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A unified view on common fixed point theorems for Ćirić quasi-contraction maps

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Abstract
The main purpose of this paper is to unify and to generalize some common fixed point results in metric spaces with a Q-function (or a w-distance) and the related results in dislocated metric spaces (also called metric-like spaces). First in the setting of dislocated quasi-metric spaces we introduce the notion of weak 0-σ-completeness which is weaker than 0-σ-completeness. By using the new type of completeness, we establish some common fixed point theorems for two self-mappings satisfying a nonlinear contractive condition of Ćirić type. Our results unify and generalize many well-known common fixed point theorems. Finally, we give some further applications of our main results.

MSC: 47H10; 54H25

Keywords: common fixed point; Ćirić quasi-contraction; dislocated quasi-metric spaces; Q-function

1 Introduction

In 1922, Banach [1] published his fixed point theorem, also known as Banach contraction principle. Over the years, various extensions and generalizations of this principle have appeared in the literature. Ćirić [2] introduced the notion of a quasi-contraction as one of the most general contractive type mappings. A well-known Ćirić result is that a quasi-contraction possesses a unique fixed point. Das and Naik [3] proved some common fixed point theorems which generalized and unified the results of Ćirić [2] and Jungck [4]. Ume [5] obtained fixed point theorems in a complete metric space using the concept of a w-distance and improved fixed point theorems of Ćirić [2]. Ilić and Rakočević [6] established some common fixed point results for maps on metric spaces with w-distance and unified the results of Das and Naik [3] and Ume [5]. Recently, Di Bari and Vetro [7] obtained common fixed points for two self-mappings satisfying nonlinear quasi-contractions of Ćirić type on metric spaces. Inspired by the ideas in [6, 7], He [8] established some common fixed point theorems for two self-mappings satisfying a nonlinear contractive condition of Ćirić type with a Q-function which generalize and unify fixed point results in [6, 7].

In 1994, Matthews [9] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow networks. Further, Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification. After that, many authors studied and generalized the results of

Matthews (see, for example, [10–14]). In particular, Hitzler and Seda [15] introduced the concept of dislocated metric spaces as a generalization of metric spaces and partial metric spaces and presented variants of Banach contraction principle in such spaces. Zeyada *et al.* [16] introduced the notion of dislocated quasi-metric space and generalized the results of Hitzler and Seda [15]. Amini-Harandi [17] re-introduced the dislocated metric spaces under the name of metric-like spaces and proved some fixed point theorems in metric-like spaces. Shukla *et al.* [18] proved some common fixed point theorems in 0-σ-complete metric-like spaces and generalized the results of Amini-Harandi [17].

In this paper, we attempt to give a unified approach to the above mentioned fixed point results. First in the framework of dislocated quasi-metric spaces we introduce the notion of weak 0-σ-completeness, which is weaker than 0-σ-completeness. In particular, we show that if p is a Q-function (or a w-distance) on complete metric spaces X then (X,p) is a weak 0-σ-complete dislocated quasi-metric space. By using the new type of completeness, we establish some common fixed point results on dislocated quasi-metric spaces, which generalize and unify the results of Ume [5], Ilić and Rakočević [6], Di Bari and Vetro [7], He [8], Amini-Harandi [17], Shukla *et al.* [18] and some others. As further applications of our results, we derive some common fixed point theorems in weak quasi-partial metric spaces, in T_0-quasi-pseudo-metric spaces and in uniform spaces.

2 Preliminaries

First we recall some definitions and results as regards the Q-function and dislocated quasi-metric spaces.

Definition 2.1 [19, 20] Let X be a metric space with metric d. Then a function $q : X \times X \to [0, +\infty)$ is called a Q-function on X if the following are satisfied:

(q1) $q(x,z) \le q(x,y) + q(y,z)$, for any $x, y, z \in X$,

(q2) if $x \in X$ and $\{y_n\}_{n\in\mathbf{N}}$ is a sequence in X which converges to a point y and $q(x,y_n) \le M$ for some $M = M(x) > 0$, then $q(x,y) \le M$,

(q3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $q(z,x) \le \delta$ and $q(z,y) \le \delta$ imply $d(x,y) \le \varepsilon$.

If the condition (q2) is replaced by the following stronger condition:

(q2′) for any $x \in X$, $q(x,\cdot) : X \to [0,+\infty)$ is lower semicontinuous,

then q is called a w-distance on X.

For some examples of Q-functions and w-distances, the reader can refer to [19, 20]. The following lemma has been presented in [19, 20].

Lemma 2.2 *Let (X,d) be a metric space and let q be a Q-function on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,+\infty)$ converging to 0, and let $x,y,z \in X$. Then the following hold*:

(i) *If $q(x_n,y) \le \alpha_n$ and $q(x_n,z) \le \beta_n$ for any $n \in \mathbf{N}$, then $y = z$. In particular, if $q(x,y) = 0$ and $q(x,z) = 0$, then $y = z$.*

(ii) *If $q(x_n,y_n) \le \alpha_n$ and $q(x_n,z) \le \beta_n$ for any $n \in \mathbf{N}$, then $\{y_n\}$ converges to z.*

(iii) *If $q(x_n,x_m) \le \alpha_n$ for any $n,m \in \mathbf{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

(iv) *If $q(y,x_n) \le \alpha_n$ for any $n \in \mathbf{N}$, then $\{x_n\}$ is a Cauchy sequence.*

Definition 2.3 [15–17, 21] A mapping $\sigma : X \times X \to [0, +\infty)$, where X is a nonempty set, is said to be a dislocated metric (or a metric-like) on X if the following conditions hold:

(σ1) if $\sigma(x,y) = \sigma(y,x) = 0$ then $x = y$,
(σ2) $\sigma(x,y) = \sigma(y,x)$,
(σ3) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$,

for all $x, y, z \in X$. The pair (X, σ) is called a dislocated metric space (or a metric-like space). If σ satisfies the conditions (σ1) and (σ3), then it is called a dislocated quasi-metric (or a quasi-metric-like) on X and the pair (X, σ) is called a dislocated quasi-metric space (or a quasi-metric-like space).

