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A general class of periodic boundary value problems for controlled nonlinear impulsive evolution equations on Banach spaces

Said Melliani*, Abdelati El Allaoui and Lalla Saadia Chadli

*Correspondence:
said.melliani@gmail.com
Laboratoire de Mathématiques
Appliquées & Calcul Scientifique,
Sultan Moulay Slimane University,
BP 523, Beni Mellal, 23000, Morocco

Abstract

This paper deals with the periodic boundary value problems for nonlinear impulsive evolution equations with controls. By using the theory of semigroup and fixed point methods, we address some conditions ensuring the existence and uniqueness. Finally, two examples are provided to prove the effectiveness of the proposed results.

MSC: 34G10; 34B37; 93C15

Keywords: impulsive evolution equations; periodic boundary value problems; control; mild solution

1 Introduction

The theory of impulsive differential equations has lately years been an object of increasing interest because of its vast applicability in several fields including mechanics, electrical engineering, biology, medicine, and so on. Therefore, it has drawn wide attention of the researchers in the recent years, among them we find JinRong Wang, Michal Feckan, Yong Zhou, and others [1–11].

For a wide bibliography and exposition on differential equations with impulses, see for instance [12–18], and there are many papers discussing the impulsive differential equations and impulsive optimal controls with the classic initial condition: $x(0) = x_0$ (see [19–24]).

In this paper, we consider the following problems for nonlinear impulsive evolution equations with periodic boundary value:

$$
\text{(IEE)} \quad
\begin{cases}
u'(t) = Au(t) + f(t, u(t), u(\rho(t))) + B(t)c(t), \\
\quad t \in (s_i, t_{i+1}], i = 0, 1, 2, \ldots, m, c \in \mathcal{U}_{\text{ad}}, \\
u(t) = T(t - t_i) g_i(t, u(t)), \quad t \in (t_i, s_i], i = 1, 2, \ldots, m, \\
u(0) = u(a) \in X.
\end{cases}
$$

The operator $A : D(A) : X \longrightarrow X$ is the generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on a Banach space X with a norm $\| \cdot \|$, and the fixed points s_i and t_i satisfying

$$0 = s_0 < t_1 \leq s_1 \leq t_2 < \cdots < t_m \leq s_m \leq t_{m+1} = a$$

are pre-fixed numbers, $f : [0, a] \times X \times X \longrightarrow X$ is continuous, $\rho : [0, a] \longrightarrow [0, a]$ is continuous, and $g_i : [t_i, s_i] \times X \longrightarrow X$ is continuous for all $i = 1, 2, \ldots, m$.

2 Preliminaries

Next, we review some basic concepts, notations, and technical results that are necessary in our study.

Throughout this paper, $I = [0, a]$, $\mathcal{C}(I, X)$ is the Banach space of all continuous functions from I into X with the norm $\|u\|_{\mathcal{C}} = \sup_{t \in I}\{\|u(t)\| : t \in I\}$ for $u \in \mathcal{C}(I, X)$, and we consider the space

$$
\mathcal{PC}(I, X) = \big\{ u : I \longrightarrow X : u \in \mathcal{C}\big((t_i, t_{i+1}], X\big), i = 0, 1, \ldots, m
$$
$$
\text{and there exist } u\big(t_i^-\big) \text{ and } u\big(t_i^+\big), i = 1, \ldots, m \text{ with } u\big(t_i^-\big) = u(t_i) \big\},
$$

endowed with the Chebyshev PC-norm $\|u\|_{\mathcal{PC}} = \sup_{t \in I}\{\|u(t)\| : t \in I\}$ for $u \in \mathcal{PC}(I, X)$. Denote $M = \sup_{t \in I} \|T(t)\|$.

Let Y be another separable reflexive Banach space where the controls c take values. Denote by $P_f(Y)$ a class of nonempty closed and convex subsets of Y. We suppose that the multivalued map $w : [0, T] \longrightarrow P_f(Y)$ is measurable, $w(\cdot) \subset E$, where E is a bounded set of Y, and the admissible control set

$$
\mathcal{U}_{\text{ad}} = \big\{ c \in L^p(E) : c(t) \in w(t), \text{a.e.} \big\}, \quad p > 1.
$$

Then $\mathcal{U}_{\text{ad}} \neq \emptyset$, which can be found in [25]. Some of our results are proved using the next well-known results.

Theorem 1 (Krasnoselskii's fixed point theorem) *Assume that K is a closed bounded convex subset of a Banach space X. Furthermore assume that Γ_1 and Γ_2 are mappings from K into X such that*:

1. *$\Gamma_1(u) + \Gamma_2(v) \in K$ for all $u, v \in K$,*
2. *Γ_1 is a contraction,*
3. *Γ_2 is continuous and compact.*

Then $\Gamma_1 + \Gamma_2$ has a fixed point in K.

