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On factorization of generalized Macdonald polynomials

Ya. Kononov[1,2], **A. Morozov**[3,4,5,a]

[1] Landau Institute for Theoretical Physics, Chernogolovka, Russia
[2] Math Department, HSE, Moscow, Russia
[3] ITEP, Moscow 117218, Russia
[4] Institute for Information Transmission Problems, Moscow 127994, Russia
[5] National Research Nuclear University MEPhI, Moscow 115409, Russia

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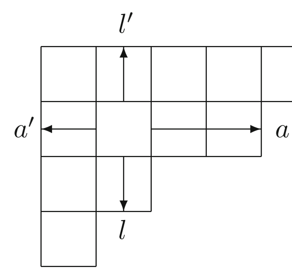
Abstract A remarkable feature of Schur functions—the common eigenfunctions of cut-and-join operators from W_∞—is that they factorize at the peculiar two-parametric topological locus in the space of time variables, which is known as the hook formula for quantum dimensions of representations of $U_q(SL_N)$ and which plays a big role in various applications. This factorization survives at the level of Macdonald polynomials. We look for its further generalization to generalized Macdonald polynomials (GMPs), associated in the same way with the toroidal Ding–Iohara–Miki algebras, which play the central role in modern studies in Seiberg-Witten–Nekrasov theory. In the simplest case of the first-coproduct eigenfunctions, where GMP depend on just two sets of time variables, we discover a weak factorization—on a one- (rather than four-) parametric slice of the topological locus, which is already a very non-trivial property, calling for proof and better understanding.

Generalized Macdonald polynomials (GMPs) [1–3] play a constantly increasing role in modern studies of the 6*d* version [4–30] of AGT relations [31–40] and spectral dualities [41–58]. At the same time they are relatively new special functions, far from being thoroughly understood and clearly described. They are deformations of the generalized Jack polynomials introduced in [59,60]. Even the simplest questions about them are yet unanswered. In this letter we address one of them—what happens to the hook formulas for classical, quantum, and Macdonald dimensions at the level of GMPs? We find that they survive, but only partly—on a one-dimensional line in the space of time variables. Lifting to the entire two-dimensional topological locus remains to be found.

We begin by recalling that the Schur functions $\chi_R\{p\}$, depend on representation (Young diagram) R and on infinitely many time variables p_k (actually, a particular χ_R depends only on p_k with $k \le |R| = \#$ boxes in R). They get nicely factorized on a peculiar two-dimensional topological locus,

$$p_k = p_k^* \equiv \frac{1-A^k}{1-t^k}, \tag{1}$$

$$\chi_R^*(A,t) = \prod_{\Box\in R} t^{l'(\Box)} \cdot \prod_{\Box\in R} \frac{1-A\cdot t^{a'(\Box)-l'(\Box)}}{1-t^{a(\Box)+l(\Box)+1}}. \tag{2}$$



Coarm a' and coleg l' are the ordinary coordinates of the box in the diagram. To keep the notation consistent throughout the text, in (1) we call the relevant parameter t, not q, from the very beginning.

It is often convenient to ignore the simple overall coefficient and substitute the product formulas like (2) by a polynomial expression for a plethystic logarithm

$$\left(\prod_{\Box\in R} t^{l'(\Box)}\right)^{-1} \cdot \chi_R^* = \mathcal{S}^\bullet\left(\sum_{\Box\in R} t^{a(\Box)+l(\Box)+1} - A\cdot t^{a'(\Box)-l'(\Box)}\right) \tag{3}$$

where we use the definition

$$\mathcal{S}^\bullet(f) = \exp\left(\sum_k \frac{f(t_0^k,\ldots,t_n^k)}{k}\right) \tag{4}$$

[a] e-mail: morozov@itep.ru

for $A = t_0$ and $t = t_1$. The plethystic exponential $\mathcal{S}^\bullet(f)$ is the character of the symmetric algebra of f as a representation of $\mathbb{C}^*_{t_0} \times \cdots \times \mathbb{C}^*_{t_n}$. By a factorization of $\mathcal{S}^\bullet(f)$ we actually mean that its plethystic logarithm f is a polynomial or, more generally, a rational function, with integer coefficients. The celebrated Gopakumar–Ooguri–Vafa hypothesis [61–68] is that such are not only the quantum dimensions, but also the HOMFLY polynomials of arbitrary knots (in this context the quantum dimensions are associated with the unknots).