Remark 2.4 Let (X, d) be a metric space and let q be a Q-function on X. Then (X, q) is a dislocated quasi-metric space. Indeed, it is obvious that the function q satisfies (σ3). Let $x, y \in X$ be such that $q(x,y) = q(y,x) = 0$. By (σ3), we have $q(x,x) = q(y,y) = 0$. From Lemma 2.2(i), $q(x,x) = 0$ and $q(x,y) = 0$ imply $x = y$. Hence (σ1) is satisfied.

Definition 2.5 [15–17, 21] Let (X, σ) be a dislocated quasi-metric space. Then:

(1) A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if

$$\lim_{n\to+\infty} \sigma(x, x_n) = \lim_{n\to+\infty} \sigma(x_n, x) = \sigma(x, x).$$

(2) A sequence $\{x_n\}$ in X is called a σ-Cauchy sequence if and only if

$$\lim_{n,m\to+\infty} \sigma(x_n, x_m) \quad \text{and} \quad \lim_{n,m\to+\infty} \sigma(x_m, x_n)$$

exist (and are finite).

(3) A sequence $\{x_n\}$ in X is called a 0-σ-Cauchy sequence if for any $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that for all $m, n > n_0$, $\sigma(x_n, x_m) < \varepsilon$, that is, $\sigma(x_n, x_m) \to 0$ and $\sigma(x_m, x_n) \to 0$ as $m, n \to +\infty$.

(4) The space (X, σ) is said to be complete if for each σ-Cauchy sequence $\{x_n\}$ in X, there exists $x \in X$ such that

$$\lim_{n\to+\infty} \sigma(x_n, x) = \lim_{n\to+\infty} \sigma(x, x_n) = \sigma(x, x) = \lim_{m,n\to\infty} \sigma(x_n, x_m).$$

(5) The space (X, σ) is said to be 0-σ-complete if every 0-σ-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $\sigma(x, x) = 0$.

It is obvious that every complete dislocated metric space is 0-σ-complete, but the converse may not be true; see Example 3 in [18].

Now we introduce the following more extensive completeness for dislocated quasi-metric spaces, which is weaker than the 0-σ-completeness.

Definition 2.6 A dislocated quasi-metric space (X, σ) is called weak 0-σ-complete, if for each 0-σ-Cauchy sequence $\{x_n\}$, there exists $x \in X$ such that $\sigma(x_n, x) \to 0$ as $n \to +\infty$.

It is not hard to see that every 0-σ-complete dislocated quasi-metric space is weak 0-σ-complete. The following example shows that the converse assertions do not hold.

Example 2.7 Let $X = \{0\} \cup \mathbf{N}$ and let σ be a dislocated quasi-metric on X defined as

$\sigma(x,x) = 0$ for all $x \in X$,

$\sigma(n,0) = 0$ for all $n \in \mathbf{N}$,

$\sigma(0,n) = 1$ for all $n \in \mathbf{N}$, and

$\sigma(n,m) = |\frac{1}{n} - \frac{1}{m}|$ for all $n,m \in \mathbf{N}$.

Clearly (X,σ) is not 0-σ-complete because a 0-σ-Cauchy sequence $\{n\}_{n\in\mathbf{N}}$ does not converge in (X,σ). However, it is easy to see that (X,σ) is weak 0-σ-complete.

Remark 2.8 In dislocated quasi-metric spaces, we should also consider some notions related to 'right' and 'left'. We illustrate by the following case. Let (X,σ) be a dislocated quasi-metric spaces.

(i) A sequence $\{x_n\}$ in X right-converges (left-converges) to $x \in X$ if and only if

$$\lim_{n\to+\infty} \sigma(x_n,x) = \sigma(x,x) \qquad \Big(\lim_{n\to+\infty} \sigma(x,x_n) = \sigma(x,x)\Big).$$

(ii) A sequence $\{x_n\}$ in X is called a 0-σ-right-Cauchy (0-σ-left-Cauchy) sequence if for any $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that for all $m > n > n_0$, $\sigma(x_m,x_n) < \varepsilon$ $(\sigma(x_n,x_m) < \varepsilon)$, that is, $\sigma(x_m,x_n) \to 0$ $(\sigma(x_n,x_m) \to 0)$ as $m > n \to +\infty$.

(iii) The space (X,σ) is said to be 0-σ-right-complete (0-σ-left-complete) if every 0-σ-right-Cauchy (0-σ-left-Cauchy) sequence $\{x_n\}$ in X right-converges (left-converges) to a point $x \in X$ such that $\sigma(x,x) = 0$.

It is obvious that every 0-σ-right-complete dislocated quasi-metric space is weak 0-σ-complete. The following example shows the converse may not hold. We can also show that the weak 0-σ-completeness and the 0-σ-left-completeness are independent notions. In fact, (X,σ) in Example 2.7 is not 0-σ-left-complete. If we replace $\sigma(n,0) = 0$ and $\sigma(0,n) = 1$ with $\sigma(n,0) = 1$ and $\sigma(0,n) = 0$ in Example 2.7, then (X,σ) is 0-σ-left-complete but is not weak 0-σ-complete.

Example 2.9 Let $X = \mathbf{N}$ and let σ be a dislocated quasi-metric on X defined as

$\sigma(n,n) = 0$ for all $n \in X$,

$\sigma(m,n) = 1$ for all $m,n \in X$ with $m < n$, and

$\sigma(m,n) = \frac{1}{n} - \frac{1}{m}$ for all $m,n \in X$ with $m > n$.

We see that (X,σ) is not 0-σ-right-complete because a 0-σ-right-Cauchy sequence $\{n\}_{n\in\mathbf{N}}$ does not right-converge in (X,σ). However, it is easy to see that (X,σ) is weak 0-σ-complete because every 0-σ-Cauchy sequence in X is constant after a certain stage.

Proposition 2.10 *Let (X,d) be a complete metric space and let q be a Q-function on X. Then (X,q) is a weak 0-q-complete dislocated quasi-metric space.*

Proof First (X,q) is a dislocated quasi-metric space; see Remark 2.4. Let $\{x_n\}$ be a 0-q-Cauchy sequence in (X,q). Then for any $\varepsilon > 0$ there exists $n_0 \in \mathbf{N}$ such that $q(x_n,x_m) < \varepsilon$ for all $m,n > n_0$. In particular, we have $q(x_n,x_m) < \varepsilon$ for all $m > n > n_0$. By Lemma 2.2(iii), we see that $\{x_n\}$ is a Cauchy sequence in (X,d). Since (X,d) is complete, there exists $x \in X$ such that $d(x_n,x) \to 0$ $(n \to +\infty)$. Let $n > n_0$ be given. Since $x_m \to x$ $(m \to +\infty)$ and $q(x_n,x_m) < \varepsilon$ $(m > n > n_0)$, by (q2) we get $q(x_n,x) \le \varepsilon$ for all $n > n_0$. This means that $q(x_n,x) \to 0$ as $n \to +\infty$. Hence (X,q) is weak 0-q-complete. □

For the following definitions and notations we can refer to [7, 8].