To begin our discussion, we need to introduce the concept of a mild solution for (IEE). Assume that $u : [0, a] \longrightarrow X$ is a solution of

$$
u'(t) = Au(t) + f\big(t, u(t), u\big(\rho(t)\big)\big) + B(t)c(t), \quad 0 \le t \le a.
$$

From the theory of strongly continuous semigroups, we get

$$
u(t) = T(t)u(0) + \int_0^t T(t - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds
$$
$$
= T(t)u(a) + \int_0^t T(t - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds
$$
$$
= T(t)\bigg[T(a - t_m)g_m\big(s_m, u(s_m)\big)
$$

$$+ \int_{s_m}^{a} T(a-s)\big(f\big(s,u(s),u(\rho(s))\big) + B(s)c(s)\big)\,ds \Bigg]$$

$$+ \int_{0}^{t} T(t-s)\big(f\big(s,u(s),u(\rho(s))\big) + B(s)c(s)\big)\,ds \quad \text{for all } t \in [0,t_1]$$

and

$$u(t) = T(t-s_i)u(s_i) + \int_{s_i}^{t} T(t-s)\big(f\big(s,u(s),u(\rho(s))\big) + B(s)c(s)\big)\,ds$$

$$= T(t-s_i)T(s_i-t_i)g_i\big(s_i,u(s_i)\big) + \int_{s_i}^{t} T(t-s)\big(f\big(s,u(s),u(\rho(s))\big) + B(s)c(s)\big)\,ds$$

$$= T(t-t_i)g_i\big(s_i,u(s_i)\big) + \int_{s_i}^{t} T(t-s)\big(f\big(s,u(s),u(\rho(s))\big) + B(s)c(s)\big)\,ds$$

for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$.

This expression motivates the following definition.

Definition 1 We say that a function $u \in \mathcal{PC}(I,X)$ is called a mild solution of the problem (IEE), if u satisfies

$$
\begin{cases}
u(t) = T(t)[T(a-t_m)g_m(s_m,u(s_m)) + \int_{s_m}^{a} T(a-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))\,ds] \\
\quad + \int_{0}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))\,ds, \quad t \in [0,t_1], \\
u(t) = T(t-t_i)g_i(t,u(t)), \quad t \in (t_i,s_i], i = 1,2,\ldots,m, \\
u(t) = T(t-t_i)g_i(s_i,u(s_i)) + \int_{s_i}^{t} T(t-s)(f(s,u(s),u(\rho(s))) + B(s)c(s))\,ds, \\
\quad t \in (s_i,t_{i+1}], i = 1,2,\ldots,m.
\end{cases}
$$

3 Existence and uniqueness of mild solutions

To establish our results, we introduce the following assumptions:

(H_0)   1. $A : D(A) \subseteq X \longrightarrow X$ is the generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on X with a norm $\|\cdot\|$.

  2. $B : [0,a] \longrightarrow \mathcal{L}(Y,X)$ is essentially bounded, *i.e.*, $B \in L^{\infty}([0,a], \mathcal{L}(Y,X))$.

(H_1) We have the functions $f \in \mathcal{C}(I \times X \times X, X)$, $g_i \in \mathcal{C}([t_i,s_i] \times X, X)$, $i = 1,2,\ldots,m$, and $\rho : I \longrightarrow I$ is continuous.

(H_2) There is a constant $C_f, L_f > 0$ such that

$$\big\| f(t,u_1,v_1) - f(t,u_2,v_2) \big\| \leq C_f \|u_1 - u_2\| + L_f \|v_1 - v_2\|$$

for each $t \in [s_i, t_{i+1}]$, $u_1, u_2, v_1, v_2 \in X$ and $i = 0,1,\ldots,m$.

(H_3) There is a constant $L > 0$ such that

$$\big\| f(t,u,v) \big\| \leq L\big(1 + \|u\|^{\mu} + \|v\|^{\nu}\big)$$

for all $t \in [s_i, t_{i+1}]$ and all $u, v \in X$, $i = 0,1,\ldots,m$, and $\mu, \nu \in [0,1]$.

(H_4) There is a constant $C_{g_i} > 0$, $i = 1,2,\ldots,m$, such that

$$\big\| g_i(t,u) - g_i(t,v) \big\| \leq C_{g_i} \|u - v\|$$

for each $t \in [t_i, s_i]$, and all $u, v \in E^n$, $i = 1,2,\ldots,m$.

(H_5) There is a function $t \longmapsto \psi_i(t)$, $i = 1, 2, \ldots, m$, such that

$$\|g_i(t, u)\| \leq \psi_i(t)$$

for each $t \in [t_i, s_i]$ and all $u \in X$.

We put $C = \max_{1 \leq i \leq m} C_{g_i}$ and $N_i = \sup_{t \in [t_i, s_i]} \psi_i(t) < +\infty$.

Remark 1 From the assumptions (H_0)-(H_2) and the definition of $\mathcal{U}_{\text{ad}}$, it is also easy to verify that $Bc \in L^p([0, a]; X)$ with $p > 1$ for all $c \in \mathcal{U}_{\text{ad}}$.

Therefore, $Bc \in L^1([0, a]; X)$ and $\|Bc\|_{L^1} < \infty$.

Now, we can establish our first existence result.

