Research Article

Some E-J Generalized Hausdorff Matrices Not of Type M

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We show that there exists a regular E-J generalized Hausdorff matrix which has no zero elements on the main diagonal and which is not of type M and establish several other related theorems.

1. Introduction

The convergence domain of an infinite matrix $A = (a_{nk})$ $(n, k = 0, 1, \ldots)$ will be denoted by (A) and is defined by $(A) := \{x = \{x_n\} \mid A_n(x) \in c\}$, where c denotes the space of convergence sequences and $A_n(x) := \sum_{k=0}^{\infty} a_{nk} x_k$. If for two matrices A and B, we have the relation $(A) \subset (B)$, we say that B is not weaker than A. The necessary and sufficient conditions of Silverman and Toeplitz for a matrix to be conservative (some authors use the word convergence-preserving instead of conservative) are as $\lim_{n \to \infty} a_{nk} = a_k$ exists for each k, $\lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = t$ exists, follows: $\|A\| := \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$. A conservative matrix A is called multiplicative if each $a_k = 0$ and regular if, in addition, $t = 1$. If A is a conservative matrix, then $\chi(A) = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk}$ is called the characteristic of A. A conservative matrix A is called coregular if $\chi(A) \neq 0$ and conull if $\chi(A) = 0$. Regular matrices are coregular, since $\chi(A) = 1$.

A matrix $A = (a_{nk})$ $(n, k = 0, 1, \ldots)$ is called triangular if $a_{nk} = 0$ for all $k > n$, and it is called a normal if it is, triangular and $a_{nn} \neq 0$ for all n.

Let $A = (a_{nk})$ $(n, k = 0, 1, \ldots)$ denote an infinite matrix. Then A is said to be of type M if the conditions

$$\sum_{n=0}^{\infty} |\alpha_n| < \infty, \qquad \sum_{n=0}^{\infty} \alpha_n a_{nk} = 0 \quad k = 0, 1, 2, \ldots \tag{1.1}$$

always imply $\alpha_n = 0 (n = 0, 1, 2, \ldots)$.

Matrices of type M were first introduced by Mazur [1] and named by Hill [2]. Hill [2] developed several sufficient conditions for a Hausdorff matrix to be of Type M. He showed that there exists a regular Hausdorff matrix which has a zero on the main diagonal, not of type M. He also posed the following question: does there exist a regular Hausdorff matrix which has no zero elements on the main diagonal and which is not of type M? Rhoades [3] answered the above question in the affirmative and established several other related theorems. In this paper, we answer the above question for E-J generalized Hausdorff matrices.

We use the words finite sequence to describe a sequence which is containing only a finite number of nonzero terms. It is clear that a triangular matrix which is not a normal cannot be of type M, since a finite sequence can be found satisfying (1.1). Also, if a matrix is a normal, there can be no finite sequence as a solution of (1.1). All diagonal matrices with nonzero diagonal elements are of type M.

Hausdorff matrices were shown by Hurwitz and Silverman [4] to be the class of triangular matrices that commute with C, the Cesáro matrix of order one. Hausdorff [5] reexamined this class, in the process of solving the moment problem over a finite interval, and developed many of the properties of the matrices that now bear his name.

Several generalizations of Hausdorff matrices have been made. In this paper we will be concerned with the generalized Hausdorff matrices as defined independently by Endl ([6, 7]) and Jakimovski [8]. A generalized Hausdorff matrix $H^{(\alpha)}$ is a lower triangular infinite matrix with entries

$$h_{nk}^{(\alpha)} = \binom{n+\alpha}{n-k} \Delta^{n-k} \mu_k, \quad 0 \le k \le n, \tag{1.2}$$

where α is real number, (μ_n) is a real sequence, and Δ is forward difference operator defined by $\Delta \mu_k = \mu_k - \mu_{k+1}, \Delta^{n+1}\mu_k = \Delta(\Delta^n \mu_k)$. We will consider here only nonnegative α. For $\alpha = 0$ one obtains an ordinary Hausdorff matrix.

From [6] or [8], a generalized Hausdorff matrix (for $\alpha > 0$) is regular if and only if there exists a function $\chi \in BV[0, 1]$ with $\chi(1) - \chi(0+) = 1$ such that

$$\mu_n^{(\alpha)} = \int_0^1 t^{n+\alpha} d\chi(t), \tag{1.3}$$

in which case $(\mu_n^{(\alpha)})$ is called the moment sequence for $H^{(\alpha)}$ and χ is called the moment generating function, or mass function, for $H^{(\alpha)}$.

For ordinary Hausdorff summability (see, e.g., [9]), the necessary and sufficient conditions for regularity are that the function $\chi \in BV[0, 1]$, $\chi(1) - \chi(0) = 1, \chi(0+) = \chi(0) = 0$, and (1.3) is satisfied with $\alpha = 0$.

The purpose of this paper is to show that there exists a regular E-J generalized Hausdorff matrix which has no zero elements on the main diagonal and which is not of type M and establish several other related theorems.

The following is a consequence of [10, Theorem 3.2.1(d)].

Theorem 1.1 (see [10]). *If A is a normal, conservative and coregular, then A is of type M if and only if $\bar{c} = (A)$.*

The following is a consequence of [11, Theorem 1((a) and (c))].