Since there is a double product/sum in (2) and (3), it is natural to consider their refinement, where a second t-parameter is introduced, usually called q (for knots this means going from HOMFLY to super- and hyper-polynomials [69–79]). The refined version of Schur is the Macdonald polynomial [80] $M_R^{q,t}\{p\}$, which—unlike Schur functions—explicitly depends on q and t. If t is the same as in (1), then (2) lifts to

$$M_R^*(A,t) = M_R^{q,t}\{p^*\} = \prod_{\square\in R} t^{l'(\square)} \cdot \prod_{\square\in R} \frac{1-Aq^{a'(\square)}t^{-l'(\square)}}{1-q^{a(\square)}t^{l(\square)+1}}, \tag{5}$$

$$\left(\prod_{\square\in R} t^{l'(\square)}\right)^{-1} \cdot M_R^* = \mathcal{S}^\bullet\left(\sum_{\square\in R} q^{a(\square)}t^{l(\square)+1} - A\cdot q^{a'(\square)}t^{-l'(\square)}\right) = \mathcal{S}^\bullet\left(\mu_R^{**} - A\cdot \nu_R^{**}\right). \tag{6}$$

We introduced special notations for the character of the Young diagram,

$$\nu_Y^{**}(q,t) = \sum_{\square\in Y} q^{-a'(\square)}t^{l'(\square)} \tag{7}$$

and

$$\mu_R^{**}(q,t) = \sum_{\square\in R} q^{a(\square)}t^{l(\square)+1}, \tag{8}$$

which fully describe the plethystic logarithm of Macdonald dimension M_R^*.

The quantities χ_R^* and M_R^* are deformations of the dimensions of representation R of SL_N-algebras, which factorize due to Weyl formulas, and are therefore called quantum and Macdonald dimensions. In the former case they can be considered as graded dimension of representation R of $U_t(SL_N)$, while in the latter case there is still no commonly accepted group-theory interpretation. In the absence of such an interpretation Macdonald polynomials are defined not as characters, but by other less straightforward methods—of which generalization to the GMP is currently only the definition as eigenvectors of Calogero-like Hamiltonians. Factorization in these terms is not straightforward and is in fact a separation-of-variables phenomenon, more or less equivalent to integrability of the associated theory. Its actual derivation on these lines is usually quite tedious. However, the very fact that factorization occurs can be observed experimentally, far before the proofs, derivations, and real understanding. Our goal in this letter is to search for such evidence in the case of a GMP.

The GMP depends on a set of Young diagrams and a set of time variables—in the following we consider the simplest non-trivial case, when there are two: two Young diagrams and two sets of times. Then just one more deformation parameter adds to t and q; we call it Q. Thus the GMP in question will be denoted by $\mathcal{M}_{A,B}^{q,t,Q}\{p,\bar{p}\}$ and, in full analogy with the Schur functions [81], they are eigenfunctions of the quantum cut-and-join operator, which is the Hamiltonian $\Delta_{\text{DIM}}(E(z))$ of the DIM algebra (see [4–30] for details of the definition):

$$H_1 = \frac{1}{t-1}\cdot \text{Res}\left[\exp\left(\sum_{n\geq 1}\frac{1-t^{-n}}{n}z^n p_n\right) \times \exp\left(\sum_{n\geq 1}(q^n-1)z^{-n}\frac{\partial}{\partial p_n}\right)\right. \tag{9}$$

$$\left. + Q^{-1}\left(\exp\left(\sum_{n\geq 1}\frac{1-t^{-n}}{n}z^n((1-t^nq^{-n})p_n+\bar{p}_n)\right)\right) \times \exp\left(\sum_{n\geq 1}(q^n-1)z^{-n}\frac{\partial}{\partial p_n}-1\right)\right]. \tag{10}$$

Somewhat remarkably, just a single Hamiltonian is needed to describe the whole set of GMPs—no higher Hamiltonians are needed, because all its eigenvalues are non-degenerate. As to the label 1 in H_1, it refers to the first coproduct in DIM, higher coproducts provide Hamiltonians for the GMP, depending on more time variables.

