Singular Isotropic Cosmologies and Bel-Robinson Energy[1]

Spiros Cotsakis† and Ifigeneia Klaoudatou‡

Research Group of Geometry, Dynamical Systems and Cosmology
Department of Information and Communication Systems Engineering
University of the Aegean
Karlovassi 83 200, Samos, Greece
E-mails: † skot@aegean.gr, ‡ iklaoud@aegean.gr

Abstract

We consider the problem of the nature and possible types of spacetime singularities that can form during the evolution of *FRW* universes in general relativity. We show that by using, in addition to the Hubble expansion rate and the scale factor, the Bel-Robinson energy of these universes we can consistently distinguish between the possible different types of singularities and arrive at a complete classification of the singularities that can occur in the isotropic case. We also use the Bel-Robinson energy to prove that known behaviours of exact flat isotropic universes with given singularities are generic in the sense that they hold true in every type of spatial geometry.

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1 Introduction

There has recently been a resurgence of interest in the old question of the existence and nature of spacetime singularities, especially in a cosmological context. Traditionally one is interested in formulating criteria, of a geometric nature, to test under what circumstances such singularities, in the form of geodesic incompleteness, will be formed during the evolution in general relativity and in other metric theories of gravity, cf. [1]. These are usually translated into sufficient conditions to be satisfied by the matter fields present and are very plausible. On the other hand, one can formulate equally plausible and generic geometric criteria for the long time existence of geodesically complete, generic spacetimes in general relativity and also other theories of gravity, cf. [2]. Such criteria assume a globally hyperbolic spacetime in the so-called sliced form (cf. [3]), require the space gradient of the lapse function as well as the extrinsic curvature not to grow without bound, and are also very plausible.

The lesson to be drawn from this state of affairs is that in general relativity (and also in other metric gravity theories), singular and complete spacetimes are equally generic in a sense. The real question is what do we mean when we say that a relativistic model develops a spacetime singularity during its evolution. In other words, what are the possible spacetime singularities which are allowed in gravity theories? These may be singularities in the form of geodesic incompleteness developing in the course of the evolution but also others which are more subtle and which are also dynamical and present themselves, for instance, in some higher derivatives of the metric functions spoiling the smoothness of global solutions to the field equations.

Obviously a recognition and complete analysis of such a program cannot be accomplished in the short run and requires complete examination of a number of different factors controlling the resulting behaviour. For instance, one needs to have control of each possible behaviour of the different families of relativistic geometry coupled to matter fields in general relativity and other theories of gravity to charter the possible singularity formation. In this sense, unraveling the nature and kinds of possible singularities in the simplest kinds of geometry becomes equally important as examining this problem in the most general solutions to the Einstein equations. In fact, an examination of the literature reveals that the more general the spacetime geometry considered (and thus the more complex the system of equations to be examined) the simplest is the type of singularities allowed to be examined. By starting with a simple cosmological spacetime we allow all possible types of singularities to come to surface and be analyzed.

In [4] we derived necessary conditions for the existence of finite time singularities in globally and regularly hyperbolic isotropic universes, and provided first evidence for their nature based entirely on the behaviour of the Hubble parameter $H = \dot{a}/a$. This result may be summarized as follows:

Theorem 1.1 *Necessary conditions for the existence of finite time singularities in globally hyperbolic, regularly hyperbolic* FRW *universes are:*

S_1 *For each finite* t*,* H *is non-integrable on* $[t_1, t]$*, or*

S_2 H *blows up in a finite future time, or*

S_3 H *is defined and integrable for only a finite proper time interval.*

Condition S_1 may describe different types of singularities. For instance, it describes a big bang type of singularity, when H blows up at t_1 since then it is not integrable on any interval of the form $[t_1, t]$, $t > t_1$ (regular hyperbolicity is violated in this case, but the scale factor is bounded from above). However, under S_1 we can have other types of singularities: Since $H(\tau)$ is integrable on an interval $[t_1, t]$, if $H(\tau)$ is defined on $[t_1, t]$, continuous on (t_1, t) and the limits $\lim_{\tau \to t_1^+} H(\tau)$ and $\lim_{\tau \to t^-} H(\tau)$ exist, the violation of *any* of these conditions leads to a singularity that is not of the big bang type discussed previously.

