J. Evol. Equ. 17 (2017), 523–536

1424-3199/17/010523-14, *published online* September 16, 2016
DOI 10.1007/s00028-016-0365-z

Journal of Evolution Equations

CrossMark

# Regularity of stochastic Volterra equations by functional calculus methods

Roland Schnaubelt and Mark Veraar

*Abstract.* We establish pathwise continuity properties of solutions to a stochastic Volterra equation with an additive noise term given by a local martingale. The deterministic part is governed by an operator with an $H^\infty$-calculus and a scalar kernel. The proof relies on the dilation theorem for positive definite operator families on a Hilbert space.

## 1. Introduction

In this paper, we investigate pathwise continuity properties of the solutions to the stochastic Volterra equation

$$u(t) = u_0 + \int_0^t a(t-s)Au(s)\,\mathrm{d}s + L(t), \quad t \geq 0. \tag{1.1}$$

Here $A$ is a closed and densely defined operator on a Hilbert space $(X, \langle\cdot,\cdot\rangle)$, the kernel $a$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}_+)$, and $L$ is an $X$-valued local $L^2$-martingale with càdlàg (or continuous) paths. Stochastic Volterra equations are widely studied, and we refer the reader to [2,5–9,13,14,22] and the references given there. In this paper, we show that for a large class of kernels $a$ and operators $A$, there exists a version of the solution $u$ of (1.1) for which the paths are càdlàg (or continuous) using dilation theory, $H^\infty$-calculus and the theory of deterministic Volterra equations.

Volterra equations arise in physical models whose constitutive laws depend on the history of the material. Such behaviour occurs in viscoelastic fluids or solids, in heat conduction with memory, or in electromagnetism. In accordance with the theory in Prüss' monograph [29], we look at an integrated formulation of such problems, which fits well to stochastic evolution equations. The stochastic term $L(t)$ can be understood as a time integral of a given external random force or a heat supply. We refer to Chapter 5 of [29] for a discussion of the underlying deterministic models.

---

*Mathematics Subject Classification*: Primary: 60H20; Secondary: 45N05, 60H15
*Keywords*: Stochastic Volterra equation, Pathwise continuity, Local martingale, Dilation, $H^\infty$-calculus.
Mark Veraar author is supported by the VIDI subsidy 639.032.427 of the Netherlands Organization for Scientific Research (NWO).
Dedicated to Jan Prüss, in admiration of his unfaltering spirit and his outstanding mathematical work.

For more regular paths, one could apply in (1.1) the theory developed in [29] pathwise (under appropriate conditions on $a$). For instance, if $L$ has Hölder continuous paths and if the deterministic part of (1.1) is of parabolic type in the sense of [29], then $u$ also has Hölder continuous paths by Theorem 3.3 of [29]. However, for general local $L^2$-martingales Hölder continuity is quite restrictive and even impossible if jumps occur. On the other hand, Chapter 8 of [29] provides a theory of maximal $L^p$-regularity for the deterministic part in the parabolic case, but it would only yield $L^p$-properties of the paths, see Theorem 8.7 of [29].

In [28], Peszat and Zabczyk found new conditions on $a$ under which the solution $u$ to (1.1) has càdlàg (or continuous) trajectories. Their method is based on dilation results, which were previously used in the case that $a = 1$ and $A$ is the generator of a semigroup which satisfies $\|T(t)\| \leq \mathrm{e}^{wt}$ for all $t \geq 0$ and a fixed $w \in \mathbb{R}$, see [20] and references therein. In Theorem 1 of [28], it is assumed that $A$ is a selfadjoint operator so that the spectral theorem provides a functional calculus for $A$. The functional calculus allows to reduce the problem to scalar Volterra equations.

However, many operators arising in applications fail to be selfadjoint. For instance, an elliptic operator $A$ with $D(A) = H^2(\mathbb{R}^d)$ in non-divergence form

$$Au = \sum_{i,j=1}^{d} a_{ij} D_i D_j u + b_i D_i u + cu$$

with space-dependent coefficients $a_{ij}$, $b_i$ and $c$ is not adjoint in general. In the system case, self-adjointness is even more problematic. Indeed, let $D(A) = H^2(\mathbb{R}^d; \mathbb{R}^N)$ and

$$Au = \begin{pmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \vdots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}$$

and assume that each $A_{mm}$ is itself an elliptic second-order differential operator. Even if the elliptic operators $A_{mn}$ have $x$-independent coefficients, the operator $A$ will only lead to a self-adjoint operator if $A_{mn} = A_{nm}$ which is rather restrictive.

Under suitable ellipticity conditions and regularity assumptions on the coefficients, the above two operators possess a bounded $H^\infty$-calculus. During the last 25 years there has been a lot of progress in the investigation of this functional calculus. Originally, it was developed by McIntosh and collaborators to solve the Kato square root problem (see [1,24]). By now the $H^\infty$-calculus is well established and has become one of the central tools in operator-theoretic approaches to PDE. Any reasonable elliptic or accretive differential operator on a Hilbert space admits a bounded $H^\infty$-calculus.

In this paper, $H^\infty$-calculus techniques allow us to show that the solution of (1.1) has the same pathwise continuity properties as the local $L^2$-martingale $L$, thereby covering the above indicated examples. Besides the $H^\infty$-calculus of the main operator $A$, we mainly assume sector conditions of the Laplace transform of the kernel $a$, see Theorem 3.3. The $H^\infty$-calculus is first used to construct the solution operator (called *resolvent*) of the deterministic Volterra equation with $L = 0$ by means of the solutions to a corresponding scalar problem with $A$ replaced by complex numbers in a suitable sector. Thanks to Laplace transform techniques from [29], we can derive the uniform estimates on these solutions which are required to apply the $H^\infty$-calculus. Second, one employs the calculus to check that the resolvent is positive definite in order to use the dilation argument from [28]. In this step, we also invoke a different dilation result taken from [23]. An important technical feature is rescaling arguments which are needed since in applications usually only a shifted operator is known to possess an $H^\infty$-calculus. We further discuss auxiliary facts, as well as examples for operators $A$ and kernels $a$ in the second and the last section.

