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 147–158, 2002.
[27] L. Erbe, "Oscillation criteria for second order linear equations on a time scale," *The Canadian Applied Mathematics Quarterly*, vol. 9, no. 4, p. 345–375 (2002), 2001.
[28] L. Erbe, A. Peterson, and P. Řehák, "Comparison theorems for linear dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 1, pp. 418–438, 2002.
[29] A. Zafer, "On oscillation and nonoscillation of second-order dynamic equations," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 136–141, 2009.
[30] L. Erbe, A. Peterson, and C. C. Tisdell, "Basic existence, uniqueness and approximation results for positive solutions to nonlinear dynamic equations on time scales," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 7, pp. 2303–2317, 2008.
[31] B. Karpuz, "Existence and uniqueness of solutions to systems of delay dynamic equations on time scales," *International Journal of Computer Mathematics*, vol. 10, no. M11, pp. 48–58, 2011.
[32] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.
[33] B. Karpuz, "Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 2009, no. 34, pp. 1–14, 2009.
[34] L. Erbe, "Oscillation results for second-order linear equations on a time scale," *Journal of Difference Equations and Applications*, vol. 8, no. 11, pp. 1061–1071, 2002.
[35] J. R. Yan, "Oscillatory properties of second-order differential equations with an "integralwise small" coefficient," *Acta Mathematica Sinica*, vol. 30, no. 2, pp. 206–215, 1987.
[36] M. Bohner and M. Ünal, "Kneser's theorem in q-calculus," *Journal of Physics A*, vol. 38, no. 30, pp. 6729–6739, 2005.
[37] L. Berezansky and E. Braverman, "On oscillation of a second order impulsive linear delay differential equation," *Journal of Mathematical Analysis and Applications*, vol. 233, no. 1, pp. 276–300, 1999.