Condition S_2 describes a future blow up singularity and condition S_3 may lead to a sudden singularity (where H remains finite), but for this to be a genuine type of singularity, in the sense of geodesic incompleteness, one needs to demonstrate that the metric is non-extendible to a larger interval.

Note that these three conditions are not overlapping, for example S_1 is not implied by S_2 for if H blows up at some finite time t_s after t_1, then it may still be integrable on $[t_1, t]$, $t_1 < t < t_s$.

There are many examples in the literature of singularities belonging to the types predicted by the above theorem; they were analyzed in [4] in detail. They usually describe flat isotropic universes with various components of matter. For example, those models containing phantom dark energy [5], or a perfect fluid and a scalar field [6], exhibit future finite time blow up singularities and therefore fall in the S_2 category. The standard big bang singularities as well as the sudden singularity of [7] fall in the S_1 category. Other sudden singularities (which do not have a blow up of H at $t = 0$, see for example [8]) and the inflation model of [9] both fall in the S_3 category.

2 Classification

As discussed in [4], Theorem (1.1) describes possible time singularities that are met in FRW universes having a Hubble parameter that behaves like S_1 or S_2 or S_3. Although such a classification is a first step to clearly distinguish between the various types of singularities that can occur in such universes, it does not bring out some of the essential features of the dynamics that differ from singularity to singularity. For instance, condition S_2 includes both a collapse singularity, where $a \to 0$ as $t \to t_s$, *and* a blow up singularity where $a \to \infty$ as $t \to t_s$. Such a degeneration is unwanted in any classification of the singularities that can occur in the model universes in question.

It is therefore necessary to refine this classification by considering also the behavior of the scale factor. Another aspect of the problem that must be taken into account is the relative behaviour of the various matter components as we approach the time singularities.

In what follows we present a summary of our most recent work on the classification of possible singularities which occur in isotropic model universes. Details will be published elsewhere, cf. [10]. Our classification is based on the introduction of an invariant geometric quantity, the Bel-Robinson energy, which takes into account precisely those features of the problem, related to the matter contribution, in which models still differ near the time singularity while having similar behaviours of a and H. In this way, we arrive at a complete classification of the possible cosmological singularities in the isotropic case.

The Bel-Robinson energy is a kind of energy of the gravitational field *projected* in a sense to a slice in spacetime. It is used in [11], [12] to prove global existence results in the case of an asymptotically flat and cosmological spacetimes respectively, and is defined as follows. Consider a sliced spacetime with metric

$$^{(n+1)}g \equiv -N^2(\theta^0)^2 + g_{ij}\,\theta^i\theta^j, \quad \theta^0 = dt, \quad \theta^i \equiv dx^i + \beta^i dt, \tag{2.1}$$

where $N = N(t, x^i)$ is the lapse function and $\beta^i(t, x^j)$ is the shift function, and the 2-covariant spatial electric and magnetic tensors

$$\begin{aligned} E_{ij} &= R^0_{i0j}, \\ D_{ij} &= \frac{1}{4}\eta_{ihk}\eta_{jlm}R^{hklm}, \\ H_{ij} &= \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j}, \\ B_{ji} &= \frac{1}{2}N^{-1}\eta_{ihk}R^{hk}_{0j}, \end{aligned}$$

where η_{ijk} is the volume element of the space metric $\bar{g}$. The *Bel-Robinson energy* is given by

$$\mathcal{B}(t) = \frac{1}{2}\int_{\mathcal{M}_t}\left(|E|^2 + |D|^2 + |B|^2 + |H|^2\right) d\mu_{\bar{g}_t}, \tag{2.2}$$