The $H^\infty$-calculus has already played an important role in several other works on stochastic partial differential equations. In [31,35] it is used to derive maximal estimates for stochastic convolutions by a dilation argument. Solutions with paths in $D((-A^{1/2}))$ almost surely are obtained via square function estimates in [3,13,15,26,27,33,34]. More indirectly, characterizations of the $H^\infty$-calculus have already been employed in Theorem 6.14 of [11] and in [4], in the form that $D_A(\theta, 2) = D((-A)^\theta)$ for some $\theta \in (0, 1)$ and that $(-A)^{is}$ is bounded for all $s \in \mathbb{R}$, respectively.

## 2. Preliminaries

### 2.1. Volterra equations

We first recall a basic definition in the theory of Volterra equations of scalar type, see Prüss' monograph [29] for details. For $\phi \in (0, \pi]$ we define the sector $\Sigma_\phi$ by

$$\Sigma_\phi = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \phi\}.$$

Let $(A, D(A))$ be a closed, densely defined and injective operator on a Hilbert space $X$ and let $\sigma(A)$ denote its spectrum. Such an operator $A$ is called *sectorial of angle* $\phi$ if $\sigma(A) \subseteq \mathbb{C}\backslash\Sigma_{\pi-\phi}$ and there is a constant $C$ such that

$$\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{C}{|\lambda|} \quad \text{whenever} \quad \arg(\lambda) < \pi - \phi.$$

We further write $\phi_A$ for the infimum of all $\phi$ such that $A$ is sectorial of angle $\phi$.

Let $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ and $(A, D(A))$ be a closed operator. We study the Volterra equation

$$u(t) = f(t) + \int_0^t a(t-s)Au(s)\,\mathrm{d}s, \quad t \geq 0, \tag{2.1}$$

for a given measurable map $f : \mathbb{R}_+ \to X$. Usually, we extend $a$, $u$ and $f$ by zero on $(-\infty, 0)$. We then write (2.1) as $u = f + aA * u$, where $*$ stands for the convolution. From [29], we recall the basic concept describing the solution operators of (2.1).

DEFINITION 2.1. A family $(S(t))_{t\geq 0}$ of bounded linear operators on $X$ is called a *resolvent* for (2.1) if it satisfies the following conditions.

(i) $S$ is strongly continuous on $[0,\infty)$ and $S(0) = I$.

(ii) We have $S(t)D(A) \subseteq D(A)$ and $AS(t)x = S(t)Ax$ for all $t \geq 0$ and $x \in D(A)$.

(iii) The *resolvent equation*

$$S(t)x = x + \int_0^t a(t-s)AS(s)x \, \mathrm{d}s \tag{2.2}$$

is valid for all $x \in D(A)$ and $\geq 0$.

By Corollary 1.1 of [29], the problem (2.1) possesses at most one resolvent.

## 2.2. Functional calculus

In this section, we briefly discuss the $H^\infty$-calculus which was developed by McIntosh [24] and many others. We also present important classes of examples below. For details we refer to [18,23] and the references therein.

Let $H^\infty(\Sigma_\phi)$ denote the space of all bounded analytic functions $f : \Sigma_\phi \to \mathbb{C}$, and $H_0^\infty(\Sigma_\phi)$ be the subspace of all $f \in H^\infty(\Sigma_\phi)$ for which there exist $\varepsilon > 0$ and $c \geq 0$ such that

$$|f(z)| \leq \frac{c\,|z|^\varepsilon}{(1+|z|)^{2\varepsilon}}, \quad z \in \Sigma_\phi.$$

If $A$ is sectorial, then for all $\phi_A < \phi' < \phi < \pi$ and $f \in H_0^\infty(\Sigma_\phi)$ we can define an operator $f(-A)$ in $\mathcal{L}(X)$ by setting

$$f(-A) = \frac{1}{2\pi i} \int_{\partial \Sigma_{\phi'}} f(z)(z+A)^{-1} \, \mathrm{d}z.$$

We say that $-A$ has a *bounded $H^\infty$-calculus* if there is a constant $C_A \geq 0$ and an angle $\phi > \phi_A$ such that for all $f \in H_0^\infty(\Sigma_\phi)$ we have

$$\|f(-A)\| \leq C_A \, \|f\|_{H^\infty(\Sigma_\phi)}.$$

We work here with $-A$ instead of $A$ to be in accordance with [29], where dissipative operators instead of accretive operators are used.

Clearly, every self-adjoint operator of negative type has a bounded $H^\infty$-calculus. We next give several examples of more general situations.

EXAMPLE 2.2. (Dissipative operators) Let $(A, D(A))$ be linear and injective such that $I - A$ is invertible. Assume that $A$ is $\phi$-dissipative for some $\phi \in [0, \pi/2]$; i.e.

$$|\arg(\langle Ax, x\rangle)| \geq \pi - \phi, \quad x \in D(A).$$

It is well known that then $A$ is sectorial with $\phi_A \leq \phi$. Moreover, $-A$ has a bounded $H^\infty$-calculus by e.g. Theorem 11.5 in [23].

A special case of this situation is normal operators with spectrum in $\mathbb{C}\setminus\Sigma_{\pi-\phi}$.

EXAMPLE 2.3. (BIP) Let $(A, D(A))$ be a sectorial operator which has bounded imaginary powers (BIP); i.e. $(-A)^{is} \in \mathcal{L}(X)$ for all $s \in \mathbb{R}$. Then, $-A$ has a bounded $H^\infty$-calculus, see e.g. Theorem 11.9 in [23]. We recall that $-A$ has bounded imaginary powers if and only if $D((-A)^\theta) = D_A(\theta, 2)$ for some $\theta \in (0, 1)$, where the latter is the real interpolation space between $X$ and $D(A)$, see Sects. 6.6.3 and 6.6.4 in [18].