Theorem 1.2 (see [11]). *Let A be conservative. c is closed in (A) if and only if A sums no bounded divergent sequences.*

Theorem 1.3. *Let*

$$\mu_n^{(\alpha)} = \frac{b(n-a)}{-(a+\alpha)(n+b+\alpha)}, a > 0, \quad b > 0, \ \alpha \geq 0 \ (n = 0, 1, 2 \ldots). \tag{1.4}$$

Then the corresponding regular E-J generalized Hausdorff matrix is not of type M.

Proof. If a is a positive integer, then $H_\mu^{(\alpha)}$ is not of type M as remarked above, since it has a zero on its diagonal.

Assume a is not a positive integer. From [12], the convergence domains for E-J generalized Hausdorff matrices with moment generating sequence as defined above are $(H_\mu^{(\alpha)}) = c \oplus x^{(\alpha)}$, where

$$x_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n-a+1)}. \tag{1.5}$$

Since $H_\mu^{(\alpha)}$ sums no bounded divergent sequences, from Theorem 1.2, c is closed in $(H_\mu^{(\alpha)})$. From Theorem 1.1, since c is not dense in the convergence domain of each $H_\mu^{(\alpha)}$, none of them is of type M. $\square$

Let A be a conservative matrix. If c is dense in (A) then A is called perfect. For certain classes of matrices, perfectness and type M are closely related. Note that, from Theorem 1.1, type M and perfectness are equivalent for normal, conservative, and coregular matrices.

If one examines the sequences of Theorem 1.3 for a an integer, then one notes that the corresponding matrices are not normal. It remains to determine if each such matrix is perfect.

Theorem 1.4. *Let*

$$\mu_n^{(\alpha)} = \frac{b(n-r)}{-(r+\alpha)(n+b+\alpha)}, \quad r \text{ is a positive integer, } b > 0, \ \alpha \geq 0 \ (n = 0, 1, 2 \ldots). \tag{1.6}$$

Then the corresponding regular E-J generalized Hausdorff matrix is not perfect.

Proof. In [12], it is proved that $H_\mu^{(\alpha)}$ is a regular E-J generalized Hausdorff matrix with $(H_\mu^{(\alpha)}) = c \oplus x^{(\alpha)}$, where

$$x_n^{(\alpha)} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n-r+1)}. \tag{1.7}$$

As mentioned in the proof of Theorem 1.3, c is not dense in the convergence domain of each $H_\mu^{(\alpha)}$. Hence each $H_\mu^{(\alpha)}$ is not perfect. $\square$

Theorem 1.5 (see [2]). *The product $AB \equiv C$ of two triangular perfect methods A and B is also a triangular perfect method.*

Theorem 1.6 (see [2]). *If the product $AB \equiv C$ of two triangular convergence-preserving methods A and B is of type M, then A must be of type M.*

Theorem 1.7 (see [2]). *If A is normal and B is triangular, then B is not weaker than A if and only if BA^{-1} is convergence preserving.*

Theorem 1.8. *If $H_\mu^{(\alpha)}$ is normal, conservative, and not of type M, and if $H_\lambda^{(\alpha)}$ is not weaker than $H_\mu^{(\alpha)}$, then $H_\lambda^{(\alpha)}$ is not of type M.*

Proof. By Theorem 1.7, $H_\varphi^{(\alpha)} = H_\lambda^{(\alpha)}(H_\mu^{(\alpha)})^{-1}$ is conservative. From the definition of E-J generalized Hausdorff matrix, it is easily shown that multiplication of an E-J generalized Hausdorff matrix is commutative and the result is again an E-J generalized Hausdorff matrix. Also the inverse of a normal E-J generalized Hausdorff matrix is also a normal E-J generalized Hausdorff matrix. Thus $H_\varphi^{(\alpha)} = (H_\mu^{(\alpha)})^{-1}H_\lambda^{(\alpha)}$ and $H_\lambda^{(\alpha)} = H_\mu^{(\alpha)}H_\varphi^{(\alpha)}$, and the result follows at once from Theorem 1.6. □

Theorem 1.9. *If $H_\lambda^{(\alpha)}$ is normal and conservative and if $H_\mu^{(\alpha)}$ is of type M and not weaker than $H_\lambda^{(\alpha)}$, then $H_\lambda^{(\alpha)}$ is of type M.*

Proof. As in the previous theorem we have $H_\varphi^{(\alpha)} = H_\mu^{(\alpha)}(H_\lambda^{(\alpha)})^{-1} = (H_\lambda^{(\alpha)})^{-1}H_\mu^{(\alpha)}$, where $H_\varphi^{(\alpha)}$ is conservative. Then $H_\mu^{(\alpha)} = H_\lambda^{(\alpha)}H_\varphi^{(\alpha)}$, and the conclusion follows again from Theorem 1.6. □

From Theorem 1.5 and the multiplication facts of E-J generalized Hausdorff matrix, we obtain the following theorem.

Theorem 1.10. *The product of a finite number of perfect E-J generalized Hausdorff methods is likewise a perfect E-J generalized Hausdorff method.*

Acknowledgment

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