Theorem 2 *Let assumptions* (H_0), (H_1), (H_2), *and* (H_4) *be satisfied. Suppose, in addition, that the following property is verified:*

$$\lambda := M \max \Big\{ \max_{1 \leq i \leq m} \big\{ C_{g_i} + (C_f + L_f)(t_{i+1} - s_i) \big\},$$

$$C, M C_{g_m} + (C_f + L_f) M(a - s_m) + (C_f + L_f) t_1 \Big\} < 1.$$

Then the problem (IEE) *has a unique mild solution.*

Proof Define a mapping $\Gamma : \mathcal{PC}(I, X) \longrightarrow \mathcal{PC}(I, X)$ by

$$(\Gamma u)(t) = \begin{cases} T(t)[T(a - t_m)g_m(s_m, u(s_m)) \\ \quad + \int_{s_m}^{a} T(a - s)(f(s, u(s), u(\rho(s))) + B(s)c(s)) \, ds] \\ \quad + \int_{0}^{t} T(t - s)(f(s, u(s), u(\rho(s))) + B(s)c(s)) \, ds, & t \in [0, t_1], \\ T(t - t_i)g_i(t, u(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ T(t - t_i)g_i(s_i, u(s_i)) \\ \quad + \int_{s_i}^{t} T(t - s)(f(s, u(s), u(\rho(s))) + B(s)c(s)) \, ds, & t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m. \end{cases}$$

Let $h > 0$ be very small and $u \in \mathcal{PC}(I, X)$, we have the following.

Case 1: For $t \in [0, t_1]$, we have

$$\big\| (\Gamma u)(t + h) - (\Gamma u)(t) \big\|$$

$$= \Big\| T(t + h) \Big[T(a - t_m)g_m\big(s_m, u(s_m)\big) + \int_{s_m}^{a} T(a - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big) \, ds \Big]$$

$$+ \int_{0}^{t+h} T(t + h - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big) \, ds$$

$$- T(t) \Big[T(a - t_m)g_m\big(s_m, u(s_m)\big) + \int_{s_m}^{a} T(a - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big) \, ds \Big]$$

$$- \int_{0}^{t} T(t - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big) \, ds \Big\|$$

$$\leq M \Big\| T(h) \Big[T(a - t_m)g_m\big(s_m, u(s_m)\big) + \int_{s_m}^{a} T(a - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big) \, ds \Big]$$

$$- \Big[T(a - t_m)g_m\big(s_m, u(s_m)\big) + \int_{s_m}^{a} T(a - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big) \, ds \Big] \Big\|$$

$$+ M \int_0^h \left\| f\big(s, u(s), u(\rho(s))\big) + B(s)c(s) \right\| ds + M \int_0^t \left\| B(s+h)c(s+h) - B(s)c(s) \right\| ds$$

$$+ M \int_0^t \left\| f\big(s+h, u(s+h), u(\rho(s+h))\big) - f\big(s, u(s), u(\rho(s))\big) \right\| ds \to 0 \quad \text{as } h \to 0.$$

Case 2: For $t \in (t_i, s_i]$, $i = 1, \ldots, m$, we have

$$\left\| (\Gamma u)(t+h) - (\Gamma u)(t) \right\| = \left\| T(t+h-t_i)g_i\big(t+h, u(t+h)\big) - T(t-t_i)g_i\big(t, u(t)\big) \right\|$$

$$\leq M \left\| T(h)g_i\big(t+h, u(t+h)\big) - g_i\big(t, u(t)\big) \right\| \to 0 \quad \text{as } h \to 0.$$

Case 3: For $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$\left\| (\Gamma u)(t+h) - (\Gamma u)(t) \right\|$$

$$= \left\| T(t+h-t_i)g_i\big(s_i, u(s_i)\big) + \int_{s_i}^{t+h} T(t+h-s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big) ds \right.$$

$$\left. - T(t-t_i)g_i\big(s_i, u(s_i)\big) - \int_{s_i}^{t} T(t-s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big) ds \right\|$$

$$\leq M \left\| T(h)g_i\big(s_i, u(s_i)\big) - g_i\big(s_i, u(s_i)\big) \right\| + M \int_{s_i}^{s_i+h} \left\| f\big(s, u(s), u(\rho(s))\big) + B(s)c(s) \right\| ds$$

$$+ M \int_{s_i}^{t} \left\| B(s+h)c(s+h) - B(s)c(s) \right\| ds$$

$$+ M \int_{s_i}^{t} \left\| f\big(s+h, u(s+h), u(\rho(s+h))\big) - f\big(s, u(s), u(\rho(s))\big) \right\| ds \to 0 \quad \text{as } h \to 0.$$

Then Γ is well defined and $\Gamma u \in \mathcal{PC}(I, X)$ for all $u \in \mathcal{PC}(I, X)$.

Now we only need to show that Γ is a contraction mapping.

Case 1: For $u, v \in \mathcal{PC}(I, X)$ and $t \in [0, t_1]$, we have

$$\left\| (\Gamma u)(t) - (\Gamma v)(t) \right\|$$

$$= \left\| T(t)\left[T(a-t_m)g_m\big(s_m, u(s_m)\big) + \int_{s_m}^{a} T(a-s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big) ds \right] \right.$$

$$+ \int_0^t T(t-s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big) ds$$

$$- T(t)\left[T(a-t_m)g_m\big(s_m, v(s_m)\big) + \int_{s_m}^{a} T(a-s)\big(f\big(s, v(s), v(\rho(s))\big) + B(s)c(s)\big) ds \right]$$

$$\left. - \int_0^t T(t-s)\big(f\big(s, v(s), v(\rho(s))\big) + B(s)c(s)\big) ds \right\|$$