EXAMPLE 2.4. (Elliptic operators) Let $A = \sum_{m,n=1}^d a_{mn}\partial_m\partial_n + \sum_{n=1}^d b_n\partial_n + c$ with coefficients $b_n, c \in L^\infty(\mathbb{R}^d)$ and $a_{mn} \in C_b^\varepsilon(\mathbb{R}^d)$ for some $\varepsilon > 0$. For some $\phi \in (0, \pi]$ we assume that

$$\sum_{m,n=1}^{N} a_{mn}(x)\xi_n\xi_m \in \{\lambda \neq 0 : |\arg(\lambda)| \geq \phi\}, \quad \xi \in \mathbb{R}^d\backslash\{0\}, \quad x \in \mathbb{R}^d.$$

Then for all $\phi' > \phi$, there exists a $w$ such that $A - w$ is sectorial of angle $\phi'$ and $-(A - w)$ has a bounded $H^\infty$-calculus, see e.g. Theorem 13.13 in [23].

The above list is far from exhaustive. For other results on operators with a bounded $H^\infty$-calculus, we refer the reader to [10,12,16] and to [1] for connections to the famous Kato square root problem.

## 3. The main result

Let $X$ be a separable Hilbert space and $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions (see [21]). We assume that $L$ is an $X$-valued local $L^2$-martingale with càdlàg paths almost surely, that the kernel $a$ belongs to $L^1_{\text{loc}}(\mathbb{R}_+)$, and that $A$ is a closed operator on $X$ with dense domain $D(A)$. We study the stochastic Volterra equation

$$u(t) = u_0 + \int_0^t a(t-s)Au(s)\,\mathrm{d}s + L(t), \quad t \geq 0, \tag{3.1}$$

for an $\mathcal{F}_0$-measurable initial function $u_0 : \Omega \to X$. We may assume that $L(0) = 0$ as we could replace $u_0$ by $u_0 + L(0)$.

In Theorem 3.3, we present the main result on the existence of solutions with regular paths. Several classes of admissible kernels $a$ will be discussed in Sect. 4.1. Finally, in Sect. 4.2 we illustrate how the results of Sects. 2.2 and 4.1 can be combined with Theorem 3.3 to obtain path properties of solutions.

Before we move to the main result, we first give the definition of a weak solution to (3.1) and we show a simple but useful lemma about shifting the operator $A$.

DEFINITION 3.1. A measurable process $u : \mathbb{R}_+ \times \Omega \to X$ is called a *weak solution* to (3.1) if almost all paths of $u$ belong to $L^1_{\text{loc}}(\mathbb{R}_+; X)$ and if for all $x^* \in D(A^*)$ and $t \in [0, \infty)$ we have almost surely

$$\langle u(t), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t a(t-s)\langle u(s), A^*x^*\rangle\,\mathrm{d}s + \langle L(t), x^*\rangle.$$

Assume that the resolvent for (2.1) exists. Proposition 2 of [28] then says that there is a unique weak solution $u$ of (3.1) given by

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\,\mathrm{d}L(s). \tag{3.2}$$

Here the stochastic integral exists since $S$ is strongly continuous and $L$ is a càdlàg local $L^2$-martingale with values in the Hilbert space $X$ (see Sects. 14.5 and 14.6 in [25], Sects. 2.2 and 2.3 in [30], and [21, Chapter 26] for the scalar case).

We start with a simple but useful lemma which allows us to replace the operator $A$ by $A - \rho$ for any $\rho \in \mathbb{C}$. In the applications of Theorem 3.3, this is quite essential since one can usually check the boundedness of the $H^\infty$-calculus only for $A - \rho$ with large $\rho \geq 0$.

LEMMA 3.2. *Assume $a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $\rho \in \mathbb{C}$. Let $s \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ solve $s - \rho a * s = a$. Then problem (3.1) has a weak solution with càdlàg/continuous paths almost surely if and only if (3.1) with $(a, A)$ replaced by $(s, A - \rho)$ has a weak solution with càdlàg/continuous paths.*

It is well known that there is a unique function $s \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ with $s - \rho a * s = a$, see Theorem 2.3.5 in [17].

*Proof.* Assume that (3.1) with $(a, A)$ replaced by $(s, A - \rho)$ has a weak solution $v$ with càdlàg/continuous paths. Set $u = v + \rho s * v$. The paths of $u$ inherit the càdlàg/continuous properties of the paths of $v$ since $f * g \in C_b(\mathbb{R})$ if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$. (The latter fact can be proved approximating $f$ by continuous functions.) Moreover, the above identities yield $s * v = a * u$. Using also that $v$ is a weak solution, we compute

$$\begin{aligned}\langle u(t), x^*\rangle &= \langle v(t), x^*\rangle + \rho s * \langle v(t), x^*\rangle = \langle u_0, x^*\rangle + s * \langle v(t), A^*x^*\rangle + \langle L(t), x^*\rangle \\ &= \langle u_0, x^*\rangle + a * \langle u(t), A^*x^*\rangle + \langle L(t), x^*\rangle\end{aligned}$$

for all $x^* \in D(A^*)$. Hence, $u$ is a weak solution to (3.1). The converse implication can be proved in a similar way. □

Our main result is the following sufficient condition for the existence and uniqueness of a solution which has càdlàg/continuous paths. We write $\hat{a}$ for the Laplace transform of $a$.

THEOREM 3.3. *Assume the following conditions.*

1. *There is a number $\rho \in \mathbb{R}$ such that $A - \rho$ is a sectorial operator of angle $\phi_{A-\rho} < \pi/2$ and $-(A - \rho)$ has a bounded $H^\infty$-calculus.*
2. *$a \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and $t \mapsto \mathrm{e}^{-w_0 t} a(t)$ is integrable on $\mathbb{R}_+$ for some $w_0 \in \mathbb{R}$.*
3. *There exist constants $\sigma, \phi, c > 0$ and $w \in \mathbb{R}$ such that $\sigma + \phi_{A-\rho} < \pi/2$, $\phi > \phi_{A-\rho}$, $\hat{a}$ is holomorphic on $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > w\}$, and for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > w$ we have*

(i) $\lambda\hat{a}(\lambda) \in \Sigma_\sigma$ *and* $\hat{a}(\lambda) \in \Sigma_{\pi-\phi}$,
(ii) $|\lambda\hat{a}'(\lambda)| \leq c\,|\hat{a}(\lambda)|$.