Definition 2.11 Let f and g be self-maps of a set X. If $fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g. The pair f, g of self-maps is weakly compatible if they commute at their coincidence points.

Let Ψ be the family of functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying

(i) ψ is nondecreasing,

(ii) $\psi(0) = 0$,

(iii) $\lim_{x\to+\infty}(x - \psi(x)) = +\infty$, and

(iv) $\lim_{t\to r^+} \psi(t) < r$ for all $r > 0$.

It is obvious that if $\psi : [0,+\infty) \to [0,+\infty)$ is defined by $\psi(t) = \lambda t$ for some $\lambda \in [0,1)$, or $\psi(t) = \ln(t + 1)$, then $\psi \in \Psi$.

Remark 2.12 (Remark 2.1 in [7]) If $\psi \in \Psi$, then we have $\psi(r) < r$ and $\lim_{n\to+\infty} \psi^n(r) = 0$ for all $r > 0$.

Let (X,σ) be a dislocated quasi-metric space and $f,g : X \to X$ be self-mappings, the pair f,g is called a (ψ,σ)-quasi-contraction if there exists $\psi : [0,+\infty) \to [0,+\infty)$ such that for all $x,y \in X$,

$$\sigma(fx,fy) \le \psi\bigl(M(x,y)\bigr), \tag{1}$$

where

$$M(x,y) = \max\bigl\{\sigma(gx,gy),\sigma(gy,gx),\sigma(gx,fx),\sigma(fx,gx),\sigma(gy,fy),\sigma(fy,gy),\\ \sigma(gx,fy),\sigma(fy,gx),\sigma(gy,fx),\sigma(fx,gy),\sigma(gx,gx),\sigma(gy,gy)\bigr\}.$$

For $E \subset X$, we define $\delta_\sigma(E) = \sup\{\sigma(x,y) : x,y \in E\}$.

If f and g satisfy $f(X) \subset g(X)$ and $x_0 \in X$, let us define $x_1 \in X$ such that $fx_0 = gx_1$. Having defined $x_n \in X$, let $x_{n+1} \in X$ be such that $fx_n = gx_{n+1}$. We say that $\{fx_n\}$ is a f-g-sequence of initial point x_0. Define

$$\mathcal{O}(x_0,n) = \{gx_0,fx_0,fx_1,\ldots,fx_n\}$$

and

$$\mathcal{O}(x_0,\infty) = \{gx_0,fx_0,fx_1,\ldots\}.$$

The following lemmas will be useful in the sequel.

Lemma 2.13 *Let (X,σ) be a dislocated quasi-metric space and let $f,g : X \to X$ be two self-mappings. If f and g are a (ψ,σ)-quasi-contraction with $\psi \in \Psi$, then, for every $z \in X$ with $fz \ne gz$,*

$$\inf\bigl\{\sigma(gx,gz) + \sigma(gz,gx) + \sigma(gx,fx) + \sigma(fx,gx) : x \in X\bigr\} > 0.$$

Proof Suppose that there exists $z \in X$ with $fz \neq gz$ and

$$\inf\bigl\{\sigma(gx,gz) + \sigma(gz,gx) + \sigma(gx,fx) + \sigma(fx,gx) : x \in X\bigr\} = 0.$$

Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty}\bigl[\sigma(gx_n,gz) + \sigma(gz,gx_n) + \sigma(gx_n,fx_n) + \sigma(fx_n,gx_n)\bigr] = 0.$$

It follows that $\sigma(gx_n,gz) \to 0$, $\sigma(gz,gx_n) \to 0$, $\sigma(gx_n,fx_n) \to 0$ and $\sigma(fx_n,gx_n) \to 0$. Define $d = \max\{\sigma(fz,gz),\sigma(gz,fz)\}$. Since $fz \neq gz$, we have $d > 0$. By the triangle inequality, we can deduce that $\sigma(gz,gz) = 0$, $\sigma(gx_n,gx_n) \to 0$, $\max\{\sigma(fx_n,gz),\sigma(gz,fx_n)\} \to 0$ and $\max\{\sigma(gx_n,fz),\sigma(fz,gx_n)\} \to d$. Thus for large enough n, using (1) we obtain

$$\begin{aligned}\sigma(gz,fz) &\le \sigma(gz,fx_n) + \sigma(fx_n,fz)\\ &\le \sigma(gz,fx_n) + \psi\bigl(\max\bigl\{\sigma(gx_n,gz),\sigma(gz,gx_n),\sigma(gx_n,fx_n),\sigma(fx_n,gx_n),\sigma(gz,fz),\\ &\quad \sigma(fz,gz),\sigma(gx_n,fz),\sigma(fz,gx_n),\sigma(gz,fx_n),\sigma(fx_n,gz),\sigma(gx_n,gx_n),\sigma(gz,gz)\bigr\}\bigr)\\ &\le \sigma(gz,fx_n) + \psi\bigl(\max\bigl\{d,\sigma(gx_n,fz),\sigma(fz,gx_n)\bigr\}\bigr)\end{aligned}$$

and, similarly,

$$\begin{aligned}\sigma(fz,gz) &\le \sigma(fx_n,gz) + \sigma(fz,fx_n)\\ &\le \sigma(fx_n,gz) + \psi\bigl(\max\bigl\{d,\sigma(gx_n,fz),\sigma(fz,gx_n)\bigr\}\bigr).\end{aligned}$$

Hence

$$d \le \max\bigl\{\sigma(gz,fx_n),\sigma(fx_n,gz)\bigr\} + \psi\bigl(\max\bigl\{d,\sigma(gx_n,fz),\sigma(fz,gx_n)\bigr\}\bigr).$$