$$\leq M\left[MC_{g_m} \left\| u(s_m) - v(s_m) \right\| + M \int_{s_m}^{a} \big(C_f \left\| u(s) - v(s) \right\| + L_f \left\| u(\rho(s)) - v(\rho(s)) \right\| \big) ds \right]$$

$$+ M \int_0^t \big(C_f \left\| u(s) - v(s) \right\| + L_f \left\| u(\rho(s)) - v(\rho(s)) \right\| \big) ds$$

$$\leq M\big[MC_{g_m} + (C_f + L_f)M(a - s_m) + (C_f + L_f)t_1 \big] \left\| u - v \right\|_{\mathcal{PC}}$$

$$\leq \lambda \left\| u - v \right\|_{\mathcal{PC}}.$$

Case 2: For $u, v \in \mathcal{PC}(I, X)$ and $t \in (t_i, s_i]$, $i = 1, \dots, m$, we have

$$
\begin{aligned}
\big\| (\Gamma u)(t) - (\Gamma v)(t) \big\| &= \big\| T(t - t_i) g_i \big(t, u(t) \big) - T(t - t_i) g_i \big(t, v(t) \big) \big\| \\
&\leq M C_{g_i} \| u - v \|_{\mathcal{PC}} \\
&\leq M C \| u - v \|_{\mathcal{PC}} \\
&\leq \lambda \| u - v \|_{\mathcal{PC}}.
\end{aligned}
$$

Case 3: For $u, v \in \mathcal{PC}(I, X)$ and $t \in (s_i, t_{i+1}]$, $i = 1, \dots, m$, we have

$$
\begin{aligned}
\big\| (\Gamma u)(t) &- (\Gamma v)(t) \big\| \\
&= \bigg\| T(t - t_i) g_i \big(s_i, u(s_i) \big) + \int_{s_i}^{t} T(t - s) \big(f \big(s, u(s), u(\rho(s)) \big) + B(s) c(s) \big) \, ds \\
&\quad - T(t - t_i) g_i \big(s_i, v(s_i) \big) - \int_{s_i}^{t} T(t - s) \big(f \big(s, v(s), v(\rho(s)) \big) + B(s) c(s) \big) \, ds \bigg\| \\
&\leq M \big[C_{g_i} + (C_f + L_f)(t_{i+1} - s_i) \big] \| u - v \|_{\mathcal{PC}} \\
&\leq M \max_{1 \leq i \leq m} \big\{ C_{g_i} + (C_f + L_f)(t_{i+1} - s_i) \big\} \| u - v \|_{\mathcal{PC}} \\
&\leq \lambda \| u - v \|_{\mathcal{PC}}.
\end{aligned}
$$

Therefore, we obtain

$$
\| \Gamma u - \Gamma v \|_{\mathcal{PC}} \leq \lambda \| u - v \|_{\mathcal{PC}}, \quad \forall u, v \in \mathcal{PC}(I, X).
$$

Finally, we find that Γ is a contraction mapping on $\mathcal{PC}(I, X)$, and there exists a unique $u \in \mathcal{PC}(I, X)$ such that $\Gamma u = u$.

So we conclude that u is the unique mild solution of (IEE). □

By using Krasnoselskii's fixed point theorem, we also obtain the existence of a mild solution.

Theorem 3 *Let assumptions* (H_0), (H_1), (H_3), *and* (H_5) *be satisfied. Suppose, in addition, that the semigroup* $\{ T(t), t \geq 0 \}$ *is compact and*

$$
\alpha := \max \big\{ LM \big(M(a - s_m) + t_1 \big), LM(t_{i+1} - s_i) \big\} < \frac{1}{2}, \quad i = 1, \dots, m,
$$

$$
\beta := \max \big\{ M^2 C_{g_m}, M C_{g_i} \big\} < 1.
$$

Then the problem (IEE) *has at least one mild solution.*

Proof Let $N = \max(N_1, N_2, \dots, N_m)$ and $B_r = \{ u \in \mathcal{PC}(I, X) : \| u \|_{\mathcal{PC}} < r \}$ the ball with radius $r > 0$.

Here

$$
r \geq \max \{ \gamma_1, \gamma_2, MN \},
$$

with

$$\gamma_1 = \frac{M^2 N_m + (M^2 + M)\|Bc\|_{L^1} + LM(M(a - s_m) + t_1)}{1 - 2\alpha}$$

and

$$\gamma_2 = \frac{MN_i + M\|Bc\|_{L^1} + ML(t_{i+1} - s_i)}{1 - 2\alpha}.$$

We introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where

$$(\Gamma_1 u)(t) = \begin{cases} T(t)T(a - t_m)g_m(s_m, u(s_m)), & t \in [0, t_1], \\ T(t - t_i)g_i(t, u(t)), & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ T(t - t_i)g_i(s_i, u(s_i)), & t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m, \end{cases}$$

and

$$(\Gamma_2 u)(t) = \begin{cases} T(t)\int_{s_m}^{a} T(a - s)(f(s, u(s), u(\rho(s))) + B(s)c(s))\,ds \\ \quad + \int_0^t T(t - s)(f(s, u(s), u(\rho(s))) + B(s)c(s))\,ds, & t \in [0, t_1], \\ 0, & t \in (t_i, s_i], i = 1, 2, \ldots, m, \\ \int_{s_i}^t T(t - s)(f(s, u(s), u(\rho(s))) + B(s)c(s))\,ds, & t \in (s_i, t_{i+1}], i = 1, 2, \ldots, m. \end{cases}$$

We distinguish in the proof several steps.