*Let* $u_0 : \Omega \to X$ *be* $\mathcal{F}_0$*-measurable. Then* (2.1) *possesses a resolvent* $S$*, and the stochastic problem* (3.1) *has a unique weak solution* $u$ *given by*

$$u(t) = S(t)u_0 + \int_0^t S(t-s)\,\mathrm{d}L(s), \quad t \geq 0,$$

*and* $u$ *has a modification with càdlàg/continuous trajectories almost surely whenever the local* $L^2$*-martingale* $L$ *has càdlàg/continuous paths almost surely.*

As announced, the required existence of an $H^\infty$-calculus plays a crucial role in our approach. The second sector condition in (3)(i) and the assumption of *1-regularity* in (3)(ii) are quite common in the theory of Volterra equations of parabolic type, see Chapters 3 and 8 of [29]. The first condition in (3)(i) is needed to derive the positive definiteness of the resolvent, as defined next.

The proof of Theorem 3.3 relies on the following result which is a slight variation of Proposition 3 in [28]. Before stating this result, we recall that a family of operators $(R(t))_{t\in\mathbb{R}}$ on $X$ is called *positive definite* if $R(t) = R(-t)^*$ and

$$\sum_{m,n=1}^{N} \langle R(t_n - t_m)x_m, x_n\rangle \geq 0.$$

for all $t, t_1, \ldots, t_N \in \mathbb{R}$, $x_1, \ldots, x_N \in X$ and $N \in \mathbb{N}$.

PROPOSITION 3.4. *Assume that* $(R(t))_{t\in\mathbb{R}}$ *is a strongly continuous family of operators on* $X$ *such that* $R(0) = I$ *and the family* $\mathrm{e}^{-w|t|}R(t)$ *is positive definite for some* $w \in \mathbb{R}$*. If* $L$ *is a càdlàg (or continuous) local* $L^2$*-martingale with values in* $X$*, then the process*

$$u(t) = R(t)u_0 + \int_0^t R(t-s)\,\mathrm{d}L(s), \quad t \geq 0,$$

*is càdlàg (or continuous) as well.*

The proof of this fact uses that the family $R$ has a dilation to a strongly continuous group by Naĭmark's theorem (see Theorem I.7.1 in [32]) and an argument from [19,20].

*REMARK* 3.5. Proposition 3.4 can be extended to larger classes of integrators $L : [0,\infty) \times \Omega \to X$. The only properties needed in the proof are:

(1) For every strongly continuous $f : \mathbb{R}_+ \to \mathcal{L}(X)$, the stochastic integral process $\int_0^\cdot f(s)\,\mathrm{d}L(s)$ exists and almost all paths are càdlàg (or continuous).
(2) For any $B \in \mathcal{L}(X)$, the following identity holds almost surely

$$B\int_0^t f(s)\,\mathrm{d}L(s) = \int_0^t Bf(s)\,\mathrm{d}L(s), \quad t \geq 0.$$

The proof of Theorem 3.3 will be divided into several steps. We first reduce the problem to the case $\rho = 0$. After that we will use the functional calculus to construct the resolvent and to show that it is positive definite. The above proposition then implies the assertions.

*Proof of Theorem 3.3. Step 1: Reduction to* $\rho = 0$. By Lemma 3.2, it suffices to prove the result with $(a, A)$ replaced by $(s, A - \rho)$, where $s \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ satisfies $s - \rho a * s = a$. Moreover, for $\mathrm{Re}(\lambda) > w \geq w_0$, we have

$$|\hat{a}(\lambda)| \leq \int_0^\infty \mathrm{e}^{-wt}|a(t)|\,\mathrm{d}t \;\longrightarrow\; 0$$

as $w \to \infty$. Hence, for all sufficiently large $w$, we can write

$$\hat{s}(\lambda) = \frac{\hat{a}(\lambda)}{1 - \rho\hat{a}(\lambda)}, \quad \mathrm{Re}(\lambda) > w.$$

It is then easy to check that also $s$ satisfies the assumptions of the theorem for a fixed (possibly larger) $w \geq 0$, but one may have to increase $\sigma$ and decrease $\phi$ a bit. In the following, we can thus assume that $\rho = 0$ and write $A$ instead of $A - \rho$.

*Step 2: Construction of the resolvent.* Choose $\beta \in (\phi_A, \phi)$ such that $\beta + \sigma < \pi/2$. Let $\alpha = \beta\frac{2}{\pi}$. It follows from Theorem 11.14 of [23] that $-A$ has a dilation to a multiplication operator $M$ on $L^2(\mathbb{R}; X)$ given by

$$Mf(\tau) = -(i\tau)^\alpha f(\tau), \quad \tau \in \mathbb{R}.$$

This means that there exists an isometric embedding $J : X \to L^2(\mathbb{R}; X)$ such that $JJ^*$ is an orthogonal projection from $L^2(\mathbb{R}; X)$ onto $J(X)$ and $J^*J = I$ on $X$ and for all $\psi > \beta$ and $f \in H^\infty(\Sigma_\psi)$ we have

$$f(-A) = J^* f(-M) J. \tag{3.3}$$

Set $a_w(t) = \mathrm{e}^{-wt}a(t)$ with $w \geq 0$ from Step 1. For each $\mu \in \mathbb{C}$, let $s_{w,\mu}$ be the unique solution to the equation

$$s_{w,\mu}(t) = \mathrm{e}^{-wt} - \mu a_w * s_{w,\mu}(t). \tag{3.4}$$

The function $s_{w,\mu}$ is continuous. (See Theorems 2.3.1 and 2.3.5 of [17].) We want to check that $\mu \mapsto s_{w,\mu}(t)$ belongs to $H^\infty(\Sigma_\psi)$ for $\psi \in (\beta, \phi)$. We first show the holomorphy of the map $\varphi_{w,t} : \mu \mapsto s_{w,\mu}(t)$ on $\mathbb{C}$ for fixed $t \geq 0$.