From $\max\{d,\sigma(gx_n,fz),\sigma(fz,gx_n)\} \to d^+$, by property (iv) of the function ψ we see that

$$d \le \lim_{n\to+\infty}\psi\bigl(\max\bigl\{d,\sigma(gx_n,fz),\sigma(fz,gx_n)\bigr\}\bigr) < d,$$

which is a contradiction. □

Lemma 2.14 *Let (X,σ) be a dislocated quasi-metric space and let $f,g : X \to X$ be two weakly compatible self-mappings. If f and g are a (ψ,σ)-quasi-contraction with $\psi \in \Psi$, then, for every $y \in g(X)$ with $fy \neq gy$,*

$$\inf\bigl\{\sigma(gx,y) + \sigma(y,gx) + \sigma(gx,fx) + \sigma(fx,gx)\bigr\} > 0.$$

Proof Suppose that there exists $y \in g(X)$ with $fy \neq gy$ and

$$\inf\bigl\{\sigma(gx,y) + \sigma(y,gx) + \sigma(gx,fx) + \sigma(fx,gx)\bigr\} = 0.$$

Since $y \in g(X)$, there is $z \in X$ such that $y = gz$, and therefore

$$\inf\bigl\{\sigma(gx,gz) + \sigma(gz,gx) + \sigma(gx,fx) + \sigma(fx,gx)\bigr\} = 0.$$

Using Lemma 2.13, we obtain $fz = gz = y$. Since f and g are weakly compatible, it follows that

$$gy = gfz = fgz = fy,$$

which is a contradiction. □

Lemma 2.15 *Let (X,σ) be a dislocated quasi-metric space and let $f,g : X \to X$ be weakly compatible self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,σ)-quasi-contraction with $\psi \in \Psi$. If f and g have a coincidence point y, i.e., $fy = gy$, then $u = fy = gy$ is the unique common fixed point of f and g and $\sigma(u,u) = 0$.*

Proof Since f and g are weakly compatible, we deduce that

$$fu = f^2y = fgy = gfy = g^2y = gu. \tag{2}$$

Using (1), we have

$$\sigma(gy,gy) = \sigma(fy,fy) \leq \psi\big(\sigma(gy,gy)\big),$$

and so $\sigma(gy,gy) = 0$. Similarly, $\sigma(g^2y,g^2y) = 0$.

Now, we show that $u = gy$ is a common fixed point of f and g. Using (1), we obtain

$$\sigma(fu,fy) \leq \psi\big(\max\big\{\sigma(fu,fy),\sigma(fy,fu)\big\}\big) \tag{3}$$

and

$$\sigma(fy,fu) \leq \psi\big(\max\big\{\sigma(fu,fy),\sigma(fy,fu)\big\}\big). \tag{4}$$

From (3) and (4), it follows that $\sigma(fu,fy) = 0$ and $\sigma(fy,fu) = 0$. By $(\sigma 1)$ We obtain $fu = fy = u$. From (2), we see that $fu = gu = u$.

To prove the uniqueness of the common fixed point of f and g, let us suppose that there exists $v \in X$ such that $fv = gv = v$. From (1), it follows that $\sigma(u,u) = \sigma(v,v) = 0$,

$$\sigma(u,v) = \sigma(fu,fv) \leq \psi\big(\max\big\{\sigma(u,v),\sigma(v,u)\big\}\big)$$

and

$$\sigma(v,u) = \sigma(fv,fu) \leq \psi\big(\max\big\{\sigma(u,v),\sigma(v,u)\big\}\big).$$

Thus $\sigma(u,v) = \sigma(v,u) = 0$. By $(\sigma 1)$, we conclude that $u = v$ and $\sigma(u,u) = 0$. □

Lemma 2.16 *Let (X,σ) be a dislocated quasi-metric space and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,σ)-quasi-contraction with $\psi \in \Psi$. For $x_0 \in X$, let $\{fx_n\}$ is a f-g-sequence of initial point x_0. Then*:

(i) *For each $x_0 \in X$ and $n \in \mathbf{N}$, there exist $k,l \in \mathbf{N}$ with $k,l \leq n$ such that*

$$\delta_\sigma\big(\mathcal{O}(x_0,n)\big) = \max\big\{\sigma(gx_0,gx_0),\sigma(gx_0,fx_k),\sigma(fx_l,gx_0)\big\}.$$

(ii) *For each* $x_0 \in X$, *there exists* $c > 0$ *such that*

$$\delta_\sigma\big(\mathcal{O}(x_0,\infty)\big) < c.$$

(iii) *For each* $x_0 \in X$, $\{fx_n\}$ *is a* 0-σ-*Cauchy sequence*.

Proof (i) Let $x_0 \in X$ and $n \in \mathbf{N}$. Without loss of generality, we may assume that $\delta_\sigma(\mathcal{O}(x_0, n)) > 0$. Since f and g are a (ψ, q)-quasi-contraction with $\psi \in \Psi$, for every $0 \le i,j \le n$, we have

$$\begin{aligned}\sigma(fx_i,fx_j) &\le \psi\big(\max\big\{\sigma(gx_i,gx_j),\sigma(gx_j,gx_i),\sigma(gx_i,fx_i),\sigma(fx_i,gx_i),\sigma(gx_j,fx_j),\sigma(fx_j,gx_j),\\ &\qquad \sigma(gx_i,fx_j),\sigma(fx_j,gx_i),\sigma(gx_j,fx_i),\sigma(fx_i,gx_j),\sigma(gx_i,gx_i),\sigma(gx_j,gx_j)\big\}\big)\\ &\le \psi\big(\delta_\sigma\big(\mathcal{O}(x_0,n)\big)\big)\\ &< \delta_\sigma\big(\mathcal{O}(x_0,n)\big).\end{aligned}$$

This implies that

$$\delta_\sigma\big(\mathcal{O}(x_0,n)\big) = \max\big\{\sigma(gx_0,gx_0),\sigma(gx_0,fx_k),\sigma(fx_l,gx_0)\big\}$$

for some $0 \le k, l \le n$.