Step 1. We prove that $\Gamma u = \Gamma_1 u + \Gamma_2 u \in B_r$ for all $u \in B_r$. Indeed:

Case 1. For $t \in [0, t_1]$, we have

$$\big\|(\Gamma_1 u + \Gamma_2 u)(t)\big\|$$

$$\leq \big\|T(t)\big\|\big\|T(a - t_m)g_m(s_m, u(s_m))\big\|$$

$$\quad + \big\|T(t)\big\|\int_{s_m}^{a}\big\|T(a - s)\big\|\big(\big\|f(s, u(s), u(\rho(s)))\big\| + \big\|B(s)c(s)\big\|\big)\,ds$$

$$\quad + \int_0^t \big\|T(t - s)\big\|\big(\big\|f(s, u(s), u(\rho(s)))\big\| + \big\|B(s)c(s)\big\|\big)\,ds$$

$$\leq M^2 N_m + LM^2 \int_{s_m}^{a}\big(1 + \big\|u(s)\big\|^{\mu} + \big\|u(\rho(s))\big\|^{\nu}\big)\,ds + (M^2 + M)\|Bc\|_{L^1}$$

$$\quad + LM \int_0^t \big(1 + \big\|u(s)\big\|^{\mu} + \big\|u(\rho(s))\big\|^{\nu}\big)\,ds$$

$$\leq M^2 N_m + (M^2 + M)\|Bc\|_{L^1} + LM^2(1 + 2r)(a - s_m) + LM(1 + 2r)t_1$$

$$= M^2 N_m + (M^2 + M)\|Bc\|_{L^1} + LM(M(a - s_m) + t_1) + 2rLM(M(a - s_m) + t_1)$$

$$\leq r(1 - 2\alpha) + 2r\alpha = r.$$

Case 2. For $t \in (t_i, s_i]$, $i = 1, \ldots, m$, we have

$$\big\|(\Gamma_1 u + \Gamma_2 u)(t)\big\| \leq \big\|T(t - t_i)\big\|\big\|g_i(t, u(t))\big\|$$

$$\leq MN_i \leq MN \leq r.$$

Case 3. For $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$
\begin{aligned}
\left\| (\Gamma_1 u + \Gamma_2 u)(t) \right\| &\leq \left\| T(t - t_i) \right\| \left\| g_i\big(s_i, u(s_i)\big) \right\| \\
&\quad + \int_{s_i}^{t} \left\| T(t - s) \right\| \left(\left\| f\big(s, u(s), u(\rho(s))\big) \right\| + \left\| B(s) c(s) \right\| \right) ds \\
&\leq M N_i + L M (1 + 2r)(t_{i+1} - s_i) + M \| B c \|_{L^1} \\
&= M N_i + M \| B c \|_{L^1} + L M (t_{i+1} - s_i) + 2r L M (t_{i+1} - s_i) \\
&\leq r(1 - 2\alpha) + 2r\alpha = r.
\end{aligned}
$$

Then we deduce that $\Gamma_1 u + \Gamma_2 u \in B_r$.

Step 2. Γ_1 is contraction on B_r. Let $u, v \in B_r$.

Case 1. For $t \in [0, t_1]$, we have

$$
\begin{aligned}
\left\| (\Gamma_1 u)(t) - (\Gamma_1 v)(t) \right\| &\leq \left\| T(t) \right\| \left\| T(a - t_m) \right\| \left\| g_m\big(s_m, u(s_m)\big) - g_m\big(s_m, v(s_m)\big) \right\| \\
&\leq M^2 C_{g_m} \left\| u(s_m) - v(s_m) \right\| \\
&\leq M^2 C_{g_m} \| u - v \|_{\mathcal{PC}} \\
&\leq \beta \| u - v \|_{\mathcal{PC}}.
\end{aligned}
$$

Case 2. For $t \in (t_i, s_i]$, $i = 1, \ldots, m$, we have

$$
\begin{aligned}
\left\| (\Gamma_1 u)(t) - (\Gamma_1 v)(t) \right\| &\leq \left\| T(t - t_i) \right\| \left\| g_i\big(t, u(t)\big) - g_i\big(t, v(t)\big) \right\| \\
&\leq M C_{g_m} \| u - v \|_{\mathcal{PC}} \\
&\leq \beta \| u - v \|_{\mathcal{PC}}.
\end{aligned}
$$

Case 3. For $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$
\begin{aligned}
\left\| (\Gamma_1 u)(t) - (\Gamma_1 v)(t) \right\| &\leq \left\| T(t - t_i) \right\| \left\| g_i\big(s_i, u(s_i)\big) - g_i\big(s_i, v(s_i)\big) \right\| \\
&\leq M C_{g_m} \| u - v \|_{\mathcal{PC}} \\
&\leq \beta \| u - v \|_{\mathcal{PC}}.
\end{aligned}
$$

This implies that Γ_1 is a contraction.

Step 3. Γ_2 is continuous.

