Signed Degree Sequences of Signed Graphs*

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Abstract: This paper gives necessary and sufficient conditions for an integral sequence to be the signed degree sequence of a signed graph or a signed tree, answering a question raised by Chartrand et al. (1994). (G. Chartrand, H. Gavlas, F. Harary, and M. Schultz, On signed degrees in signed graphs, *Czech. Math. J.* **44** (1994), 677–690). J Graph Theory **26:** 111–117, 1997

1. INTRODUCTION

All graphs in this paper, except those discussed in the last section, are finite, undirected, without loops and multiple edges.

The concept of a *signed graph* was first introduced by Harary [4]. A signed graph is a graph in which every edge is labeled with a ''$+$'' or a ''$-$''. An edge uv labeled with a ''$+$'' (respectively, ''$-$'') is called a *positive edge* (respectively, *negative edge*) and is denoted by uv^+ (respectively, uv^-). In a signed graph $G=(V,E)$,

the *positive degree* of a vertex u is $deg^+(u) = |\{uv : uv^+ \in E\}|$,
the *negative degree* of u is $deg^-(u) = |\{uv : uv^- \in E\}|$,
the *signed degree* of u is $sdeg(u) = deg^+(u) - deg^-(u)$, and
the *degree* of u is $deg(u) = deg^+(u) + deg^-(u)$.

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An integral sequence $\sigma : d_1, d_2, \ldots, d_p$ is the *signed degree sequence* (respectively *degree sequence*) of a signed graph (respectively graph) $G = (V, E)$ if $V = \{v_1, v_2, \ldots, v_p\}$ and $sdeg(v_i) = d_i$ (respectively $deg(v_i) = d_i$) for $1 \leq i \leq p$.

Chartrand et al. [2] initiated a systematic study of signed degrees of signed graphs. They gave characterizations of signed degree sequences of signed paths, signed stars, signed double stars, and complete signed graphs. They also gave a necessary and sufficient condition under which an integral sequence is a signed degree sequence similar to Hakimi's result for degree sequences [3]. They questioned whether Hakimi's procedure for degree sequences also works for signed degree sequences. In Section 2, we answer this question with a modification of Hakimi's procedure.

It is well-known that a sequence of $n \geq 2$ positive integers is the degree sequence of a tree if and only if their sum is $2n - 2$ ([1], p. 27). Section 3 characterizes signed degree sequences of signed trees. In Section 4, signed degree sequences for signed graphs with loops or multiple edges are discussed.

2. GENERAL SIGNED GRAPHS

This section gives a new characterization of signed degree sequences that yields an efficient algorithm for recognizing signed degree sequences.

An integral sequence is *s-graphical* if it is the signed degree sequence of a signed graph. An integral sequence $\sigma : d_1, d_2, \ldots, d_p$ is *standard* if $p - 1 \geq d_1 \geq d_2 \geq \cdots \geq d_p$ and $d_1 \geq |d_p|$. The following obvious lemma demonstrates that a signed degree sequence can be modified and rearranged into an equivalent standard form.

Lemma 1. *If $\sigma : d_1, d_2, \ldots, d_p$ is the signed degree sequence of a signed graph G, then $-\sigma : -d_1, -d_2, \ldots, -d_p$ is the signed degree sequence of the signed graph G' obtained from G by interchanging positive edges with negative edges.*

The most important result in Chartrand et al. [2] is the following necessary and sufficient condition under which an integral sequence is s-graphical.

Theorem 2. *A standard integral sequence $\sigma : d_1, d_2, \ldots, d_p$ is s-graphical if and only if*

$$\sigma' : d_2 - 1, \ldots, d_{d_1+s+1} - 1, d_{d_1+s+2}, \ldots, d_{p-s}, d_{p-s+1} + 1, \ldots, d_p + 1$$

is s-graphical for some $s, 0 \leq s \leq (p - 1 - d_1)/2$.

