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# The dynamics of a stage structure population model with fixed-time birth pulse and state feedback control strategy

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**Abstract**

In this paper, we study a stage structure population model with fixed-time birth pulse and state feedback control strategy. The stability of the trivial solution and the existence of periodic solutions are investigated. Sufficient conditions for the permanence of the system are obtained. Furthermore, some numerical simulations are given to illustrate our results. The superiority of the mixed control strategy is also discussed.

**Keywords:** stage structure; birth pulse; state feedback control strategy; periodic solution; permanence

## 1 Introduction

Stage structure models have attracted much attention in recent years. In most cases, ordinary differential equations are used to build stage structure models [1, 2]. However, impulsive differential equations [3, 4] are also suitable for the mathematical simulation of evolutionary processes in which the parameters state variables undergo relatively long periods of smooth variation followed by a short-term rapid change in their values. Many results have been obtained for stage structure models described by impulsive differential equations [5–7].

In most models, the increases in population due to births are assumed to be time-independent. However, many species give birth in a very short time. Caughley [8] termed this growth pattern a birth pulse. Thus, the continuous reproduction of population should be replaced with a birth pulse. Liu and Chen [9] investigated a two-species competitive system with toxicant and birth pulse and obtained the existence of positive periodic solutions. On the other hand, different kinds of impulsive effects were assumed to occur simultaneously in building models in most cases for simplicity. But different kinds of impulsive effects occur at different moments in many practical problems. Recently, many authors considered different kinds of impulsive effects that occur at different moments

[10–15]. In [16], the authors considered the following model:

$$\begin{cases} \left.\begin{aligned} x'(t) &= -dx - \delta x, \\ y'(t) &= \delta x - dy, \end{aligned}\right\} & t \neq nT, \\ \left.\begin{aligned} \Delta x(t) &= -(1-p)x, \\ \Delta y(t) &= -(1-p)y, \end{aligned}\right\} & t = 2nT, \\ \left.\begin{aligned} \Delta x(t) &= (b - c(x+y))y, \\ \Delta y(t) &= 0, \end{aligned}\right\} & t = (2n+1)T, \end{cases} \tag{1.1}$$

where $x(t)$ and $y(t)$ denote the densities of the immature and mature pests at time $t$, respectively, and $\delta > 0$ is the maturity rate that determines the mean length of the juvenile period. A constant fraction $1-p$ $(0 < p < 1)$ of pest population is killed under the impulsive strategy at time $t = 2nT$, $b > 0$ is the maximum birth rate, $c = r(b-d)$, $d$ is the maximum death rate, and $r$ is a parameter reflecting the relative importance of density-dependent population regulation through birth and death. If $r = 0$, then all density dependence acts through the death rate, and if $r = 1$, then all density dependence acts through the birth rate.

In [16], the authors addressed some problems on system (1.1) such as the existence and stability of positive period-$2T$ solutions and the existence of flip bifurcations by means of bifurcation theory.

In pest management, the pest population can be controlled by many methods, among which the fixed-time impulsive control strategy is widely performed in practice. However, this measure has some shortcomings, regardless of the growth rules of the pest and the cost of management. Another measure based on the state feedback control strategy is proposed in which the pesticide is sprayed only when the observed pest population reaches a certain threshold size. The latter measure is obviously more reasonable and suitable for pest control. Motivated by [16], we consider the following model with fixed-time birth pulse and state feedback control strategy:

$$\begin{cases} \left.\begin{aligned} x'(t) &= -dx - \delta x, \\ y'(t) &= \delta x - dy, \end{aligned}\right\} & t \neq nT, y(t) < h, \\ \left.\begin{aligned} \Delta x(t) &= (b - c(x+y))y, \\ \Delta y(t) &= 0, \end{aligned}\right\} & t = nT, \\ \left.\begin{aligned} \Delta x(t) &= -(1-p_1)x, \\ \Delta y(t) &= -(1-p_2)y, \end{aligned}\right\} & y(t) = h, \end{cases} \tag{1.2}$$

where the threshold $h > 0$ is a constant. When the amount of mature pests reaches the threshold $h$ at time $t_i(h)$, controlling measures are taken, and the amounts of the immature and mature pests abruptly turn to $p_1 x(t_i(h))$ and $p_2 y(t_i(h))$, respectively.