(ii) By property (iii) of the function ψ, for $h = \max\{\sigma(gx_0,gx_0),\sigma(gx_0,fx_0),\sigma(fx_0,gx_0)\}$, there is a $c > h$ such that $t - \psi(t) > h$ for all $t > c$. For each $n \ge 1$,

$$\begin{aligned}\delta_\sigma\big(\mathcal{O}(x_0,n)\big) &= \max\big\{\sigma(gx_0,gx_0),\sigma(gx_0,fx_k),\sigma(fx_l,gx_0)\big\}\\ &\le \max\big\{\sigma(gx_0,gx_0),\sigma(gx_0,fx_0)+\sigma(fx_0,fx_k),\sigma(fx_l,fx_0)+\sigma(fx_0,gx_0)\big\}\\ &\le h + \max\big\{\sigma(fx_0,fx_k),\sigma(fx_l,fx_0)\big\}\\ &\le h + \psi\big(\delta_\sigma\big(\mathcal{O}(x_0,n)\big)\big).\end{aligned}$$

Thus,

$$\delta_\sigma\big(\mathcal{O}(x_0,n)\big) - \psi\big(\delta_\sigma\big(\mathcal{O}(x_0,n)\big)\big) \le h$$

for all $n \in \mathbf{N}$. It follows that

$$\delta_\sigma\big(\mathcal{O}(x_0,n)\big) \le c \quad \text{and} \quad \delta_\sigma\big(\mathcal{O}(x_0,\infty)\big) \le c.$$

(iii) Define $\mathcal{O}(fx_k) = \{fx_k, fx_{k+1}, \ldots\}$ for every $k \ge 0$. It is obvious that for all $k \ge 1$, $\delta_\sigma(\mathcal{O}(fx_k)) \le \psi(\delta_\sigma(\mathcal{O}(fx_{k-1})))$. Let $m, n \in \mathbf{N}$. If $m > n$,

$$\begin{aligned}\sigma(fx_n,fx_m) &\le \delta_\sigma\big(\mathcal{O}(fx_n)\big) \le \psi\big(\delta_\sigma\big(\mathcal{O}(fx_{n-1})\big)\big)\\ &\le \cdots \le \psi^n\big(\delta_\sigma\big(\mathcal{O}(fx_0)\big)\big)\\ &\le \psi^n\big(\delta_\sigma\big(\mathcal{O}(x_0,\infty)\big)\big)\\ &\le \psi^n(c).\end{aligned} \tag{5}$$

If $m < n$, similarly,

$$\sigma(fx_n, fx_m) \leq \delta_\sigma\big(\mathcal{O}(fx_m)\big) \leq \psi^m(c). \tag{6}$$

Now that $\psi^n(r) \to 0$ $(n \to +\infty)$ for all $r > 0$, it follows that $\{fx_n\}$ is a 0-σ-Cauchy sequence. □

3 Main results

Now we begin to state our main result.

Theorem 3.1 *Let (X,σ) be a weak 0-σ-complete dislocated quasi-metric space and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,σ)-quasi-contraction with $\psi \in \Psi$. Let*

(D1) *for every $y \in X$ with $fy \neq gy$,*

$$\inf\big\{\sigma(gx, y) + \sigma(gx, fx) + \sigma(fx, gx) : x \in X\big\} > 0.$$

Then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $\sigma(u,u) = 0$.

Proof Let $x_0 \in X$ be fixed. As $f(X) \subset g(X)$, we may choose $x_1 \in X$ such that $fx_0 = gx_1$. If $x_n \in X$ is given, we may choose $x_{n+1} \in X$ such that $fx_n = gx_{n+1}$. In this way we construct a f-g-sequence $\{fx_n\}$ of initial point x_0. Using Lemma 2.16(iii), we see that $\{fx_n\}$ is a 0-σ-Cauchy sequence. Since X is weak 0-σ-complete, there exists $y \in X$ such that $\sigma(fx_n, y) \to 0$.

Now, we prove that y is a coincidence point of f and g, *i.e.*, $fy = gy$. If $fy \neq gy$, then (D1), (5), and (6) imply

$$\begin{aligned}
0 &< \inf\big\{\sigma(gx, y) + \sigma(gx, fx) + \sigma(fx, gx) : x \in X\big\} \\
&\leq \inf\big\{\sigma(gx_n, y) + \sigma(gx_n, fx_n) + \sigma(fx_n, gx_n) : n \in \mathbf{N}\big\} \\
&= \inf\big\{\sigma(fx_{n-1}, y) + \sigma(fx_{n-1}, fx_n) + \sigma(fx_n, fx_{n-1}) : n \in \mathbf{N}\big\} \\
&\leq \inf\big\{\sigma(fx_{n-1}, y) + 2\psi^{n-1}(c) : n \in \mathbf{N}\big\} \\
&= 0.
\end{aligned}$$

This is a contradiction. Hence $fy = gy$.

If f and g are weakly compatible, then by Lemma 2.15 we obtain $u = fy$ is a unique common fixed point of f and g and $\sigma(u,u) = 0$. □

Theorem 3.2 *Let (X,σ) be a dislocated quasi-metric space and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,σ)-quasi-contraction with $\psi \in \Psi$. Let*

(D2) *for every $z \in X$ with $fz \neq gz$,*

$$\inf\big\{\sigma(gx, gz) + \sigma(gx, fx) + \sigma(fx, gx) : x \in X\big\} > 0.$$

If $f(X)$ or $g(X)$ is a weak 0-σ-complete subspace of X, then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $\sigma(u,u) = 0$.

Proof Let $x_0 \in X$ be fixed. By using the same way as in the proof Theorem 3.1, we may construct a f-g-sequence $\{fx_n\}$ of initial point x_0. Using Lemma 2.16(iii), we see that $\{fx_n\}$ is a 0-σ-Cauchy sequence. Since $f(X)$ or $g(X)$ is a weak 0-σ-complete subspace of X, there exists $y \in g(X)$ such that $\sigma(fx_n, y) \to 0$. Let $z \in X$ be such that $gz = y$.

Now, we prove that z is a coincidence point of f and g, *i.e.*, $fz = gz$. If $fz \neq gz$, then (D2), (5), and (6) imply

$$\begin{aligned} 0 &< \inf\bigl\{\sigma(gx,gz) + \sigma(gx,fx) + \sigma(fx,gx) : x \in X\bigr\} \\ &\leq \inf\bigl\{\sigma(fx_{n-1},gz) + \sigma(gx_n,fx_n) + \sigma(fx_n,gx_n) : n \in \mathbf{N}\bigr\} \\ &= \inf\bigl\{\sigma(fx_{n-1},y) + \sigma(fx_{n-1},fx_n) + \sigma(fx_n,fx_{n-1}) : n \in \mathbf{N}\bigr\} \\ &\leq \inf\bigl\{\sigma(fx_{n-1},y) + 2\psi^{n-1}(c) : n \in \mathbf{N}\bigr\} \\ &= 0. \end{aligned}$$

This is a contradiction. Hence $fz = gz$.