Note that Hakimi's theorem for degree sequence is the particular case $s = 0$ of Theorem 2. This leads to an efficient algorithm for recognizing the degree sequence of a graph. However, the wide degree of latitude for choosing s in Theorem 2 makes it harder to devise an efficient algorithmic implementation. Chartrand et al. conjectured that Theorem 2 is also true for signed degree sequences if $s = 0$. This is not an unreasonable proposition since signed degree sequences are closely related to degree sequences. For instance, one can make the following correspondence between signed degree sequences and degree sequences. Suppose $G = (V, E)$ is a complete signed graph. Consider the graph $G' = (V, E')$ for which $E' = \{uv : uv^+ \in E\}$. Then $deg_{G'}(u) = (|V| + sdeg_G(u) - 1)/2$ for all $u \in V$. So an integral sequence $\sigma : d_1, d_2, \ldots, d_p$ is the signed degree sequence of a complete signed graph if and only if $\sigma' : (p + d_1 - 1)/2, (p + d_2 - 1)/2, \ldots, (p + d_p - 1)/2$ is the degree sequence of a graph.

However, the following is a counterexample to the conjecture of Chartrand et al. Let G be the signed graph in Figure 1. An edge is negative if and only if it is a thick segment. The signed degree sequence of G is $4, 4, 4, 4, 4, 4, -4, -4, -4$. On the other hand, sequential application of

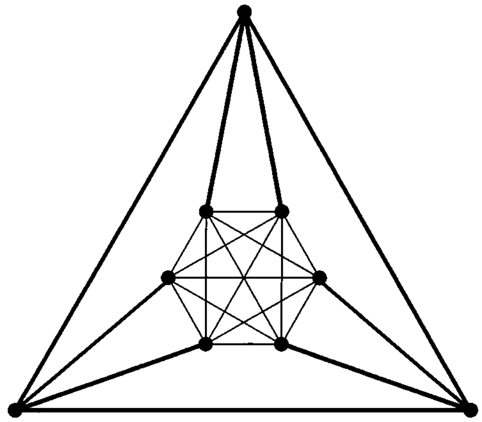

FIGURE 1. A signed graph G of 9 vertices.

Hakimi's procedure and standardization leads to the following list.

$$\begin{aligned}
&\sigma_1\colon\ 4,4,4,4,4,4,-4,-4,-4;\\
&\sigma_1'\colon\ 3,3,3,3,4,-4,-4,-4;\\
&\sigma_2\colon\ 4,3,3,3,3,-4,-4,-4;\\
&\sigma_2'\colon\ 2,2,2,2,-4,-4,-4;\\
&\sigma_3\colon\ 4,4,4,-2,-2,-2,-2;\\
&\sigma_3'\colon\ 3,3,-3,-3,-2,-2;\\
&\sigma_4\colon\ 3,3,-2,-2,-3,-3;\\
&\sigma_4'\colon\ 2,-3,-3,-3,-3;\\
&\sigma_5\colon\ 3,3,3,3,-2;\\
&\sigma_5'\colon\ 2,2,2,-2;\\
&\sigma_6\colon\ 2,2,2,-2;\\
&\sigma_6'\colon\ 1,1,-2;\\
&\sigma_7\colon\ 2,-1,-1;\\
&\sigma_7'\colon\ -2,-2.
\end{aligned}$$

Note that σ_7' is not a signed degree sequence.

The following theorem provides a good candidate for parameter s in Theorem 2. It leads to a polynomial-time algorithm for recognizing signed degree sequences.

Theorem 3. *A standard integral sequence $\sigma : d_1, d_2, \ldots, d_p$ is s-graphical if and only if*

$$\sigma_m'\colon d_2-1,\ldots,d_{d_1+m+1}-1,d_{d_1+m+2},\ldots,d_{p-m},d_{p-m+1}+1,\ldots,d_p+1$$

is s-graphical, where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{p-m+1}$.