Explicitly, the simplest GMPs in a "natural" normalization —which will appear consistent with the factorization property —are:

$$M([\,],[\,]) = 1,$$

$$M([\,],[1]) = \bar{p}_1 - \frac{p_1(q-t)}{q(Q-1)},$$

$$M([1],[\,]) = p_1,$$

$$M([\,],[2]) = \frac{p_1(q+1)(t-1)\bar{p}_1(q-t)}{(q-Q)(qt-1)} + \frac{(q+1)(t-1)\bar{p}_1^2}{2(qt-1)}$$

$$+\frac{(q-1)(t+1)\bar{p}_2}{2(qt-1)}$$
$$-\frac{p_1^2(q+1)(t-1)(q-t)\left(q^2+qQt-qt-Qt\right)}{2q^2(Q-1)(q-Q)(qt-1)}$$
$$-\frac{p_2(q-1)(t+1)(q-t)\left(q^2-qQt+qt-Qt\right)}{2q^2(Q-1)(q-Q)(qt-1)},$$
$$M([\,],[1,1]) = -\frac{p_1\bar{p}_1(q-t)}{q(Qt-1)}$$
$$+\frac{\bar{p}_1^2}{2}-\frac{\bar{p}_2}{2}+\frac{p_1^2(q-t)\left(q-Qt^2+Qt-t\right)}{2q^2(Q-1)(Qt-1)}$$
$$-\frac{p_2(q-t)\left(q-Qt^2-Qt+t\right)}{2q^2(Q-1)(Qt-1)},$$
$$M([1],[1]) = p_1\bar{p}_1$$
$$-\frac{p_1^2(q-t)(qQt+qQ-Qt+Q-2t)}{2q(qQ-1)(Q-t)}$$
$$+\frac{p_2(q-1)Q(t+1)(q-t)}{2q(qQ-1)(Q-t)},$$

$$M([2],[\,]) = \frac{p_1^2(q+1)(t-1)}{2(qt-1)}$$
$$+\frac{p_2(q-1)(t+1)}{2(qt-1)}$$
$$M([1,1],[\,]) = \frac{p_1^2}{2}-\frac{p_2}{2}.$$

When all $p_k = 0$, the GMPs become just ordinary Macdonald polynomials of $\bar{p}_k$:

$$M_{Y_1Y_2}\big|_{p_k=0} = \delta_{Y_1,\emptyset}\cdot M_{Y_2}(\bar{p}_k). \tag{11}$$

However, for $\bar{p}_k = 0$, they remain quite complicated functions of p_k:

$$M_{[1],[1]}\big|_{\bar{p}_i=0} = \frac{p_2Q\,(-1+q)\,(-t+q)\,(t+1)}{2\,(Q-t)\,(Qq-1)\,q} - \frac{{p_1}^2\,(-t+q)\,(Qqt+Qq-Qt+Q-2\,t)}{2\,(Q-t)\,(Qq-1)\,q}$$
$$= \frac{q-t}{q(1-Qq)}M_{[2]} + \frac{(1-q)(q-t)(1+t)}{(qt-1)q(Q-t)}M_{[1,1]}, \tag{12}$$

which do not look much simpler than the general expressions for $\bar{p}_k \neq 0$. Already from these examples it is clear that the GMPs are non-trivial functions of Q. Moreover, this complexity may seem to persist in restriction to the topological locus.

However, this is not quite the case if one looks at the right quantities—our claim is that at $A = 0$ and $\bar{p} = 0$ plethystic logarithm is just linear in Q with a further factorized coefficient:

$$\boxed{\left(\prod_{\square\in Y_1}(-q^{a'(\square)}t)\prod_{\square\in Y_2}(-q^{a'(\square)}t)\right)^{-1}\cdot M^{**}_{Y_1,Y_2} = \mathcal{S}^{\bullet}\left(\mu^{**}_{Y_1}+\mu^{**}_{Y_2}-tq^{-1}\nu^{**}_{Y_2}+Q\cdot\Upsilon^{**}_{Y_1}\cdot\nu^{**}_{Y_2}\right)} \tag{13}$$

where $**$ denotes the special locus

$$p_i^{**} = \frac{1}{1-t^{-i}},\quad \bar{p}_i = 0, \tag{14}$$

and Υ is made from the dual character $\bar{\nu}^{**}_{q\to q^{-1},t\to t^{-1}}$:

$$\boxed{\Upsilon^{**}_{Y_1} = 1-(1-q)(1-t^{-1})\cdot\bar{\nu}^{**}_{Y_1}}. \tag{15}$$