If f and g are weakly compatible, then by Lemma 2.15 we obtain $u = fz$ is a unique common fixed point of f and g and $\sigma(u,u) = 0$. □

The following theorem shows that in Theorem 3.2 if the weak 0-σ-completeness is replaced by the 0-σ-completeness then the condition (D2) can be omitted.

Theorem 3.3 *Let (X,σ) be a dislocated quasi-metric space and let $f, g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,σ)-quasi-contraction with $\psi \in \Psi$. Let $f(X)$ or $g(X)$ is a 0-σ-complete subspace of X. Then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $\sigma(u,u) = 0$.*

Proof Let $x_0 \in X$ be fixed. By using the same way as in the proof Theorem 3.1, we may construct a f-g-sequence $\{fx_n\}$ of initial point x_0. Using Lemma 2.16(iii), we see that $\{fx_n\}$ is a 0-σ-Cauchy sequence. Since $f(X)$ or $g(X)$ is 0-σ-complete, there exists $y \in g(X)$ such that $\sigma(fx_n, y) \to 0$ and $\sigma(y, fx_n) \to 0$.

Now, we prove that y is a coincidence point of f and g, *i.e.*, $fy = gy$. If $fy \neq gy$, then by Lemma 2.14, (5), and (6) we obtain

$$\begin{aligned} 0 &< \inf\bigl\{\sigma(gx,y) + \sigma(y,gx) + \sigma(gx,fx) + \sigma(fx,gx) : x \in X\bigr\} \\ &\leq \inf\bigl\{\sigma(gx_n,y) + \sigma(y,gx_n) + \sigma(gx_n,fx_n) + \sigma(fx_n,gx_n) : n \in \mathbf{N}\bigr\} \\ &= \inf\bigl\{\sigma(fx_{n-1},y) + \sigma(y,fx_{n-1}) + \sigma(fx_{n-1},fx_n) + \sigma(fx_n,fx_{n-1}) : n \in \mathbf{N}\bigr\} \\ &\leq \inf\bigl\{\sigma(fx_{n-1},y) + \sigma(y,fx_{n-1}) + 2\psi^{n-1}(c) : n \in \mathbf{N}\bigr\} \\ &= 0. \end{aligned}$$

This is a contradiction. Hence $fy = gy$.

If f and g are weakly compatible, then by Lemma 2.15 we obtain $u = fy$ is a unique common fixed point of f and g and $\sigma(u,u) = 0$. □

Now, we present two examples to illustrate the obtained results.

Example 3.4 Let $X = \{0,1,2\}$ and let σ be a dislocated quasi-metric on X defined as $\sigma(x,y) = y$ for all $x,y \in X$. We easily see that (X,σ) is weak 0-σ-complete. Let f and $g : X \to X$ be defined by $f0 = 0$, $f1 = 0$, $f2 = 1$ and $gx = x$ for all $x \in X$. Then $fX = \{0,1\} \subset gX = X$ and

$$\sigma(fx,fy) = fy \leq \frac{1}{2}gy = \frac{1}{2}\sigma(gx,gy)$$

for all $x,y \in X$. We also see that if $y \in X$ with $fy \neq gy$, then $y \neq 0$ and

$$\inf\{\sigma(gx,y) + \sigma(gx,fx) + \sigma(fx,gx) : x \in X\} = y > 0.$$

Consider $\psi(t) = \frac{1}{2}t$ for all $t > 0$. Then all conditions of Theorem 3.1 hold and $u = 0$ is a unique common fixed point of f and g.

Example 3.5 Let $X = [0,1)$ and let σ be a dislocated quasi-metric on X defined as $\sigma(x,y) = y$ for all $x,y \in X$. Let f and $g : X \to X$ be defined by

$$f(x) = \begin{cases} x^2, & x \in [0,\frac{1}{2}], \\ \frac{1}{4}, & x \in (\frac{1}{2},1) \end{cases}$$

and $gx = x$ for all $x \in X$. Then $fX = [0,1/4] \subset gX = X$ and fX is 0-σ-complete. Let x, y be arbitrary points in X.

If $y \in [0,1/2]$, then

$$\sigma(fx,fy) = y^2 \leq \frac{1}{2}y = \frac{1}{2}\sigma(gx,gy).$$

If $y \in (1/2,1)$, then

$$\sigma(fx,fy) = \frac{1}{4} < \frac{1}{2}y = \frac{1}{2}\sigma(gx,gy).$$

Consider $\psi(t) = \frac{1}{2}t$ for all $t > 0$. Then all conditions of Theorem 3.3 hold and $u = 0$ is a unique common fixed point of f and g.

4 Applications

4.1 Applications to metric spaces with *Q*-function

From Theorem 3.1, Theorem 3.2 and Proposition 2.10, we can obtain the following results, which generalize Theorem 13 and Theorem 14 in [8].

Corollary 4.1 *Let (X,d) be a complete metric space with a Q-function q and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,q)-quasi-contraction with $\psi \in \Psi$. Let*

(D1′) *for every $y \in X$ with $fy \neq gy$,*

$$\inf\{q(gx,y) + q(gx,fx) + q(fx,gx) : x \in X\} > 0.$$

Then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $\sigma(u,u) = 0$.

Corollary 4.2 *Let (X,d) be a metric space with a Q-function q and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,q)-quasi-contraction with $\psi \in \Psi$. Let*

(D2′) *for every $z \in X$ with $fz \neq gz$,*

$$\inf\{q(gx,gz) + q(gx,fx) + q(fx,gx) : x \in X\} > 0.$$

If $f(X)$ or $g(X)$ is a complete subspace of X, then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $\sigma(u,u) = 0$.

Remark 4.3 It is clear that the condition (D1) in Theorem 13 of [8], that is,

(D1″) for every $y \in X$ with $fy \neq gy$,

$$\inf\{q(gx,y) + q(gx,fx) : x \in X\} > 0,$$

implies the condition (D1′) in Corollary 4.1. Moreover, the contractive condition in Corollary 4.1 is more general than one in Theorem 13 of [8]. Consequently, Corollary 4.1 is an improvement of Theorem 13 in [8].