To this aim, take $\mu_0 \in \mathbb{C}$ and $\varepsilon > 0$. Set $B = \{\mu \in \mathbb{C} : |\mu - \mu_0| < \varepsilon\}$. It is enough to prove that $\mu \mapsto s_{\tilde{w},\mu}(t)$ is holomorphic on $B$ for a sufficiently large $\tilde{w} \geq 0$. Indeed, the uniqueness of (3.4) yields $s_{w,\mu}(t) = \mathrm{e}^{(\tilde{w}-w)t}s_{\tilde{w},\mu}(t)$band; thus, $\varphi_{w,t}$ will also be holomorphic on $B$. Since $B$ is arbitrary, the holomorphy of $\varphi_{w,t}$ on $\mathbb{C}$ will then follow. Take now $\tilde{w}$ such that $|\mu_0 + \varepsilon|\,\|a_{\tilde{w}}\|_{L^1(\mathbb{R}_+)} < 1$. By the proof of Theorem 2.3.1 of [17] the function $r_\mu = \sum_{j=1}^\infty (-1)^{j-1}(\mu a_{\tilde{w}})^{*j}$ converges in $L^1(\mathbb{R}_+)$ uniformly for

$\mu \in B$ and $r_\mu$ solves $r_\mu + \mu a_{\tilde{w}} * r_\mu = \mu a_{\tilde{w}}$. Hence, the map $B \ni \mu \mapsto r_\mu \in L^1(\mathbb{R}_+)$ is holomorphic. Theorem 2.3.5 of [17] also yields that

$$s_{\tilde{w},\mu}(t) = \mathrm{e}^{-\tilde{w}t} - \int_0^t r_\mu(t-\tau)\mathrm{e}^{-\tilde{w}\tau}\,\mathrm{d}\tau, \quad t \in \mathbb{R}_+.$$

The right-hand side is holomorphic in $\mu \in B$ for each $t \in \mathbb{R}_+$ because integration with respect to the measure $\mathrm{e}^{-\tilde{w}\tau}\,\mathrm{d}\tau$ is a bounded linear functional on $L^1(\mathbb{R}_+)$.

We next claim that there exists a constant $C > 0$ such that $|s_{w,\mu}(t)| \le C$ for all $t \ge 0$ and $\mu \in \Sigma_\psi$, where $\psi \in (\beta, \phi)$. Thanks to Corollary 0.1 and (the proof of) Proposition 0.1 of [29] it suffices to find a constant $K$ independent of $\mu$ such that

$$|\lambda \hat{s}_{w,\mu}(\lambda)| + \left|\lambda^2 \hat{s}'_{w,\mu}(\lambda)\right| \le K, \quad \lambda \in \mathbb{C}_+.$$

(These results at first give the bound on $s_{w,\mu}(t)$ only for a.e. $t$, but $s_{w,\mu}$ is continuous.) Since $\hat{s}_{w,\mu}(\lambda) = \frac{1}{\lambda+w}\frac{1}{1+\mu\hat{a}(\lambda+w)}$ by (3.4), we can compute

$$\begin{aligned}
|\lambda \hat{s}_{w,\mu}(\lambda)| &= \left|\frac{\lambda}{\lambda+w}\right|\left|\frac{1}{1+\mu\hat{a}(\lambda+w)}\right| \le \sup\left\{|1+z|^{-1} : z \in \Sigma_{\pi-(\phi-\psi)}\right\} =: M_1,\\
|\lambda^2 \hat{s}'_{w,\mu}(\lambda)| &= \left|\frac{\lambda^2}{(\lambda+w)^2}\right|\left|\frac{(1+\mu\hat{a}(\lambda+w)) + (\lambda+w)\mu\hat{a}'(\lambda+w)}{(1+\mu\hat{a}(\lambda+w))^2}\right|\\
&\le \left|\frac{1}{1+\mu\hat{a}(\lambda+w)}\right| + \frac{c\,|\mu\hat{a}(\lambda+w)|}{|1+\mu\hat{a}(\lambda+w)|^2}\\
&\le M_1 + \sup\left\{\left|\frac{cz}{(1+z)^2}\right| : z \in \Sigma_{\pi-(\phi-\psi)}\right\} =: M_1 + M_2
\end{aligned}$$

for all $\lambda \in \mathbb{C}_+$. Here, we employed the second part of condition (3)(i) several times and (3)(ii) in the penultimate estimate. The claim follows.

We conclude that the map $\mu \mapsto s_{w,\mu}(t)$ belongs to $H^\infty(\Sigma_\psi)$ for each $t \ge 0$. Using the $H^\infty$-calculus of $-A$, we define $S_{w,A}(t) = s_{w,-A}(t)$ in $\mathcal{L}(X)$ with norm less or equal $C_A C$. To relate these operators to the desired resolvent, we further let $S_{w,M}(t)$ be the (multiplication) operator on $L^2(\mathbb{R}; X)$ which is given by the map $\mu \mapsto s_{w,\mu}(t)$ and the functional calculus of $-M$. The norm of $S_{w,M}(t)$ is bounded by $C$. Since the maps $s_{w,\mu}$ are continuous, $S_{w,M}(t)f$ is continuous in $L^2(\mathbb{R}; X)$ for $t \ge 0$ if $f$ is a simple function. By density and uniform boundedness, we infer that $t \mapsto S_{w,M}(t)$ is strongly continuous. Equation (3.3) further yields

$$S_{w,A}(t) = J^* S_{w,M}(t) J, \quad t \ge 0. \tag{3.5}$$

This identity and the strong continuity of $S_{w,M}$ imply that $S_{w,A}$ is strongly continuous. The operators $S_{w,A}(t)$ and $A$ commute on $D(A)$ by the functional calculus, see e.g. Theorem 2.3.3 in [18]. To derive the resolvent equation, we observe

$$S_{w,M}(t)g = \mathrm{e}^{-wt}g + a_w * (MS_{w,M}g)(t)$$

for $g \in L^2(\mathbb{R}; X)$ and $t \geq 0$ due to the definition of $S_{w,M}$ and (3.4). This identity for $g = nR(n, M)Jx$ with $x \in D(A)$ and Eq. (3.3) then imply

$$\begin{aligned}S_{w,A}(t)nR(n, A)x &= J^* S_{w,M}(t)nR(n, M)Jx \\ &= J^* \mathrm{e}^{-wt} nR(n, M)Jx + J^* a_w * (S_{w,M} nMR(n, M)Jx)(t) \\ &= \mathrm{e}^{-wt} nR(n, A)x + a_w * (S_{w,A} nAR(n, A)x)(t).\end{aligned}$$

Letting $n \to \infty$ and using that $A$ and $S_{w,A}(t)$ commute on $D(A)$, we find

$$S_{w,A}(t)x = \mathrm{e}^{-wt} x + a_w * (S_{w,A}Ax)(t) = \mathrm{e}^{-wt} x + a_w * (AS_{w,A}x)(t)\, \mathrm{d}s \tag{3.6}$$

for $t \geq 0$. One now easily sees that $(\mathrm{e}^{wt} S_{w,A}(t))_{t\geq 0}$ is the resolvent of (2.1).