Proof. Suppose σ is the signed degree sequence of a signed graph $G=(V,E)$ with $V=\{v_1,v_2,\ldots,v_p\}$ and $sdeg(v_i)=d_i$ for $1\le i\le p$. For each $s, 0\le s\le (p-1-d_1)/2$, consider the sequence

$$\sigma_s'\colon d_2-1,\ldots,d_{d_1+s+1}-1,d_{d_1+s+2},\ldots,d_{p-s},d_{p-s+1}+1,\ldots,d_p+1.$$

By Theorem 2, σ_s' is s-graphical for some s. We may choose s such that $|s-m|$ is minimum. Suppose $G'=(V',E')$ is a signed graph with $V'=\{v_2,v_3,\ldots,v_p\}$ whose signed degree sequence is σ_s'.

If $s < m$, then $d_a > d_b$ by the choice of m, where $a = d_1 + s + 2$ and $b = p - s$. Since $d_a > d_b$, there exists some vertex v_k of G' different from v_a and v_b that satisfies one of the following conditions:

(1) $v_a v_k^+$ is a positive edge and $v_b v_k^-$ is a negative edge.
(2) $v_a v_k^+$ is a positive edge and v_b is not adjacent to v_k.
(3) v_a is not adjacent to v_k and $v_b v_k^-$ is a negative edge.

For (1), remove $v_a v_k^+$ and $v_b v_k^-$ from G'; for (2), remove $v_a v_k^+$ from G' and add a new positive edge $v_b v_k^+$ to G'; for (3), remove $v_b v_k^-$ from G' and add a new negative edge $v_a v_k^-$ to G'. These modifications result in a signed graph G'' whose signed degree sequence is σ'_{s+1}. This contradicts the minimality of $|s - m|$.

If $s > m$, then $d_{d_1+s+1} = d_{p-s+1}$ and so $d_{d_1+s+1} - 1 < d_{p-s+1} + 1$. An argument similar to that above leads to a contradiction in the choice of s. Therefore, $s = m$ and σ'_m is s-graphical.

Conversely, suppose σ'_m is the signed degree sequence of a signed graph $G' = (V', E')$ in which $V' = \{v_2, v_3, \ldots, v_p\}$. If G is the signed graph obtained from G' by adding a new vertex v_1 and new positive edges $v_1 v_i^+$ for $2 \le i \le d_1 + m + 1$, and new negative edges $v_1 v_j^-$ for $p - m + 1 \le j \le p$, then σ is the signed degree sequence of G. ∎

3. SIGNED TREES

In this section, we study the signed degree sequences of signed trees. We give characterizations of integral sequences that are signed degree sequences of signed trees.

To give necessary conditions for the signed degree sequence of a signed tree, we in fact establish properties for general signed graphs in Lemmas 4 to 6. For Lemmas 4 to 6, we assume that $G = (V, E)$ is a signed graph of p vertices and q edges. We use q^+ and q^- to denote respectively the numbers of positive edges and negative edges of G. We also use p_+, p_0, and p_- to denote respectively the numbers of vertices with positive, zero, and negative signed degrees.

Lemma 4. (Chartrand et al.). *If $G = (V, E)$ is a signed graph, then $k = \sum_{v \in V} sdeg(v) \equiv 2q \pmod 4$, $q^+ = \frac{1}{4}(2q + k)$, and $q^- = \frac{1}{4}(2q - k)$.*

Lemma 5. *For any signed graph $G = (V, E)$ without isolated vertices, $\sum_{v \in V} |sdeg(v)| + 2p_0 \le 2q$.*

Proof. First, each $|sdeg(v)| = |deg^+(v) - deg^-(v)| \le deg^+(v) + deg^-(v)$. Since G has no isolated vertices, $2 \le deg^+(v) + deg^-(v)$ when $sdeg(v) = 0$. Therefore,

$$\sum_{v \in V} |sdeg(v)| + 2p_0 \le \sum_{v \in V} (deg^+(v) + deg^-(v)) = 2q^+ + 2q^- = 2q.$$