The next is an example where we can apply Theorem 3.1 but cannot apply Theorem 13 in [8].

Example 4.4 Let $X = \{0\} \cup \mathbf{N}$ and let d be a metric on X defined by
$d(x,x) = 0$ for all $x \in X$,
$d(n,0) = d(0,n) = 1$ for all $n \in \mathbf{N}$, and
$d(n,m) = |\frac{1}{n} - \frac{1}{m}|$ for all $n,m \in \mathbf{N}$.
Clearly (X,d) is not complete because a Cauchy sequence $\{n\}_{n\in\mathbf{N}}$ does not converge in (X,d). Let q be a Q-function on X defined by $q(x,y) = y$. It can verify that (X,q) is a weak 0-σ-complete dislocated quasi-metric space. Let f and $g : X \to X$ be defined by

$$f(x) = \begin{cases} \frac{x}{2}, & x = 0,2,4,\ldots, \\ \frac{x-1}{2}, & x = 1,3,5,\ldots \end{cases}$$

and $gx = x$ for all $x \in X$. Then $fX = gX = X$. If $y \in X$ with $fy \neq gy$, then $y \geq 1$ and

$$\inf\{q(gx,y) + q(gx,fx) + q(fx,gx) : x \in X\} = y > 0.$$

Let x, y be arbitrary points in X.

If $y = 0,2,4,\ldots$, then

$$q(fx,fy) = \frac{1}{2}y = \frac{1}{2}q(gx,gy).$$

If $y = 1, 3, 5, \dots$, then

$$q(fx, fy) = \frac{1}{2}(y-1) < \frac{1}{2}y = \frac{1}{2}\sigma(gx, gy).$$

Consider $\psi(t) = \frac{1}{2}t$ for all $t > 0$. Applying Theorem 3.1 to (X, q), we see that f and g have a unique common fixed point. In fact, $u = 0$ is a unique common fixed point of f and g. However, we cannot apply Theorem 13 of [8] since (X, d) is not complete.

4.2 Application to dislocated metric spaces

If (X, σ) is a dislocated metric space, then the weak 0-σ-completeness and the 0-σ-completeness are exactly the same. In this case, the following statements are equivalent.

(i) For every $z \in X$ with $fz \neq gz$,

$$\inf\bigl\{\sigma(gx, gz) + \sigma(gx, fx) : x \in X\bigr\} > 0.$$

(ii) For every $z \in X$ with $fz \neq gz$,

$$\inf\bigl\{\sigma(gx, gz) + \sigma(gx, fx) + \sigma(fx, gx) : x \in X\bigr\} > 0.$$

(iii) For every $z \in X$ with $fz \neq gz$,

$$\inf\bigl\{\sigma(gx, gz) + \sigma(gz, gx) + \sigma(gx, fx) + \sigma(fx, gx) : x \in X\bigr\} > 0.$$

Consequently, by Lemma 2.13 and Theorem 3.2 or by Theorem 3.3 we obtain the following result.

Corollary 4.5 (Theorem 1 in [18]) *Let (X, σ) be a dislocated metric space (or metric-like space) and let $f, g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that the mappings f and g are a (ψ, σ)-quasi-contraction with $\psi \in \Psi$. Let $f(X)$ or $g(X)$ is a 0-σ-complete subspace of X. Then f and g have a coincidence point in X. If f and g are weakly compatible, then the mappings f and g have a unique common fixed point u in X and $\sigma(u, u) = 0$.*

The following example shows that Theorem 3.3 is indeed a proper generalization of Theorem 1 in [18] (that is, Corollary 4.5).

Example 4.6 Let X, σ be the same as in Example 2.7. Define $f : X \to X$ as $fx = 0$ for all $x \in X$. Define $g : X \to X$ as $g0 = 0$ and $gn = n - 1$ for all $n \in \mathbf{N}$. Take any $\psi \in \Psi$. Then $fX = \{0\} \subset gX = X$ and

$$\sigma(fx, fy) = 0 \le \psi\bigl(\sigma(gx, gy)\bigr)$$

for all $x, y \in X$. Consequently, we see that all conditions of Theorem 3.3 are satisfied. In fact $u = 0$ is a unique common fixed point of f and g. However, we cannot apply Corollary 4.5 because (X, σ) is not a dislocated metric space.

4.3 Application to weak quasi-partial spaces

First, let us briefly recall some definitions and facts about partial metric spaces. For more details, we refer to [9, 12, 13].

Definition 4.7 Let $p : X \times X \to [0, +\infty)$ be a function where X is a nonempty set. If the function p satisfies the following conditions for all $x, y, z \in X$:

(pm1) $x = y$ if and only if $p(x,x) = p(y,y) = p(x,y)$,
(pm2) $p(x,x) \leq p(x,y)$,
(pm3) $p(x,y) = p(y,x)$,
(pm4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

then p is called a partial metric on X and the pair (X,p) is called a partial metric space. If the function p satisfies (pm1), (pm3), and (pm4), then p is called a weak partial metric on X and the pair (X,p) is called a weak partial metric space. If the function p satisfies (pm1), (pm2), (pm4), and

(pm2′) $p(y,y) \leq p(x,y)$, for all $x, y \in X$,

then p is called a quasi-partial metric on X and the pair (X,p) is called a quasi-partial metric space.

It is clear that the partial metric space is a weak partial metric space, as well as a quasi-partial metric space. But the converse may not be true; see Example 12 in [13] and Example 2.2 in [12].

Now, we introduce a new type of partial metric space as follows.

Definition 4.8 Let $p : X \times X \to [0, +\infty)$ be a function where X is a nonempty set. If the function p satisfies (pm1) and (pm4) in Definition 4.7, then p is called a weak quasi-partial metric on X and the pair (X,p) is called a weak quasi-partial metric space.

It is clear that both weak partial metric space and quasi-partial metric space are a weak quasi-partial metric space, but the converse may not be true. A basic example of a weak quasi-partial metric space but not a weak partial metric space or a quasi-partial metric space is the pair (X,p), where $X = [0,1]$, $p(x,y) = y$ for all $x, y \in X$.