Let $(u_n)_{n \geq 0}$ be a sequence such that $\lim_{n \to +\infty} \| u_n - u \|_{\mathcal{PC}} = 0$.

Case 1. For $t \in [0, t_1]$, we have

$$
\begin{aligned}
&\left\| (\Gamma_2 u_n)(t) - (\Gamma_2 u)(t) \right\| \\
&\quad \leq \left\| T(t) \right\| \int_{s_m}^{a} \left\| T(a - s) \right\| \left\| f\big(s, u_n(s), u_n(\rho(s))\big) - f\big(s, u(s), u(\rho(s))\big) \right\| ds \\
&\qquad + \int_{0}^{t} \left\| T(t - s) \right\| \left\| f\big(s, u_n(s), u_n(\rho(s))\big) - f\big(s, u(s), u(\rho(s))\big) \right\| ds \\
&\quad \leq M^2 (a - s_m) \left\| f\big(\cdot, u_n(\cdot), u_n(\rho(\cdot))\big) - f\big(\cdot, u(\cdot), u(\rho(\cdot))\big) \right\|_{\mathcal{PC}}
\end{aligned}
$$

$$+ Mt_1 \big\| f\big(\cdot, u_n(\cdot), u_n(\rho(\cdot))\big) - f\big(\cdot, u(\cdot), u(\rho(\cdot))\big)\big\|_{\mathcal{PC}}$$
$$= M\big[M(a - s_m) + t_1\big]\big\| f\big(\cdot, u_n(\cdot), u_n(\rho(\cdot))\big) - f\big(\cdot, u(\cdot), u(\rho(\cdot))\big)\big\|_{\mathcal{PC}}.$$

Case 2. For $t \in (t_i, s_i]$, $i = 1, \ldots, m$, we have

$$\big\|(\Gamma_2 u_n)(t) - (\Gamma_2 u)(t)\big\| = 0.$$

Case 3. For $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$\big\|(\Gamma_2 u_n)(t) - (\Gamma_2 u)(t)\big\| \leq \int_{s_i}^{t} \big\| T(t - s)\big\|\big\| f\big(s, u_n(s), u_n(\rho(s))\big) - f\big(s, u(s), u(\rho(s))\big)\big\| \, ds$$
$$= M(t_{i+1} - s_i)\big\| f\big(\cdot, u_n(\cdot), u_n(\rho(\cdot))\big) - f\big(\cdot, u(\cdot), u(\rho(\cdot))\big)\big\|_{\mathcal{PC}}.$$

This implies that $\lim_{n \to +\infty} \|\Gamma_2 u_n - \Gamma_2 u\|_{\mathcal{PC}} = 0$, then we deduce that Γ_2 is continuous.

Step 4. Γ_2 is compact.

1. We have $\Gamma_2 B_r \subseteq B_r$, then Γ_2 is uniformly bounded on B_r.

2. For $u \in B_r$, we have the following.

Case 1. For $0 \leq l_1 < l_2 \leq t_1$, we have

$$\big\|(\Gamma_2 u)(l_2) - (\Gamma_2 u)(l_1)\big\|$$
$$\leq \big\| T(l_2) - T(l_1)\big\| \int_{s_m}^{a} \big\| T(a - s)\big\|\big(\big\| f\big(s, u(s), u(\rho(s))\big)\big\| + \big\| B(s)c(s)\big\|\big) \, ds$$
$$+ \int_{0}^{l_1} \big\| T(l_2 - s) - T(l_1 - s)\big\|\big(\big\| f\big(s, u(s), u(\rho(s))\big)\big\| + \big\| B(s)c(s)\big\|\big) \, ds$$
$$+ \int_{l_1}^{l_2} \big\| T(l_2 - s)\big\|\big(\big\| f\big(s, u(s), u(\rho(s))\big)\big\| + \big\| B(s)c(s)\big\|\big) \, ds$$
$$\leq M^2\big(L(1 + 2r)(a - s_m) + \|Bc\|_{L^1}\big)\big\| T(l_2 - l_1) - I\big\|$$
$$+ M\big(L(1 + 2r)t_1 + \|Bc\|_{L^1}\big)\big\| T(l_2 - l_1) - I\big\|$$
$$+ LM(1 + 2r)(l_2 - l_1) + M \int_{l_1}^{l_2} \big\| B(s)c(s)\big\| \, ds$$
$$= \big(LM(1 + 2r)\big[M(a - s_m) + t_1\big] + \big(M^2 + M\big)\|Bc\|_{L^1}\big)\big\| T(l_2 - l_1) - I\big\|$$
$$+ LM(1 + 2r)(l_2 - l_1) + M \int_{l_1}^{l_2} \big\| B(s)c(s)\big\| \, ds \to 0 \quad \text{as } l_2 \to l_1.$$

Since $\{T(t), t \geq 0\}$ is compact, $\| T(l_2 - l_1) - I\| \to 0$ as $l_2 \to l_1$.