∎

Lemma 6. *For any connected signed graph $G = (V, E)$, $\sum_{v \in V} |sdeg(v)| + 2 \sum_{sdeg(v) < 0} |sdeg(v)| \le 6q + 4 - 4\delta - 4p_+ - 4p_0$, where $\delta = 1$ if $p_+ p_- > 0$ and $\delta = 0$ otherwise.*

Proof. Consider the subgraph $G' = (V', E')$ of G induced by those edges incident to vertices with non-negative signed degrees. We have

$$\sum_{sdeg(v) > 0} |sdeg(v)| \le 2(\text{number of positive edges in } G') - (\text{number of negative edges in } G') \le 3q^+ - |E'|.$$

Since G is connected, each component of G' contains at least one vertex of negative signed degree except for the case of $G' = G$. Therefore, $p_+ + p_0 - 1 + \delta \leq |E'|$. And so,

$$\sum_{sdeg(v)>0} |sdeg(v)| + p_+ + p_0 - 1 + \delta \leq 3q^+ = 3\left(\frac{1}{2}q + \frac{1}{4}\sum_{v\in V} sdeg(v)\right).$$

Therefore

$$\sum_{v\in V} |sdeg(v)| + 2\sum_{sdeg(v)<0} |sdeg(v)| \leq 6q + 4 - 4\delta - 4p_+ - 4p_0.$$

∎

Theorem 7. *Suppose $\sigma : d_1, d_2, \ldots, d_p$ is an integral sequence of $p \geq 2$ terms. Suppose σ has p_+ positive terms, p_0 zero terms, and p_- negative terms. Let $\delta = 1$ if $p_+p_- > 0$ and $\delta = 0$ otherwise. Then σ is the signed degree sequence of a signed tree if and only if* (T1) *to* (T4) *hold.*

(T1) $\sum_{i=1}^{p} d_i \equiv 2p - 2 (\bmod 4)$.
(T2) $\sum_{i=1}^{p} |d_i| \leq 2p - 2 - 2p_0$.
(T3) $\sum_{i=1}^{p} |d_i| + 2\sum_{d_i<0} |d_i| \leq 2p - 2 - 4\delta + 4p_-$.
(T4) $\sum_{i=1}^{p} |d_i| + 2\sum_{d_i>0} |d_i| \leq 2p - 2 - 4\delta + 4p_+$.

Proof. Note that Condition (T4) for σ is the same as Condition (T3) for $-\sigma$. The necessity of the theorem follows from the fact that $q = p - 1$, and Lemmas 4 to 6.

We shall prove the sufficiency by induction on p. For $p = 2$, by (T1) and (T2), $d_1 = d_2 = 1$ or -1. So σ is the signed degree sequence of K_2 with a positive edge or a negative edge. Suppose the theorem is true for $p - 1$. Now consider the case of $p \geq 3$.

By (T2), σ has at least two terms in which $|d_i| = 1$. After rearranging the terms in σ or taking $-\sigma$, we may assume without loss of generality that $d_p = 1$ and one of the following holds.

(1) $|d_i| \leq 1$ for $1 \leq i \leq p, d_1 \geq 0$, and $d_1 = 0$ if $p_0 > 0$.
(2) $d_1 \geq 2$.
(3) $d_i \leq 1$ but $d_i \neq -1$ for $1 \leq i \leq p$ and $d_1 = 0$ and $\delta = 1$.
(4) $d_i = 1$ or $d_i \leq -2$ for $1 \leq i \leq p$ and $d_1 = \delta = 1$.

For any of the above, we consider the sequence $\sigma' : d'_1, d'_2, \ldots, d'_{p'}$, where $p' = p - 1$ and $d'_1 = d_1 - 1$ and $d'_i = d_i$ for $2 \leq i \leq p - 1$.

Note that $\sum_{i=1}^{p'} d'_i = (\sum_{i=1}^{p} d_i) - 2 \equiv (2p - 2) - 2 \equiv 2p' - 2$ (mod 4), i.e., (T1) holds for σ'. We shall check Conditions (T2) to (T4) for σ' according to the four cases above.