*Step 3: Positive definiteness.* Let $\tilde{S}_{w,A}$ be the extension of $S_{w,A}$ given by $\tilde{S}_{w,A}(t) = S_{w,A}(-t)^*$. Analogously, we extend $\tilde{S}_{w,M}$ and $\tilde{s}_{w,\mu}$ to functions on $\mathbb{R}$. Fix $t_1, \ldots, t_N \geq 0$ and $x_1, \ldots, x_N \in X$. Setting $f_n = Jx_n$, we infer from (3.5) that

$$\sum_{m,n=1}^{N} \langle \tilde{S}_{w,A}(t_n - t_m)x_m, x_n\rangle = \int_{\mathbb{R}} \sum_{m,n=1}^{N} \langle \tilde{s}_{w,(i\tau)^\alpha}(t_n - t_m) f_m(\tau), f_n(\tau)\rangle \, \mathrm{d}\tau.$$

Therefore, for the positive definiteness of $S_{w,A}(t)$ is suffices to prove that the function $\tilde{s}_{w,(i\tau)^\alpha}$ is positive definite for a.e. $\tau \in \mathbb{R}$.

By the easy direction of Bochner's characterization, it is enough to check that $\mathcal{F}(\tilde{s}_{w,(i\tau)^\alpha})(\xi) \geq 0$ for all $\xi \geq 0$, where $\mathcal{F}$ denotes the Fourier transform. The Fourier transform of $\tilde{s}_{w,(i\tau)^\alpha}$ satisfies

$$\begin{aligned}\mathcal{F}(\tilde{s}_{w,(i\tau)^\alpha})(\xi) &= \int_0^\infty \tilde{s}_{w,(i\tau)^\alpha}(t)\mathrm{e}^{-it\xi}\, \mathrm{d}t + \int_{-\infty}^0 \overline{\tilde{s}_{w,(i\tau)^\alpha}(-t)}\mathrm{e}^{-it\xi}\, \mathrm{d}t \\ &= 2\,\mathrm{Re} \int_0^\infty s_{w,(i\tau)^\alpha}(t)\mathrm{e}^{-it\xi}\, \mathrm{d}t = 2\,\mathrm{Re}(\widehat{s}_{w,(i\tau)^\alpha}(i\xi)).\end{aligned}$$

If we extend $s$ and $a$ by zero to $t < 0$, Eq. (3.4) yields

$$\begin{aligned}\mathrm{Re}\,\widehat{s}_{w,(i\tau)^\alpha}(i\xi) &= \mathrm{Re}\left((1 + (i\tau)^\alpha \hat{a}(w + i\xi))^{-1}(w + i\xi)^{-1}\right) \\ &= \mathrm{Re}\left((1 + |\tau|^\alpha \mathrm{e}^{\pm i\alpha\pi/2}\hat{a}(w + i\xi))^{-1}(w + i\xi)^{-1}\right),\end{aligned}$$

where $\pm$ is the sign of $\tau \in \mathbb{R}$. Clearly, $\mathrm{Re}(z^{-1}) \geq 0$ if and only if $\mathrm{Re}(z) \geq 0$. The number $z_0 = \hat{a}(w + i\xi))(w + i\xi)$ belongs to $\Sigma_\sigma$ by assumption (3). Since $\mathrm{e}^{\pm i\alpha\pi/2} = \mathrm{e}^{\pm i\beta}$, the condition $\sigma + \beta < \pi/2$ implies that $\mathrm{e}^{\pm i\alpha\pi/2} z_0$ has a nonnegative real part. Hence, $\mathrm{Re}\,\widehat{s}_{w,(i\tau)^\alpha}(i\xi)$ is nonnegative as required.

*Step 4: Conclusion.* Since the resolvent $S(t) := \mathrm{e}^{wt} S_{w,A}(t)$ exists, the solution $u$ of (3.1) is given by (3.2). Now as $\tilde{S}_{w,A}$ is positive definite, Proposition 3.4 shows that $u$ has a version with the required properties. $\square$

*REMARK* 3.6. In Theorem 1 of [28], it is assumed that $A$ is self-adjoint and nonpositive. In this paper, the crucial property of the kernel $a$ is the inequality $\mathrm{Re}(\lambda\hat{a}(\lambda)) \geq 0$ for all $\lambda \in \mathbb{C}_+$ with $\mathrm{Re}(\lambda) \geq w$ for some $w$, which corresponds to $\sigma = \pi/2$ in our theorem. This sharp case is needed for the kernel $a(t) = t$ which leads to second-order Cauchy problems such as the wave equation. We cannot treat this case since we have to slightly enlarge sectors when working with the $H^\infty$-calculus instead of the functional calculus for self-adjoint operators.

However, if one wants to use our Lemma 3.2 to extend Theorem 1 of [28] to self-adjoint operators with $\langle Ax, x\rangle \leq \rho\|x\|^2$ for $x \in D(A)$ and some $\rho > 0$, then one has to impose the slightly stronger sector condition $\lambda\hat{a}(\lambda) \in \Sigma_{\frac{\pi}{2}-\varepsilon}$ for some $\varepsilon > 0$. In fact, by Step 1 of our proof the shifting procedure requires this extra angle.