We checked the factorization conjecture (13) up to level five, i.e. for $|Y_1| + |Y_2| \leq 5$. Here are particular examples:

Y_1	Y_2	Plethystic logarithm of $M^{**}_{Y_1,Y_2}$	$M^{**}_{Y_1,Y_2}$
[ ]	[1]	$-\frac{t}{q}+Q+t$	$-\frac{t(q-t)}{q(Q-1)(t-1)}$
[ ]	[2]	$-\frac{t}{q^2}+\frac{Q}{q}+qt-\frac{t}{q}+Q+t$	$-\frac{t^2(q-t)\left(q^2-t\right)}{q(Q-1)(t-1)(q-Q)(qt-1)}$
[ ]	[1,1]	$-\frac{t^2}{q}-\frac{t}{q}+Qt+Q+t^2+t$	$\frac{t^2(q-t)\left(q-t^2\right)}{q^2(Q-1)(t-1)^2(t+1)(Qt-1)}$
[1]	[1]	$-\frac{qQ}{t}+qQ-\frac{t}{q}+\frac{Q}{t}+2t$	$-\frac{t^2(q-t)(qQ-t)}{q(t-1)^2(qQ-1)(Q-t)}$
[ ]	[3]	$-\frac{t}{q^3}+\frac{Q}{q^2}+q^2t-\frac{t}{q^2}+\frac{Q}{q}+qt-\frac{t}{q}+Q+t$	$-\frac{t^3(q-t)\left(q^2-t\right)\left(q^3-t\right)}{(Q-1)(t-1)(q-Q)\left(q^2-Q\right)(qt-1)\left(q^2t-1\right)}$
[ ]	[2,1]	$-\frac{t}{q^2}+\frac{Q}{q}+qt^2-\frac{t^2}{q}-\frac{t}{q}+Qt+Q+2t$	$\frac{t^3(q-t)\left(q^2-t\right)\left(q-t^2\right)}{q^2(Q-1)(t-1)^2(q-Q)\left(qt^2-1\right)(Qt-1)}$
[ ]	[1,1,1]	$-\frac{t^3}{q}-\frac{t^2}{q}-\frac{t}{q}+Qt^2+Qt+Q+t^3+t^2+t$	$-\frac{t^3(q-t)\left(q-t^2\right)\left(q-t^3\right)}{q^3(Q-1)(t-1)^3(t+1)\left(t^2+t+1\right)(Qt-1)\left(Qt^2-1\right)}$
[1]	[2]	$-\frac{t}{q^2}-\frac{qQ}{t}+\frac{Q}{qt}+qQ+qt-\frac{t}{q}+Q+2t$	$-\frac{t^3(q-t)\left(q^2-t\right)(qQ-t)}{q(Q-1)(t-1)^2(qQ-1)(qt-1)(qt-Q)}$
[1]	[1,1]	$qQt-\frac{qQ}{t}-\frac{t^2}{q}-\frac{t}{q}+\frac{Q}{t}+Q+t^2+2t$	$\frac{t^3(q-t)\left(q-t^2\right)(qQ-t)}{q^2(Q-1)(t-1)^3(t+1)(Q-t)(qQt-1)}$
[2]	[1]	$-\frac{q^2Q}{t}+q^2Q+qt-\frac{t}{q}+\frac{Q}{t}+2t$	$-\frac{t^3(q-t)\left(q^2Q-t\right)}{(t-1)^2\left(q^2Q-1\right)(qt-1)(Q-t)}$
[1,1]	[1]	$-\frac{qQ}{t^2}+qQ-\frac{t}{q}+\frac{Q}{t^2}+t^2+2t$	$-\frac{t^3(q-t)\left(qQ-t^2\right)}{q(t-1)^3(t+1)(qQ-1)\left(Q-t^2\right)}$
[3]	[1]	$-\frac{q^3Q}{t}+q^3Q+q^2t+qt-\frac{t}{q}+\frac{Q}{t}+2t$	$-\frac{q^2t^4(q-t)\left(q^3Q-t\right)}{(t-1)^2\left(q^3Q-1\right)(qt-1)\left(q^2t-1\right)(Q-t)}$
[2,1]	[1]	$-\frac{q^2Q}{t}+q^2Q-\frac{qQ}{t^2}+\frac{qQ}{t}+qt^2-\frac{t}{q}+\frac{Q}{t^2}+3t$	$-\frac{t^4(q-t)\left(q^2Q-t\right)\left(qQ-t^2\right)}{(t-1)^3\left(q^2Q-1\right)\left(qt^2-1\right)\left(Q-t^2\right)(qQ-t)}$