Definition 4.9 (Definition 6 in [13]) Let (X,p) be a weak quasi-partial metric space. Then

(i) a sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x_n, x) \to p(x,x)$ and $p(x, x_n) \to p(x,x)$ as $n \to \infty$;
(ii) a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_n, x_m)$ and $\lim_{m,n\to\infty} p(x_m, x_n)$ exist (and are finite);
(iii) a sequence $\{x_n\}$ in X is called a 0-Cauchy sequence if and only if $p(x_n, x_m) \to 0$ and $p(x_m, x_n) \to 0$ as $m, n \to \infty$;
(iv) the space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges;
(v) the space (X,p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x,x) = 0$.

It is not hard to see that if (X,p) is a 0-complete weak quasi-partial metric space, then (X,p) is a 0-p-complete dislocated quasi-metric space. Consequently, using Theorem 3.3

we obtain the following result which generalizes Theorem 3.3 in [10] and Theorem 1 in [14].

Corollary 4.10 *Let (X,p) be a weak quasi-partial metric space and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,p)-quasi-contraction with $\psi \in \Psi$. Let $f(X)$ or $g(X)$ is a 0-complete subspace of X. Then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $p(u,u) = 0$.*

4.4 Applications to T_0-quasi-pseudo-metric spaces

We recall some basic concept and facts about T_0-quasi-pseudo-metric spaces. For more details, see, for example [22, 23].

Let X be a nonempty set. A real valued function $d : X \times X \to [0,+\infty)$ is said to be T_0-quasi-pseudo-metric (in short, T_0-qpm) on X if the following conditions are satisfied:

(i) $d(x,y) = d(y,x) = 0$ if and only if $x = y$;

(ii) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x,y,z \in X$.

The pair (X,d) is said to be a T_0-qpm space.

Given a T_0-qpm d on X, the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$ for all $x,y \in X$, is also a T_0-qpm, and the function d^s defined by $d^s(x,y) = \max\{d(x,y),d(y,x)\}$ for all $x,y \in X$, is a metric on X.

A T_0-qpm space (X,d) is called complete if every Cauchy sequence $\{x_n\}$ in the metric space (X,d^s) converges with respect to the topology $\tau_{d^{-1}}$ (*i.e.*, there exists $z \in X$ such that $d(x_n,z) \to 0$). We see that the completeness for a T_0-qpm space is very general.

It can be verified that if (X,d) is a complete T_0-quasi-pseudo-metric space, then (X,d) is a weak 0-d-complete dislocated quasi-metric spaces. Consequently, using Theorem 3.1 and Theorem 3.2 we obtain the following results.

Corollary 4.11 *Let (X,d) be a complete T_0-qpm space and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,d)-quasi-contraction with $\psi \in \Psi$. Let, for every $y \in X$ with $fy \neq gy$,*

$$\inf\bigl\{d(gx,y) + d(gx,fx) + d(fx,gx) : x \in X\bigr\} > 0.$$

Then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X.

Corollary 4.12 *Let (X,d) be a T_0-qpm space and let $f,g : X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,d)-quasi-contraction with $\psi \in \Psi$. Let, for every $z \in X$ with $fz \neq gz$,*

$$\inf\bigl\{d(gx,gz) + d(gx,fx) + d(fx,gx) : x \in X\bigr\} > 0.$$

If $f(X)$ or $g(X)$ is a complete subspace of (X,d), then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X.

4.5 Applications to uniform spaces

Now, we recall the notions of E-distance and S-completeness in uniform spaces; see [24, 25].

Let $(X,\mathcal{U})$ be a uniform space. A function $p: X \times X \to [0,+\infty)$ is called an A-distance on X if for any $V \in \mathcal{U}$ there exists $\delta > 0$ such that if $p(x,z) \leq \delta$ and $p(z,y) \leq \delta$ for some $z \in X$, then $(x,y) \in V$. If an A-distance p satisfies the triangle inequality, that is,

$$p(x,y) \leq p(x,z) + p(z,y), \quad \forall x,y,z \in X,$$

then p is called a E-distance.

Let $(X,\mathcal{U})$ be a uniform space and let p be an A-distance on X. Then the space $(X,\mathcal{U})$ is called S-complete if for any sequence $\{x_n\}$ in X with $p(x_n,x_m) \to 0$ as $m,n \to +\infty$, there exists $\bar{x} \in X$ such that $p(x_n,\bar{x}) \to 0$ as $n \to +\infty$.

Lemma 4.13 (see [24, 25]) *Let $(X,\mathcal{U})$ be a separated uniform space and let p be an A-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,+\infty)$ converging to 0, and let $x,y,z \in X$. Then the following hold.*

(a) *If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for all $n \in \mathbf{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.*
(b) *If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for all $n \in \mathbf{N}$, then $\{y_n\}$ converges to z.*
(c) *If $p(x_n,x_m) \leq \alpha_n$ for all $n,m \in \mathbf{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence.*

From Lemma 4.13, we can easily show that if $(X,\mathcal{U})$ be a S-complete separated uniform space such that p is an E-distance on X, then (X,p) is a weak 0-p-complete dislocated quasi-metric space. Consequently, using Theorem 3.1 and Theorem 3.2, we easily deduce the following corollaries.

Corollary 4.14 *Let $(X,\mathcal{U})$ be a S-complete separated uniform space such that p is an E-distance on X and let $f,g: X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,p)-quasi-contraction with $\psi \in \Psi$. Let, for every $y \in X$ with $fy \neq gy$,*

$$\inf\bigl\{p(gx,y) + p(gx,fx) + p(fx,gx) : x \in X\bigr\} > 0.$$

Then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $p(u,u) = 0$.

Corollary 4.15 *Let $(X,\mathcal{U})$ be a separated uniform space such that p is an E-distance on X and let $f,g: X \to X$ be two self-mappings such that $f(X) \subset g(X)$. Suppose that f and g are a (ψ,p)-quasi-contraction with $\psi \in \Psi$. Let, for every $z \in X$ with $fz \neq gz$,*

$$\inf\bigl\{p(gx,gz) + p(gx,fx) + p(fx,gx) : x \in X\bigr\} > 0.$$

If $f(X)$ or $g(X)$ is a S-complete subspace of X, then f and g have a coincidence point in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point u in X and $p(u,u) = 0$.

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The authors declare that they have no competing interests.

Authors' contributions

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