Case 1. $|d_i| \leq 1$ for $1 \leq i \leq p, d_1 \geq 0$, and $d_1 = 0$ if $p_0 > 0$. In this case, since $|d'_i| \leq 1$ for $1 \leq i \leq p - 1$, we have

$$\sum_{i=1}^{p'} |d'_i| = p'_+ + p'_-, \sum_{d'_i>0} |d'_i| = p'_+, \sum_{d'_i<0} |d'_i| = p'_-.$$

Thus, (T2) to (T4) hold for σ' as $p'_+ + p'_- \geq 2$.

Case 2. $d_1 \geq 2$. In this case, since $d_1 \geq 2$ and $d_p = 1$, we have

$$p' = p - 1,\ p'_+ = p_+ - 1,\ p'_0 = p_0,\ p'_- = p_-,\ \delta' = \delta,$$

$$\sum_{i=1}^{p'} |d'_i| = \sum_{i=1}^{p} |d_i| - 2, \sum_{d'_i>0} |d'_i| = \sum_{d_i>0} |d_i| - 2, \sum_{d'_i<0} |d'_i| = \sum_{d_i<0} |d_i|.$$

(T2) to (T4) for σ imply that (T2) to (T4) for σ' hold.

Case 3. $d_i \le 1$ but $d_i \ne -1$ for $1 \le i \le p$ and $d_1 = 0$ and $\delta = 1$. In this case, since $d_1 = 0$ and $d_p = 1$, we have

$$p' = p - 1,\ p'_+ = p_+ - 1,\ p'_0 = p_0 - 1,\ p'_- = p_- + 1,\ \delta' \le \delta,$$

$$\sum_{i=1}^{p'} |d'_i| = \sum_{i=1}^{p} |d_i|,\ \sum_{d'_i>0} |d'_i| = \sum_{d_i>0} |d_i| - 1,\ \sum_{d'_i<0} |d'_i| = \sum_{d_i<0} |d_i| + 1.$$

(T2) and (T3) for σ imply that (T2) and (T3) for σ' hold. Since $d'_i \le 1$ for $1 \le i \le p - 1, \sum_{d'_i>0} |d'_i| = p'_+$. By (T3) for σ and the fact that $d_i \le -2$ when $d_i < 0$,

$$p_+ + 6p_- \le \sum_{i=1}^{p} |d_i| + 2\sum_{d_i<0} |d_i| \le 2p - 2 - 4\delta + 4p_- = 2p_+ + 2p_0 + 6p_- - 6$$

and so $6 \le p_+ + 2p_0$. Therefore, $3 \le p'_+ + 2p'_0$ and then $4 \le 2p'_+ + 2p'_0$. This together with (T2) for σ' and $\sum_{d'_i>0} |d'_i| = p'_+$ implies (T4) for σ'.

Case 4. $d_i = 1$ or $d_i \le -2$ for $1 \le i \le p$ and $d_1 = \delta = 1$. In this case, since $d_1 = d_p = 1$, we have

$$p' = p - 1,\ p'_+ = p_+ - 2,\ p'_0 = p_0 + 1 = 1,\ p'_- = p_-,\ \delta' \le \delta,$$

$$\sum_{i=1}^{p'} |d'_i| = \sum_{i=1}^{p} |d_i| - 2,\ \sum_{d'_i>0} |d'_i| = \sum_{d_i>0} |d_i| - 2,\ \sum_{d'_i<0} |d'_i| = \sum_{d_i<0} |d_i|.$$

(T3) for σ implies that (T3) for σ' holds. As in the argument for Case 3, we have $\sum_{d'_i>0} |d'_i| = p'_+$ and $6 \le p_+ + 2p_0$. Therefore, $4 \le p'_+$. Adding $2\sum_{d'_i>0} |d'_i| = 2p'_+$ to the equality in (T3) for σ' and dividing the resulting equality by 3, we get (T2) for σ' as $2p'_0 \le 2p'_+$. Adding $2\sum_{d'_i>0} |d'_i| = 2p'_+$ to the equality in (T2) for σ', we get (T4) for σ' as $4\delta' \le 2p'_0 + 2p'_+$.