where by $|X|^2 = g^{ij}g^{kl}X_{ik}X_{jl}$ we denote the spatial norm of the 2-covariant tensor X. In the following, we exclusively use an FRW universe filled with various forms of matter with metric given by

$$ds^2 = -dt^2 + a^2(t)d\sigma^2, \tag{2.3}$$

where $d\sigma^2$ denotes the usual time-independent metric on the 3-slices of constant curvature k. For this spacetime, we find that the norms of the magnetic parts, $|H|, |B|$, are identically zero while $|E|$ and $|D|$, the norms of the electric parts, reduce to the forms

$$|E|^2 = 3\left(\ddot{a}/a\right)^2 \quad \text{and} \quad |D|^2 = 3\left((\dot{a}/a)^2 + k/a^2\right)^2. \tag{2.4}$$

Therefore the Bel-Robinson energy becomes

$$\mathcal{B}(t) = \frac{C}{2}\left(|E|^2 + |D|^2\right), \tag{2.5}$$

where C is the constant volume of (or *in* in the case of a non-closed space) the 3-dimensional slice at time t.

We can now classify the possible types of singularities that are formed in an FRW geometry during its cosmic evolution and enumerate the possible types that result from the different combinations of the three main functions in the problem, namely, the scale factor a, the Hubble expansion rate H and the Bel Robinson energy $\mathcal{B}$. If we suppose that the model has a finite time singularity at $t = t_s$, then the possible behaviours of the functions in the triplet $(H, a, (|E|, |D|))$ in accordance with Theorem (1.1) are as follows:

S_1 H non-integrable on $[t_1, t]$ for every $t > t_1$

S_2 $H \to \infty$ at $t_s > t_1$

S_3 H otherwise pathological

N_1 $a \to 0$

N_2 $a \to a_s \neq 0$

N_3 $a \to \infty$

B_1 $|E| \to \infty$, $|D| \to \infty$

B_2 $|E| < \infty$, $|D| \to \infty$

B_3 $|E| \to \infty$, $|D| < \infty$

B_4 $|E| < \infty$, $|D| < \infty$.

There are a few types that cannot occur. For instance, we cannot have an (S_2, N_2, B_3) singularity because that would imply having $a < \infty$ (N_2) and $H \to \infty$ (S_2), while $3\left((\dot{a}/a)^2 + k/a^2\right)^2 < \infty$ (B_3), at t_s which is impossible since $|D|^2 \to \infty$ at t_s (k arbitrary). A complete list of impossible singularities is given by (S_i, N_j, B_l), where the indices in the case of a $k = 0, \pm 1$ universe take the values $i = 1, 2,\ j = 1, 2, 3,\ l = 3, 4$, whereas in the case of a $k = -1$ universe the indices take the values $i = 1, 2,\ j = 2, 3,\ l = 3, 4$ (here by S_1 we denote for simplicity only the big bang case in the S_1 category). We thus see that some singularities which are impossible for a flat or a closed universe become possible for a hyperbolic universe. Consider for example the triplet (S_2, N_1, B_3) which means having $H \to \infty$, $a \to 0$ and

$$|D|^2 = 3\left((\dot{a}/a)^2 + k/a^2\right)^2 < \infty$$

at t_s. This behaviour is valid only for some cases of a hyperbolic universe.

All other types of finite time singularities can in principle be formed during the evolution of FRW, matter-filled models, in general relativity or other metric theories of gravity.

It is interesting to note that all the standard dust or radiation-filled big-bang singularities fall under the *strongest* singularity type, namely, the type (S_1, N_1, B_1). For example, in a flat universe filled with dust, at $t = 0$ we have

$$a(t) \propto t^{2/3} \to 0, \quad (N_1), \tag{2.6}$$

$$H \propto t^{-1} \to \infty, \quad (S_1), \tag{2.7}$$

$$|E|^2 = 3/4H^4 \to \infty, \quad |D|^2 = 3H^4 \to \infty, \quad (B_1). \tag{2.8}$$

Note that the classification is organized in such a way that the character of the singularities becomes milder as the indices of S, N and B increase. Milder singularities are thus met as one proceeds down the list.