*REMARK* 3.7. Let $H$ be a separable Hilbert space and let $W_H$ be a cylindrical Brownian motion on $H$. An important special case is given by $u_0 = 0$ and

$$L(t) = \int_0^t g\, dW_H,$$

where $g \in L^2_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{L}_2(H, X))$ a.s. is measurable and adapted. This process $L$ is a continuous local martingale. If the conditions of Theorem 3.3 hold, then the solution $u$ has a version with continuous paths. Moreover, the solution formula (3.2) and the Burkholder–Davis–Gundy estimate imply that

$$\left(\mathbb{E}\left(\sup_{t\in[0,T]} \|u(t)\|^p\right)\right)^{1/p} \leq C\|g\|_{L^p(\Omega; L^2(0,T;\mathcal{L}_2(H,X)))}$$

for every $T < \infty$ and $p \in (0, \infty)$, whenever the right-hand side is finite. (See also [19,20].) Here, $C$ is a constant independent of $g$. In [19,20], it has been shown how one can use this result to obtain results on exponential integrability of $\sup_{t\in[0,T]} \|u(t)\|^2$, and their methods extend to our setting.

## 4. Applications

### 4.1. Examples of kernels $a$

In this section, we present examples of kernels which satisfy the conditions of Theorem 3.3. Examples of sectorial operators $A$ with an $H^\infty$-calculus have been given in Sect. 2.2. We start with the arguably most prominent class of scalar kernels $a$.

EXAMPLE 4.1. Let $a : (0, \infty) \to \mathbb{R}$ be given by $a(t) = t^{\beta-1}/\Gamma(\beta)$ with $\beta \in (0, 2)$. Assume that the operator $A$ is sectorial with

$$\phi_{A-\rho} < \min\left\{\tfrac{\pi}{2}(2-\beta), \tfrac{\pi}{2}\beta\right\}$$

for some $\rho$ and that $-(A-\rho)$ has an $H^\infty$-calculus. Then, the conditions of Theorem 3.3 are fulfilled with $w = 0$.

To check this claim, let $\lambda \in \mathbb{C}_+$. Since $\hat{a}(\lambda) = \lambda^{-\beta}$, we can compute $|\lambda \hat{a}'(\lambda)| = \beta|\lambda|^{-\beta} = \beta|\hat{a}(\lambda)|$. Moreover, $\lambda\hat{a}(\lambda) = \lambda^{1-\beta}$ belongs to $\Sigma_\sigma$ and $\hat{a}(\lambda) = \lambda^{-\beta}$ to $\Sigma_{\pi-\phi}$ with $\sigma = |\beta - 1|\frac{\pi}{2}$ and $\phi = \frac{\pi}{2}(2-\beta)$. Our assumption on $\phi_{A-\rho}$ then yields $\phi_{A-\rho} + \sigma < \pi/2$ and $\phi > \phi_{A-\rho}$.

We add a basic example from viscoelasticity discussed in Section 5.2 of [29].

EXAMPLE 4.2. Let $a(t) = \nu + \mu t$ with $\nu, \mu > 0$ be the kernel arising in a Kelvin–Voigt solid. Let $A$ be sectorial with $\phi_A < \pi/2$ and let $-A$ possess an $H^\infty$-calculus. We show the conditions of Theorem 3.3.

Let $\lambda \in \mathbb{C}_+$ with $\operatorname{Re}\lambda > w$. We first observe that $\hat{a}(\lambda) = \frac{\nu}{\lambda} + \frac{\mu}{\lambda^2}$. It suffices to check (3). One has $|\lambda\hat{a}'(\lambda)| \le 2\,|\hat{a}(\lambda)|$ for any choice $w \ge 0$. Take $\sigma > 0$ with $\phi_A + \sigma < \pi/2$ and set $\phi = \frac{\pi}{2} - \sigma > \phi_A$. Notice that $\lambda\hat{a}(\lambda) = \nu + \frac{\mu}{\lambda}$ belongs to $\nu + (B(0, \mu/w) \cap \mathbb{C}_+)$. Hence, $\lambda\hat{a}(\lambda) \in \Sigma_\sigma$ for a fixed sufficiently large $w$. This fact then implies that $\hat{a}(\lambda) \in \Sigma_{\sigma+\pi/2} = \Sigma_{\pi-\phi}$.

Our final example cannot be treated within our setting.

EXAMPLE 4.3. Let $a(t) = t$. Then $\hat{a}(\lambda) = \frac{1}{\lambda^2}$ and so $\lambda\hat{a}(\lambda) = \lambda^{-1}$. Hence, we have to take $\sigma = \pi/2$, which contradicts the assumption $\phi_A + \sigma < \pi/2$.

### 4.2. Illustration

In this section, we present an example of a stochastic Volterra equation with all details. This is an illustration how the results from the previous sections can be combined. One can easily treat much larger classes of examples. We study the equation

$$u(t) = u_0 + \int_0^t a(t-s)Au(s)\,\mathrm{d}s + L(t). \tag{4.1}$$

with $a(t) = t^{\beta-1}/\Gamma(\beta)$ for any fixed $\beta \in (0, 2)$ and

$$Au = \sum_{m,n=1}^{d} a_{m,n} u_{x_m,x_n} + \sum_{n=1}^{d} b_n u_{x_n} + cu.$$

We assume that

- $b_n, c \in L^\infty(\mathbb{R}^d)$,
- $a_{m,n} \in C_b^\varepsilon(\mathbb{R}^d)$ for some $\varepsilon > 0$,
- $a_{mn} = a_{nm}$ are real valued and $\sum_{m,n=1}^N a_{m,n}(x)\xi_n\xi_m \ge \delta|\xi|^2$.

Let $D(A) = H^2(\mathbb{R}^d)$. The next result follows from Theorem 3.3.

PROPOSITION 4.4. *Assume the above conditions and that $L$ is a local $L^2$-martingale with càdlàg/continuous paths almost surely. Then,* (4.1) *has a unique weak solution $u$ and $u$ has a modification with càdlàg/continuous trajectories almost surely.*

*Proof.* Theorem 13.13 of [23] shows that $\lim_{\rho\to\infty}\phi_{A-\rho} = 0$ and that $-(A-\rho)$ has a bounded $H^\infty$-calculus for all sufficiently large $\rho$. We choose $\rho \geq 0$ so that $\phi_{A-\rho} < \min\{\frac{\pi}{2}(2-\beta), \frac{\pi}{2}\beta\}$. Setting $\sigma = |\beta-1|\frac{\pi}{2}$ and $\phi = \frac{\pi}{2}(2-\beta)$, the conditions of Theorem 3.3 hold due to Example 4.1. □



## REFERENCES

[1] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and Ph. Tchamitchian. The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$. *Ann. of Math., (2)*, 156(2):633–654, 2002.