[1,1,1]	[1]	$-\frac{qQ}{t^3}+qQ-\frac{t}{q}+\frac{Q}{t^3}+t^3+t^2+2t$	$-\frac{t^4(q-t)\left(qQ-t^3\right)}{q(t-1)^4(t+1)\left(t^2+t+1\right)(qQ-1)\left(Q-t^3\right)}$
[2]	[2]	$-\frac{q^2Q}{t}+q^2Q-\frac{t}{q^2}-\frac{qQ}{t}+\frac{Q}{qt}+qQ+2qt-\frac{t}{q}+\frac{Q}{t}+2t$	$-\frac{t^4(q-t)\left(q^2-t\right)(qQ-t)\left(q^2Q-t\right)}{(t-1)^2(qQ-1)\left(q^2Q-1\right)(qt-1)^2(Q-t)(qt-Q)}$
[2]	[1,1]	$q^2Qt-\frac{q^2Q}{t}-\frac{t^2}{q}+qt-\frac{t}{q}+\frac{Q}{t}+Q+t^2+2t$	$\frac{t^4(q-t)\left(q-t^2\right)\left(q^2Q-t\right)}{q(Q-1)(t-1)^3(t+1)(qt-1)(Q-t)\left(q^2Qt-1\right)}$
[1,1]	[2]	$-\frac{t}{q^2}-\frac{qQ}{t^2}+\frac{Q}{qt^2}+qQ+qt-\frac{t}{q}+Q+t^2+2t$	$-\frac{t^4(q-t)\left(q^2-t\right)\left(qQ-t^2\right)}{q(Q-1)(t-1)^3(t+1)(qQ-1)(qt-1)\left(qt^2-Q\right)}$
[1,1]	[1,1]	$-\frac{qQ}{t^2}+qQt-\frac{qQ}{t}+qQ-\frac{t^2}{q}-\frac{t}{q}+\frac{Q}{t^2}+\frac{Q}{t}+2t^2+2t$	$\frac{t^4(q-t)\left(q-t^2\right)(qQ-t)\left(qQ-t^2\right)}{q^2(t-1)^4(t+1)^2(qQ-1)(Q-t)\left(Q-t^2\right)(qQt-1)}$
[1]	[3]	$-\frac{t}{q^3}+\frac{Q}{q^2t}+q^2t-\frac{t}{q^2}-\frac{qQ}{t}+qQ+\frac{Q}{q}+qt-\frac{t}{q}+Q+2t$	$-\frac{t^4(q-t)\left(q^2-t\right)\left(q^3-t\right)(qQ-t)}{(Q-1)(t-1)^2(q-Q)(qQ-1)(qt-1)\left(q^2t-1\right)\left(q^2t-Q\right)}$
[1]	[2,1]	$-\frac{t}{q^2}+qQt-\frac{qQ}{t}+\frac{Q}{qt}+qt^2-\frac{t^2}{q}-\frac{t}{q}+2Q+3t$	$\frac{t^4(q-t)\left(q^2-t\right)\left(q-t^2\right)(qQ-t)}{q^2(Q-1)^2(t-1)^3\left(qt^2-1\right)(qt-Q)(qQt-1)}$
[1]	[1,1,1]	$qQt^2-\frac{qQ}{t}-\frac{t^3}{q}-\frac{t^2}{q}-\frac{t}{q}+Qt+\frac{Q}{t}+Q+t^3+t^2+2t$	$-\frac{t^4(q-t)\left(q-t^2\right)\left(q-t^3\right)(qQ-t)}{q^3(Q-1)(t-1)^4(t+1)\left(t^2+t+1\right)(Q-t)(Qt-1)\left(qQt^2-1\right)}$
[2,1,1]	[1]	$-\frac{q^2Q}{t}+q^2Q-\frac{qQ}{t^3}+\frac{qQ}{t}+qt^3-\frac{t}{q}+\frac{Q}{t^3}+t^2+3t$	$-\frac{t^5(q-t)\left(q^2Q-t\right)\left(qQ-t^3\right)}{(t-1)^4(t+1)\left(q^2Q-1\right)\left(qt^3-1\right)\left(Q-t^3\right)(qQ-t)}$
[2,2]	[1]	$-\frac{q^2Q}{t^2}+q^2Q+qt^2+qt-\frac{t}{q}+\frac{Q}{t^2}+t^2+2t$	$-\frac{qt^5(q-t)\left(q^2Q-t^2\right)}{(t-1)^3(t+1)\left(q^2Q-1\right)(qt-1)\left(qt^2-1\right)\left(Q-t^2\right)}$