Case 2. For $t_i \leq l_1 < l_2 \leq s_i$, $i = 1, \ldots, m$, we have

$$\big\|(\Gamma_2 u)(l_2) - (\Gamma_2 u)(l_1)\big\| = 0.$$

Case 3. For $s_i \leq l_1 < l_2 \leq t_{i+1}$, $i = 1, \ldots, m$, we have

$$\big\|(\Gamma_2 u)(l_2) - (\Gamma_2 u)(l_1)\big\|$$
$$= \bigg\| \int_{s_i}^{l_2} T(l_2 - s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big) \, ds$$

$$- \int_{s_i}^{l_1} T(l_1 - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds \bigg\|$$

$$\leq \int_{l_1}^{l_2} \big\| T(l_2 - s)\big\| \big(\big\| f\big(s, u(s), u\big(\rho(s)\big)\big)\big\| + \big\| B(s)c(s)\big\|\big)\, ds$$

$$+ \int_{s_i}^{l_1} \big\| T(l_1 - s)\big\| \big\| T(l_2 - l_1) - I\big\| \big(\big\| f\big(s, u(s), u\big(\rho(s)\big)\big)\big\| + \big\| B(s)c(s)\big\|\big)\, ds$$

$$\leq LM(1 + 2r)(l_2 - l_1) + M \int_{l_1}^{l_2} \big\| B(s)c(s)\big\|\, ds$$

$$+ M\big(L(1 + 2r)(t_{i+1} - s_i) + \|Bc\|_{L^1}\big)\big\| T(l_2 - l_1) - I\big\| \to 0 \quad \text{as } l_2 \to l_1.$$

This permits us to conclude that Γ_2 is equicontinuous.

We have $\Gamma_2 B_r \subseteq B_r$, let $\Theta := \Gamma_2 B_r$, $\Theta(t) := \Gamma_2 B_r(t) = \{(\Gamma_2 u)(t) : u \in B_r\}$ for $t \in [0, a]$.

3. $\Theta(t)$ is relatively compact. Indeed:

$T(t)$ is compact, hence

$$\Theta(0) = \left\{ \int_{s_m}^{a} T(a - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds \right\},$$

is relatively compact. For $0 < \varepsilon < t \leq a$, define

$$\Theta_\varepsilon(t) := \Gamma_2^\varepsilon B_r(t) = \big\{ T(\varepsilon)(\Gamma_2 u)(t - \varepsilon) : u \in B_r \big\}.$$

Clearly, $\Theta_\varepsilon(t)$ is relatively compact for $t \in (\varepsilon, a]$, since $T(t)$ is compact.

Case 1. For $t \in (0, t_1]$, we have

$$\Theta_\varepsilon(t) := \big(\Gamma_2^\varepsilon u\big)(t) = T(\varepsilon)(\Gamma_2 u)(t - \varepsilon)$$

$$= \left\{ T(\varepsilon)T(t - \varepsilon)\left[\int_{s_m}^{a} T(a - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds \right] \right.$$

$$\left. + T(\varepsilon)\int_{0}^{t-\varepsilon} T(t - \varepsilon - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds : u \in B_r \right\}$$

$$= \left\{ T(t)\left[\int_{s_m}^{a} T(a - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds \right] \right.$$

$$\left. + \int_{0}^{t-\varepsilon} T(t - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds : u \in B_r \right\},$$

and we get

$$\big\|(\Gamma_2 u)(t) - \big(\Gamma_2^\varepsilon u\big)(t)\big\|$$

$$= \left\| \int_{0}^{t} T(t - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds \right.$$

$$\left. - \int_{0}^{t-\varepsilon} T(t - s)\big(f\big(s, u(s), u\big(\rho(s)\big)\big) + B(s)c(s)\big)\, ds \right\|$$

$$\leq \int_{t-\varepsilon}^{t} \big\| T(t - s)\big\| \big(\big\| f\big(s, u(s), u\big(\rho(s)\big)\big)\big\| + \big\| B(s)c(s)\big\|\big)\, ds$$

$$\leq LM(1 + 2r)\varepsilon + M \int_{t-\varepsilon}^{t} \big\| B(s)c(s)\big\|\, ds \to 0 \quad \text{as } \varepsilon \to 0.$$

Case 2. For $t \in (t_i, s_i]$, $i = 1, \ldots, m$, we have

$$\Theta_\varepsilon(t) := \{0, u \in B_r\},$$

in this case $\|(\Gamma_2 u)(t) - (\Gamma_2^\varepsilon u)(t)\| = 0$.

Case 3. For $t \in (s_i, t_{i+1}]$, $i = 1, \ldots, m$, we have

$$\begin{aligned}
\Theta_\varepsilon(t) &:= \left(\Gamma_2^\varepsilon u\right)(t) \\
&= \left\{ T(\varepsilon) \int_{s_i}^{t-\varepsilon} T(t - \varepsilon - s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big)\, ds : u \in B_r \right\} \\
&= \left\{ \int_{s_i}^{t-\varepsilon} T(t - s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big)\, ds : u \in B_r \right\},
\end{aligned}$$

and we get

$$\begin{aligned}
\big\|(\Gamma_2 u)(t) &- \left(\Gamma_2^\varepsilon u\right)(t)\big\| \\
&= \bigg\| \int_{s_i}^{t} T(t - s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big)\, ds \\
&\quad - \int_{s_i}^{t-\varepsilon} T(t - s)\big(f\big(s, u(s), u(\rho(s))\big) + B(s)c(s)\big)\, ds \bigg\| \\
&\leq \int_{t-\varepsilon}^{t} \big\| T(t - s)\big\| \big(\big\|f\big(s, u(s), u(\rho(s))\big)\big\| + \big\|B(s)c(s)\big\|\big)\, ds \\
&\leq LM(1 + 2r)\varepsilon + M \int_{t-\varepsilon}^{t} \big\|B(s)c(s)\big\|\, ds \to 0 \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Now, from the Arzela-Ascoli theorem we can conclude that $\Gamma_2 : B_r \longrightarrow B_r$ is completely continuous. The existence of a mild solution for (IEE) is now a consequence of Krasnoselskii's fixed point theorem. □

4 Examples

In this section, we give examples to illustrate our abstract results in the previous section.