In fluid-filled models, the various behaviours of the Bel-Robinson energy density can be related to four conditions imposed on the density and pressure of the cosmological fluid:

$B_1 \Leftrightarrow \mu \to \infty$ and $|\mu + 3p| \to \infty$

$B_2 \Leftrightarrow \mu \to \infty$ and $|\mu + 3p| < \infty$

$B_3 \Leftrightarrow \mu < \infty$ and $|\mu + 3p| \to \infty \Leftrightarrow \mu < \infty$ and $|p| \to \infty$

$B_4 \Leftrightarrow \mu < \infty$ and $|\mu + 3p| < \infty \Leftrightarrow \mu < \infty$ and $|p| < \infty$.

Of course we can translate these conditions to asymptotic behaviours in terms of a, H, depending on the value of k, for example,

1. If $k = 0$, $\mu < \infty \Rightarrow H^2 < \infty$, a arbitrary
2. If $k = 1$, $\mu < \infty \Rightarrow H^2 < \infty$ and $a \neq 0$
3. If $k = -1$, $\mu < \infty \Rightarrow H^2 - 1/a^2 < \infty$.

As an example, we consider the sudden singularity introduced in [7]. This has a finite a (condition N_2), finite H (condition S_3), finite μ but a divergent p (condition B_3) at t_s. As another example, consider the flat FRW model containing dust and a scalar field studied in [6]. The scale factor collapses at both an initial (big bang) and a final time (big crunch). The Hubble parameter and $\ddot{a}/a$ both blow up at the times of the big bang and big crunch (cf. [4]) leading to an (S_1, N_1, B_1) big bang singularity and an (S_2, N_1, B_1) big crunch singularity, respectively.

3 Generic results and examples

In this Section we provide necessary and sufficient conditions for the occurrence of some of the singularities introduced above. These conditions are motivated from studies of cosmological models described by exact solutions in the recent literature. By exact solutions we mean those in which all arbitrary constants have been given fixed values. We expect the proofs of these results to be all quite straightforward, for we have now already identified the type of singularity that we are looking for in accordance with our classification. Proving such results *without* this knowledge would have been a problem of quite a different order.

The usefulness of the results proved below lies in that they answer the question of whether the behaviours met in known models described by exact solutions (which as a rule have a flat spatial metric ($k = 0$)) continue to valid in universes having all possible values of k described by solutions which are more general than exact in the sense that some or all of the arbitrary constants still remain arbitrary. See [10] for more results of this type.

Theorem 3.1 *Necessary and sufficient conditions for an (S_2, N_3, B_1) singularity occurring at the finite future time t_s in an isotropic universe filled with a fluid with equation of state $p = w\mu$, are that $w < -1$ and $|p| \to \infty$ at t_s.*

Proof.

Substituting the equation of state $p = w\mu$ in the continuity equation $\dot{\mu}+3H(\mu+p) = 0$, we have

$$\mu \propto a^{-3(w+1)}, \tag{3.1}$$

and so if $w < -1$ and p blows up at t_s, a also blows up at t_s. Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2}, \quad |D|^2 = \frac{\mu^2}{3}, \quad |E|^2 = \frac{1}{12}\mu^2(1+3w)^2, \tag{3.2}$$

we conclude that at t_s, H, a, $|D|$ and $|E|$ are divergent.