$M_{Y_1,[\,]} = M_{Y_1}$ are just the ordinary Macdonald polynomials, factorized according to (5), and they are omitted from the table.

Despite (14) being a relatively small slice of what one could expect from a refinement of the topological locus, the property (13) looks quite spectacular and mysterious. Its understanding can provide new insights about both the GMP and the integrality conjectures. The GMPs are clearly simpler and closer to conventional group theory than knot (super)polynomials—and (13) is naturally more structured than one can expect for generic knots, however, its continuation to $A \neq 0$ may already be of more general type.

Factorization does not survive the deformation to non-vanishing $\bar{p}^*_i = \frac{1-\bar{A}^i}{1-\bar{t}^{-i}}$ for any $\bar{t}$, nor the A-deformation to $p_i = \frac{1-A^i}{1-t^{-i}}$, $\bar{p}_i = 0$:

$$
\begin{aligned}
&M^*([1],[2]) \\
&\quad = (A-1)t^3(q-t)\Big(A^2q^3Q^2t^2 - A^2q^3Qt^2 + A^2q^3Qt \\
&\quad -A^2q^2Q^2t^2 + A^2q^2Q^2t - A^2q^2Qt^2 + A^2q^2t^2 - A^2q^2t \\
&\quad -A^2qQ^2t^2 + A^2qQt^2 - 2A^2qQt + A^2qQ + A^2Q^2t^2 \\
&\quad -A^2Q^2t + A^2Qt^2 - Aq^4Qt - Aq^3Q^2t + Aq^3Qt^2 - Aq^3Q \\
&\quad +Aq^3t + Aq^2Qt^2 + 2Aq^2Qt - Aq^2Q - Aq^2t^2 + Aq^2t \\
&\quad +AqQ^2t - AqQt^2 + AqQt - Aqt^2 - AQt^2 + q^4Q - q^3t \\
&\quad -q^2Qt + qt^2\Big) / (q^2(Q-1)(t-1)^2(qQ-1)(qt-1)(qt-Q)),
\end{aligned}
\tag{16}
$$

while at $A = 0$ we return to a nicely factorized

$$
M^{**}([1],[2]) = -\frac{t^3(q-t)\left(q^2-t\right)(qQ-t)}{q(Q-1)(t-1)^2(qQ-1)(qt-1)(qt-Q)}.
\tag{17}
$$

As an A-series, the plethystic logarithm of the ratio $M^*_{[1],[1]}/M^{**}_{[1],[1]}$ is

$$
-\frac{qQt + qQ - Qt + Q - 2t}{qQ - t}A + \frac{(q-1)Q(t-1)t(qQ-1)(Q-t)}{(qQ-t)^2(qQ+t)}A^2
$$

$$+\frac{(q-1)Q(t-1)t(qQ-1)(Q-t)\left(q^2Q^2t+qQ^2-Qt^2-t^2\right)}{(qQ-t)^3\left(q^2Q^2+qQt+t^2\right)}A^3+\cdots \tag{18}$$

The first term at the r.h.s. resembles the Q-linear term in (13). An additional surprise is that the second term also factorizes, and really bad things happen only in the order A^3.