Let $X = L^2(0, 1)$, $I = [0, 3]$, $0 = t_0 = s_0$, $t_1 = 1$, $s_1 = 2$, and $a = 3$. Define $Av = \frac{\partial^2}{\partial^2 x} v$ for

$$v \in D(A) = \left\{ v \in X : \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial^2 x} \in X, v(0) = v(1) = 0 \right\}.$$

Then A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on X. In addition $T(t)$ is compact and $\|T(t)\| \leq 1$, for all $t \geq 0$.

Example 1 Consider

$$\begin{cases}
\frac{\partial}{\partial t} u(t, x) = \frac{\partial^2}{\partial^2 x} u(t, x) + \frac{1}{12} \cos(u(t, x) + u(t^2, x)) + c(t, x), & x \in (0, 1), t \in [0, 1) \cup (2, 3], \\
\frac{\partial}{\partial x} u(t, 0) = \frac{\partial}{\partial x} u(t, 1) = 0, & t \in [0, 1) \cup (2, 3], \\
u(0, x) = u(3, x), & x \in (0, 1), \\
u(t, x) = T(t - 1)\frac{1}{4} \sin(u(t, x)), & x \in (0, 1), t \in (1, 2].
\end{cases}$$

Denote $v(t)(x) = u(t, x)$ and $B(t)c(t)(x) = c(t, x)$, this problem can be abstracted into

$$\begin{cases} v'(t) = Av(t) + f(t, v(t), v(\rho(t))) + B(t)c(t), & t \in [s_0, t_1) \cup (s_1, a], \\ v(t) = T(t - t_i)g_1(t, v(t)), & t \in (t_1, s_1], \\ v(0) = v(a) \in X, \end{cases} \tag{1}$$

where $\rho(t) = t^2, f(t, v(t), v(\rho(t)))(x) = \frac{1}{12} \cos(v(t)(x) + v(t^2)(x))$ and

$$g_1(t, v(t))(x) = \frac{1}{4} \sin(v(t)(x)).$$

In this case, we have $M = 1$, $C_f = L_f = \frac{1}{12}$, $C_{g_1} = \frac{1}{4}$, and

$$\lambda = M[MC_{g_1} + (C_f + L_f)(a - s_1) + (C_f + L_f)t_1] = \frac{7}{12} < 1.$$

This implies that all assumptions in Theorem 2 are satisfied. Then there exists a unique mild solution for this problem.

Example 2 Consider

$$\begin{cases} \frac{\partial}{\partial t}u(t, x) = \frac{\partial^2}{\partial^2 x}u(t, x) + \frac{1}{8e^t}\frac{|u(t,x)+u(t^2,x)|}{1+|u(t,x)+u(t^2,x)|} + c(t, x), & x \in (0, 1), t \in [0, 1) \cup (2, 3], \\ \frac{\partial}{\partial x}u(t, 0) = \frac{\partial}{\partial x}u(t, 1) = 0, & t \in [0, 1) \cup (2, 3], \\ u(0, x) = u(3, x), & x \in (0, 1), \\ u(t, x) = T(t - 1)\frac{1}{8e^{t-1}}\frac{|u(t,x)|}{1+|u(t,x)|}, & x \in (0, 1), t \in (1, 2]. \end{cases}$$

This problem can be abstracted into (1), with $\rho(t) = t^2$,

$$f(t, v(t), v(\rho(t)))(x) = \frac{1}{8e^t}\frac{|v(t)(x) + v(t^2)(x)|}{1 + |v(t)(x) + v(t^2)(x)|},$$

$$g_1(t, v(t))(x) = \frac{1}{8e^{t-1}}\frac{|v(t)(x)|}{1 + |v(t)(x)|},$$

and $B(t)c(t)(x) = c(t, x)$.

In this case, we have $L = \frac{1}{8}$, $C_{g_1} = \frac{1}{8}$, $\alpha = \frac{1}{4} < \frac{1}{2}$, and $\beta = \frac{1}{8} < 1$.

This implies that all assumptions in Theorem 3 are satisfied. Then this problem has at least one mild solution.

5 Conclusion

In order to describe the evolution of the temperature using a control, we consider periodic boundary value problems for controlled nonlinear impulsive evolution equations. By using operator semigroup theory, impulsive conditions, and fixed point methods, we overcome some difficulties from the proof of equicontinuity and compactness and obtain new existence results. In addition, future work includes expanding the idea signalized in this work and introducing observability. This is a fertile field with vast research projects, which can lead to numerous theories and applications. We plan to devote significant attention to this field of research.

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