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Aequationes Mathematicae

Orthogonally additive bijections are additive

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Abstract. Any orthogonally additive injection of a real inner product space of dimension at least 2 onto an abelian group is additive.

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Let E be a real inner product space of dimension at least 2.

A function f mapping E into an abelian group is called orthogonally additive, if

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in E \text{ with } x \perp y.$$

It is well known, see [3, Corollary 10] and [2, Theorem 1], that every orthogonally additive function f defined on E has the form

$$f(x) = a(\|x\|^2) + b(x) \quad \text{for } x \in E, \tag{1}$$

where a and b are additive functions uniquely determined by f.

According to [1, Theorem 1], if $f : E \to E$ is orthogonally additive and $f(f(x)) = x$ for $x \in E$, then f is additive. We generalize this theorem and [1, Remark 3] as follows.

Theorem. *Any orthogonally additive injection of E onto an abelian group is additive.*

Proof. Assume f is an orthogonally additive injection of E onto an abelian group G and let $a : \mathbb{R} \to G$ and $b : E \to G$ be additive functions such that (1) holds. Suppose that $a \neq 0$. □

Consider first the case where

$$a(\mathbb{R}) \cap b(E) = \{0\}, \tag{2}$$

fix $y \in a(\mathbb{R})\backslash\{0\}$ and put $x = f^{-1}(y)$. Since, according to (1),

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$$b(x) = y - a(||x||^2) \in a(\mathbb{R}),$$

it follows from (2) that $b(x) = 0$. Consequently $f(x) = a(||x||^2)$ and

$$f(-x) = a(||x||^2) - b(x) = a(||x||^2) = f(x)$$

which contradicts the injectivity of f.

Assume now that (2) does not hold and fix $y \in a(\mathbb{R}) \cap b(E)\backslash\{0\}$. Then $y = a(\alpha)$ for $\alpha \in \mathbb{R}$ and $y = b(x)$ for $x \in E\backslash\{0\}$ and there is u in E such that $(x|u) = -\alpha$. Putting

$$v = \frac{u-x}{2}, \quad w = \frac{u+x}{2}$$

we see that $v \neq w$ and

$$\begin{aligned} a(||v||^2) - a(||w||^2) &= a(||v||^2 - ||w||^2) = a((v-w|v+w)) = a((-x|u)) \\ &= a(\alpha) = y = b(x) = b(w-v) = b(w) - b(v), \end{aligned}$$

whence

$$f(v) = a(||v||^2) + b(v) = a(||w||^2) + b(w) = f(w)$$

which contradicts the injectivity of f and ends the proof.

Remark 1 from [1] provides an example of an orthogonally additive function $f : E \to E$ which is injective and not additive and Remark 2 from [1] provides an example of an orthogonally additive function which maps E onto E and is not additive.

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