Along with factorization, Schur functions satisfy the Cauchy identity:

$$\sum_R \chi_R\{p\}\chi_R\{\bar p\} = \exp\left(\sum_{k=1}^\infty \frac{p_n\bar p_n}{n}\right). \tag{19}$$

Already in the simplest case of Schur polynomials (where only single-line Young diagrams $R=[n]$, i.e. purely symmetric representations, contribute), combination of this identity and factorization provides a remarkable product formula, identifying the t-exponent and the Pochhammer symbol:

$$\sum_{n=0}\frac{z^n}{[n]!}=\sum_{n\ge 0}\frac{z^n}{\prod_{1\le i\le n}(1-t^i)}=\prod_{n\ge 0}\frac{1}{1-zt^n}. \tag{20}$$

This can also be considered as a formula for the Euler characteristic of the structure sheaf of Hilb $(\mathbb{C},n)$. In Miwa coordinates $p_k=\sum_i x_i^k$, the Cauchy identity (19) can be considered as following from the decomposition $\mathcal{S}^\bullet(V\otimes W)=\oplus_R\chi_R V\otimes\chi_R W$ as GL $(V)\times$ GL (W)-modules:

$$\sum_R \chi_R[x]\chi_R[y]=\prod_{i,j}\frac{1}{1-x_iy_j}. \tag{21}$$

If we restrict both p_i and $\bar p_i$ to the topological locus,

$$p_n^*=\frac{1-A^n}{1-t^{-n}},\qquad \bar p_n^*=\frac{1-B^n}{1-q^{-n}}, \tag{22}$$

then

$$\sum_R z^{|R|}\prod_{\square\in R}\frac{t^{l'(\square)}-At^{a'(\square)}}{1-t^{a(\square)+l(\square)+1}}\frac{q^{l'(\square)}-Aq^{a'(\square)}}{1-q^{a(\square)+l(\square)+1}}$$
$$=\exp\left(\sum_{n=1}^\infty\frac{z^n}{n}\frac{1-A^n}{1-t^n}\frac{1-B^n}{1-q^n}\right). \tag{23}$$

For Macdonald polynomials it merits to take the proper form of the Cauchy identity:

$$\sum_n z^n\sum_{|R|=n}M_R^{q,t}M_{\bar R}^{t^{-1},q^{-1}}=\exp\left(\sum_{n\ge 1}-\frac{(-z)^n}{n}p_n\bar p_n\right). \tag{24}$$

It is related to a more complicated but better-known version

$$\sum_n z^n\sum_{|R|=n}\left(\prod_{\square\in R}\frac{1-q^{a(\square)+1}t^{l(\square)}}{1-q^{a(\square)}t^{l(\square)+1}}\right)^{-1}M_R(p)M_R(\bar p)$$
$$=\exp\left(\sum_k\frac{1-t^k}{1-q^k}\frac{p_k\bar p_k}{k}\right) \tag{25}$$

through the transposition identity [1–3,82]

$$M_R^{q,t}\left(-\frac{1-q^i}{1-t^i}p_i\right)$$
$$=\prod_{\square\in R}\left(-\frac{1-q^{a(\square)+1}t^{l(\square)}}{1-q^{a(\square)}t^{l(\square)+1}}\right)M_{R'}^{t,q}(p_i). \tag{26}$$

It actually simplifies greatly at the topological locus:

$$\sum_R z^{|R|}\prod_{\square\in R}\left(t^{2l'(\square)}\frac{(1-Aq^{a'(\square)}t^{-l'(\square)})(1-Bq^{a'(\square)}t^{-l'(\square)})}{(1-q^{a(\square)}t^{l(\square)+1})(1-q^{a(\square)+1}t^{l(\square)})}\right)$$
$$=\exp\left(\sum_{k\ge 1}\frac{z^n}{n}\frac{(1-A^n)(1-B^n)}{(1-t^n)(1-q^n)}\right). \tag{27}$$

This formula can be seen as an equivariant Euler characteristic of a certain tautological bundle over the Hilbert scheme. References [83–86] provide factorization formulas for the equivariant Euler characteristic of the sheaf of differential forms:

$$\sum_R z^{|R|}\prod_{\square\in R}\frac{(1-Mt_1^{-a(\square))}t_2^{l(\square)+1})(1-Mt_1^{a(\square)+1}t_2^{-l(\square)})}{(1-t_1^{-a(\square)}t_2^{l(\square)+1})(1-t_1^{a(\square)+1}t_2^{-l(\square)})}$$
$$=\mathcal{S}^\bullet\left(\frac{z}{1-Mz}\frac{(1-Mt_1)(1-Mt_2)}{(1-t_1)(1-t_2)}\right). \tag{28}$$

In the case $A=B=0$ and $M=0$ both (27) and (28) represent the equivariant Euler characteristic of the structure sheaf, and the formulas become identical.