[2] B. Baeumer, M. Geissert, and M. Kovács. Existence, uniqueness and regularity for a class of semilinear stochastic Volterra equations with multiplicative noise. *J. Differential Equations*, 258(2):535–554, 2015.

[3] S. Bonaccorsi and G. Desch. Volterra equations in Banach spaces with completely monotone kernels. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):557–594, 2013.

[4] Z. Brzeźniak. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastics Rep.*, 61(3–4):245–295, 1997.

[5] Z.-Q. Chen, K.-H. Kim, and P. Kim. Fractional time stochastic partial differential equations. *Stochastic Process. Appl.*, 125(4):1470–1499, 2015.

[6] P. Clément and G. Da Prato. Some results on stochastic convolutions arising in Volterra equations perturbed by noise. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 7(3):147–153, 1996.

[7] P. Clément and G. Da Prato. White noise perturbation of the heat equation in materials with memory. *Dynam. Systems Appl.*, 6(4):441–460, 1997.

[8] P. Clément and G. Da Prato. Stochastic convolutions with kernels arising in some Volterra equations. In *Volterra equations and applications (Arlington, TX, 1996)*, volume 10 of *Stability Control Theory Methods Appl.*, pages 57–65. Gordon and Breach, Amsterdam, 2000.

[9] P. Clément, G. Da Prato, and J. Prüss. White noise perturbation of the equations of linear parabolic viscoelasticity. *Rend. Istit. Mat. Univ. Trieste*, 29(1–2):207–220 (1998), 1997.

[10] M. Cowling, I. Doust, A. McIntosh, and A. Yagi. Banach space operators with a bounded $H^\infty$ functional calculus. *J. Austral. Math. Soc. Ser. A*, 60(1):51–89, 1996.

[11] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[12] R. Denk, G. Dore, M. Hieber, J. Prüss, and A. Venni. New thoughts on old results of R. T. Seeley. *Math. Ann.*, 328(4):545–583, 2004.

[13] G. Desch and S.-O. Londen. Maximal regularity for stochastic integral equations. *J. Appl. Anal.*, 19(1):125–140, 2013.

[14] W. Desch and S.-O. Londen. A generalization of an inequality by N. V. Krylov. *J. Evol. Equ.*, 9(3):525–560, 2009.

[15] J. Dettweiler, L.W. Weis, and J.M.A.M. van Neerven. Space-time regularity of solutions of the parabolic stochastic Cauchy problem. *Stoch. Anal. Appl.*, 24(4):843–869, 2006.

[16] X.T. Duong and G. Simonett. $H_\infty$-calculus for elliptic operators with nonsmooth coefficients. *Differential Integral Equations*, 10(2):201–217, 1997.

[17] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations*, volume 34 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990.

[18] M.H.A. Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[19] E. Hausenblas and J. Seidler. A note on maximal inequality for stochastic convolutions. *Czechoslovak Math. J.*, 51(126)(4):785–790, 2001.
[20] E. Hausenblas and J. Seidler. Stochastic convolutions driven by martingales: maximal inequalities and exponential integrability. *Stoch. Anal. Appl.*, 26(1):98–119, 2008.
[21] O. Kallenberg. *Foundations of Modern Probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
[22] A. Karczewska. *Convolution type stochastic Volterra equations*, volume 10 of *Lecture Notes in Nonlinear Analysis*. Juliusz Schauder Center for Nonlinear Studies, Toruń, 2007.
[23] P.C. Kunstmann and L.W. Weis. Maximal $L_p$-regularity for parabolic equations, Fourier multiplier theorems and $H^\infty$-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.
[24] A. McIntosh. Operators which have an $H_\infty$ functional calculus. In *Miniconference on operator theory and partial differential equations (North Ryde, 1986)*, volume 14 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 210–231. Austral. Nat. Univ., Canberra, 1986.
[25] M. Métivier and J. Pellaumail. *Stochastic integration*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1980.
[26] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Stochastic maximal $L^p$-regularity. *Ann. Probab.*, 40:788–812, 2012.
[27] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. Maximal $\gamma$-regularity. *J. Evol. Equ.*, 15(2):361–402, 2015.
[28] S. Peszat and J. Zabczyk. Time regularity for stochastic Volterra equations by the dilation theorem. *J. Math. Anal. Appl.*, 409(2):676–683, 2014.
[29] J. Prüss. *Evolutionary integral equations and applications*, volume 87 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1993.
[30] B.L. Rozovskiĭ. *Stochastic evolution systems*, volume 35 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1990. Linear theory and applications to nonlinear filtering, Translated from the Russian by A. Yarkho.
[31] J. Seidler. Exponential estimates for stochastic convolutions in 2-smooth Banach spaces. *Electron. J. Probab.*, 15:no. 50, 1556–1573, 2010.
[32] B. Sz.-Nagy, C. Foias, H. Bercovici, and L. Kérchy. *Harmonic analysis of operators on Hilbert space*. Universitext. Springer, New York, second edition, 2010.
[33] M.C. Veraar. Non-autonomous stochastic evolution equations and applications to stochastic partial differential equations. *J. Evol. Equ.*, 10(1):85–127, 2010.
[34] M.C. Veraar and J. Zimmerschied. Non-autonomous stochastic Cauchy problems in Banach spaces. *Studia Math.*, 185(1):1–34, 2008.
[35] M.C. Veraar and L.W. Weis. A note on maximal estimates for stochastic convolutions. *Czechoslovak Math. J.*, 61 (136):743–758, 2011.

*Roland Schnaubelt*
*Department of Mathematics*
*Karlsruhe Institute of Technology*
*76128 Karlsruhe, Germany*
*E-mail: schnaubelt@kit.edu*

*Mark Veraar*
*Delft Institute of Applied Mathematics*
*Delft University of Technology*
*P.O. Box 5031*
*2600 GA Delft, The Netherlands*
*E-mail: M.C.Veraar@tudelft.nl*