Conversely, assuming an (S_2, N_3, B_1) singularity at t_s in an FRW universe with the equation of state $p = w\mu$, we have from the (B_1) hypothesis that $\mu \to \infty$ at t_s and so p also blows up at t_s. Since a is divergent as well, we see from (3.1) that $w < -1$. ■

As an example, consider an exact solution which describes a flat, isotropic phantom dark energy filled universe, studied in [5] given by

$$\alpha = [{\alpha_0}^{3(1+w)/2} + \frac{3(1+w)\sqrt{A}}{2}(t-t_0)]^{\frac{2}{3(1+w)}}, \tag{3.3}$$

where A is a constant. From (3.1) we see that the scale factor, and consequently H, blows up at the finite time

$$t_s = t_0 + \frac{2}{3\sqrt{A}(|w|-1)\alpha_0^{3(|w|-1)/2}}.$$

Then it follows that $|E|^2 = \frac{3}{4}H^4(1+3w)^2$ and $|D|^2 = 3H^4$ also blow up at t_s. Therefore in this model the finite time singularity is of type (S_2, N_3, B_1).

The following result says that the strongest big bang type singularities that can occur in universes with a massless scalar field are due to the kinetic term.

Theorem 3.2 *A necessary and sufficient condition for an (S_1, N_1, B_1) singularity at t_1 in an isotropic universe with a massless scalar field is that $\dot{\phi} \to \infty$ at t_1.*

PROOF. From the continuity equation, $\ddot{\phi} + 3H\dot{\phi} = 0$, we have $\dot{\phi} \propto a^{-3}$, and if $\dot{\phi} \to \infty$ then $a \to 0$. Since

$$H^2 = \frac{\mu}{3} - \frac{k}{a^2} \to \infty \tag{3.4}$$

H becomes unbounded at t_1. In addition, since

$$|D|^2 = \frac{\mu^2}{3} = \frac{\dot{\phi}^4}{12} \to \infty, \tag{3.5}$$

and

$$|E|^2 = \frac{1}{12}(\mu + 3p)^2 = \frac{\dot{\phi}^4}{3} \to \infty, \tag{3.6}$$

at t_1, both $|D|$ and $|E|$ diverge there.

Conversely, assuming an (S_1, N_1, B_1) singularity at t_1, we have (from B_1) that $\mu \to \infty$ and so $\dot{\phi}^2 \to \infty$ at t_1. ■

As an example of the above behaviour we can use the exact solution for a flat isotropic universe given in [13]. The solution is given by

$$H = \frac{1}{3t}, \quad \phi = \pm\sqrt{\frac{2}{3}} \ln \frac{t}{c}.$$

Since $a \propto t^{1/3}$ we have that at $t = 0$, $a \to 0$ (N_1), $H = \frac{1}{3t} \to \infty$ (S_1), $|E|^2 = \frac{2}{9}\dot{\phi}^4$ and $|D|^2 = 3H^4 \to \infty$ (B_1). As it follows from [13], this exact solution represents the asymptotic behaviour of a scalar field model if

$$\lim_{\phi \to \pm\infty} e^{-\sqrt{6}|\phi|} V(\phi) = 0. \tag{3.7}$$

4 Discussion

We have reviewed some of the main results of our recent work [10] regarding the classification of singularities in isotropic universes. The possible behaviours of the three functions, H, a and $\mathcal{B}$ exhaust the types of singularities that are possible in an isotropic universe. We have used known exact solutions as examples to illustrate the possible singularities that can be accommodated in the categories included in the classification. We have also shown that the Bel-Robinson energy can be used invariable as a tool to test the nature of singularities described by given exact solutions and decide whether such behaviour is generic and independent of the spatial geometry.

It would be interesting to further investigate how this classification will change when we consider other types of model universes, matter fields different from the fluid ones we considered in this paper, or other metric theories of gravity.

We note that the "crudest" singularity type (S_1, N_1, B_1), seems to accommodate all standard big bang singularities known. However, the precise analytical nature of the other new singularity types presented here, although certainly possible, remains a mystery. It is only through a combination of analytic and geometrical techniques that more light will be shed in such interesting questions of principle.

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