For the GMP the Cauchy identity is [1–3]

$$\sum_{Y_1,Y_2}z^{|Y_1|+|Y_2|}M_{Y_1,Y_2}^{q,t}M_{Y_2^t,Y_1^t}^{t^{-1},q^{-1}}$$
$$=\exp\sum_{n\ge 1}\left[\frac{(-z)^n}{n}\left(\left(1-\frac{t^n}{q^n}\right)p_nq_n+\bar p_nq_n+p_n\bar q_n\right)\right], \tag{29}$$

and our factorization conjecture (13) implies the following analog of (20):

$$\boxed{\sum_{Y_1,Y_2}z^{|Y_1|+|Y_2|}M_{Y_1,Y_2}^{q,t\,**}M_{Y_2^t,Y_1^t}^{t^{-1},q^{-1}\,**}=\exp\left[\sum_{n\ge 1}\frac{(-z)^n}{n}\left(\frac{1-t^n/q^n}{(1-t^{-n})(1-q^n)}\right)\right]} \tag{30}$$

where

$$M^{q,t\,**}_{Y_1,Y_2} = \prod_{Y_1} \frac{-q^{a'} t}{1 - q^a t^{l+1}} \prod_{Y_2} \frac{(-q^{a'} t)(1 - q^{-a'-1} t^{l'+1})}{(1 - q^a t^{l+1})(1 - Q q^{-a'} t^{l'})} \prod_{Y_1,Y_2} \frac{(1 - q Q q^{a'_1 - a'_2} t^{l'_2 - l'_1})(1 - t^{-1} Q q^{a'_1 - a'_2} t^{l'_2 - l'_1})}{(1 - Q q^{a'_1 - a'_2} t^{l'_2 - l'_1})(1 - q t^{-1} Q q^{a'_1 - a'_2} t^{l'_2 - l'_1})} \tag{31}$$

and, actually, $M^{t^{-1},q^{-1}}_{Y_1^t,Y_2^t}(p_k, \bar{p}_k) = (-1)^{|Y_1|+|Y_2|} \Big(\prod_{\square\in Y_1} \frac{1-q^{a(\square)+1}t^{l(\square)}}{1-q^{a(\square)}t^{l(\square)+1}} \prod_{\square\in Y_2} \frac{1-q^{a(\square)+1}t^{l(\square)}}{1-q^{a(\square)}t^{l(\square)+1}}\Big)^{-1} \cdot M^{q,t}_{Y_1,Y_2}\Big(-\frac{1-q^k}{1-t^k} p_k, -\frac{1-q^k}{1-t^k} \bar{p}_k\Big)$.

Equation (30) is a new identity, which is a generalization of (20) with two additional parameters. This formula expresses the Euler characteristic of a certain tautological bundle on the moduli space of framed sheaves on $\mathbb{P}^2$. The left hand side is the sum of localization contributions of the fixed points (which are parameterized by pairs of Young diagrams) due to the K-theoretic Lefschetz fixed point formula.

To conclude, we investigated the possibility that the generalized Macdonald polynomials (GMPs) satisfy some kind of hook factorization formula on some kind of a topological locus—and we discovered that this is indeed the case. The main observation (conjecture) is a rather beautiful expression, (13), where Q enters only linearly—like A in (3), and the coefficient in front of it is further decomposed into Y_1- and Y_2-dependent factors. This result is twice surprising: it is remarkable that factorization at all persists for GMP, but, if it does, one wonders: why only on the restricted one-parametric subspace (14) and not on the naive four-parametric extension of (1),

$$p_i = \frac{1 - A^i}{1 - t^{-i}}, \quad \bar{p}_i = \frac{1 - \bar{A}^i}{1 - \bar{t}^{-i}} \tag{32}$$

at least with some $\bar{A}$ and $\bar{t}$? Hopefully, this small discovery will cause interest in the problem and this will help to clarify the structure of Weyl formulas for DIM and double Hekke algebras, which stand behind the generalized Macdonald polynomials and their properties. Immediate straightforward questions concern extension to the GMP, depending on many time variables (eigenfunctions of the higher coproducts of DIM) and extension to more general representations of DIM, labeled by 3d (plane) partitions.

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