From the above discussions, σ' satisfies (T1) to (T4). By the induction hypothesis, there exists a signed tree T' with the vertex set $\{v_1, v_2, \ldots, v_{p-1}\}$ and $sdeg_{T'}(v_i) = d'_i$ for $1 \le i \le p - 1$. Suppose T is the signed tree obtained from T' by adding a new vertex v_p and a new positive edge $v_1v_p^+$, then T has signed degree sequence σ. ∎

Corollary 8. *Suppose $\sigma : d_1, d_2, \ldots, d_p$ is an integral sequence of $p \ge 3$ terms. Suppose σ has at least two terms in which $|d_i| = 1, d_p = 1$, and one of the following conditions holds*:

(1) $|d_i| \le 1$ *for* $1 \le i \le p, d_1 \ge 0,$ *and* $d_1 = 0$ *if* $p_0 > 0$.

(2) $d_1 \ge 2$.

(3) $d_i \le 1$ *but* $d_i \ne -1$ *for* $1 \le i \le p$ *and* $d_1 = 0$ *and* $\delta = 1$.

(4) $d_i = 1$ *or* $d_i \le -2$ *for* $1 \le i \le p$ *and* $d_1 = \delta = 1$.

Then σ is the signed degree sequence of a signed tree if and only if $\sigma' : d_1 - 1, d_2, \ldots, d_{p-1}$ is the signed degree sequence of a signed tree.

4. CONCLUDING REMARKS

The main results of this paper are conditions under which an integral sequence is the signed degree sequence of a signed graph or a signed tree. We can also get results for signed graphs with loops or multiple edges.

In the proofs of the following theorems, for any integer k, by ''k copies of v_iv_j'' we mean ''k copies of positive edges $v_iv_j^+$'' when $k > 0$, ''no edges'' when $k = 0$, and ''$-k$ copies of negative edges $v_iv_j^-$'' when $k < 0$.

Theorem 9. *An integral sequence $d_1, d_2, \ldots, d_p$ is the signed degree sequence of a signed graph with loops if and only if $\sum_{i=1}^{p} d_i$ is even.*

Proof. The necessity follows from Lemma 4, the sufficiency from the observation that since $\sum_{i=1}^{p} d_i$ is even, the number of odd terms is even. Say $d_i = 2e_i + 1$ for $1 \le i \le 2k$ and $d_i = 2e_i$ for $2k + 1 \le i \le p$. Then $d_1, d_2, \ldots, d_p$ is the signed degree sequence of the signed graph with vertex set $\{v_1, v_2, \ldots, v_p\}$ and edge set

$$\{v_iv_{i+k}^+ \colon 1 \le i \le k\} \cup \{e_i \text{ copies } v_iv_i \colon 1 \le i \le p\}.$$

∎

Theorem 10. *For $p \ge 3$, an integral sequence $d_1, d_2, \ldots, d_p$ is the signed degree sequence of a signed graph with multiple edges if and only if $\sum_{i=1}^{p} d_i$ is even.*

Proof. The necessity follows from Lemma 4, the sufficiency from the observation that since $\sum_{i=1}^{p} d_i$ is even, $d_1, d_2, \ldots, d_p$ is the signed degree sequence of the signed graph with vertex set $\{v_1, v_2, \ldots, v_p\}$ and edge set

$$\begin{aligned}
&\left\{-d_3 + \frac{1}{2}\sum_{i=1}^{p} d_i \text{ copies of } v_1v_2\right\} \\
&\quad \cup \left\{d_2 + d_3 - \frac{1}{2}\sum_{i=1}^{p} d_i \text{ copies of } v_2v_3\right\} \\
&\quad \cup \left\{d_1 + d_3 - \frac{1}{2}\sum_{i=1}^{p} d_i \text{ copies of } v_1v_3\right\} \\
&\quad \cup \{d_i \text{ copies of } v_3v_i \colon 4 \le i \le p\}.
\end{aligned}$$

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