Jiang *et al.* [17] investigated the periodic solutions and their relationship in an SIS epidemic model with fixed-time birth pulses and state feedback pulse treatments. Lin *et al.* [18] considered an SIS epidemic model with fixed-time birth pulses and state feedback pulse treatments. They investigated the existence of positive periodic solutions and permanence of the system. However, stage structure models with fixed-time birth pulses and state feedback control strategy have not been discussed. Motivated by this, we seek to analyze this problem in detail.

The remaining part of this paper is organized as follows. In the next section, we discuss the existence of positive periodic solutions of system (1.2). In Section 3, the stability of the trivial solution is considered. We study the permanence of system (1.2) in Section 4. In Section 5, some numerical simulations are given to illustrate our results. Finally, some concluding remarks are given.

## 2 The existence of periodic solutions

In this section, we investigate the existence of periodic solutions.

Set the initial point of system (1.2) as $A_0(x_0, y_0)$ and suppose that the trajectory originating from the initial point $A_0$ reaches the line $y(t) = h$ at the point $A_1(x_1, y_1)$ at time $t = t_1$, where $0 < t_1 < T$. Then the pesticide is spayed, and the trajectory jumps to the point $A_1^+(x_1^+, y_1^+)$, where $x_1^+ = p_1x_1$, $y_1^+ = p_2y_1$. The trajectory reaches the point $A_2(x_2, y_2)$ at $t = T$ and jumps to $A_2^+(x_2^+, y_2^+)$ due to the effect of birth pulse. Thus, it follows from system (1.2) that

$$\begin{cases} x_1^+ = p_1x_1 = p_1x_0\exp(-(d+\delta)t_1), \\ y_1^+ = p_2y_1 = p_2(x_0+y_0)\exp(-dt_1) - p_2x_0\exp(-(d+\delta)t_1), \end{cases}$$

$$\begin{cases} x_2 = x_1^+\exp(-(d+\delta)(T-t_1)), \\ y_2 = (x_1^+ + y_1^+)\exp(-d(T-t_1)) - x_1^+\exp(-(d+\delta)(T-t_1)), \end{cases}$$

$$\begin{cases} x_2^+ = x_2 + (b - c(x_2+y_2))y_2, \\ y_2^+ = y_2. \end{cases}$$

Hence,

$$y_1 = (x_0+y_0)\exp(-dt_1) - x_0\exp\bigl(-(d+\delta)t_1\bigr) = h, \tag{2.1}$$

$$\begin{cases} x_2^+ = p_1x_0\exp(-(d+\delta)T) + bp_1x_0\exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T)) \\ \qquad + bp_2h\exp(-d(T-t_1)) - cp_1^2x_0^2\exp(-2dT-\delta t_1)(\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_1p_2hx_0\exp(-2dT+dt_1)(2\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_2^2h^2\exp(-2d(T-t_1)), \\ y_2^+ = p_1x_0\exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T)) + p_2h\exp(-d(T-t_1)). \end{cases} \tag{2.2}$$

If $x_2^+ = x_0, y_2^+ = y_0$, then the evolution of the dynamics repeats itself. For this to hold, from Eq. (2.1) and system (2.2) we have

$$\begin{cases} x_0 = p_1x_0\exp(-(d+\delta)T) + bp_1x_0\exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T)) \\ \qquad + bp_2h\exp(-d(T-t_1)) - cp_1^2x_0^2\exp(-2dT-\delta t_1)(\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_1p_2hx_0\exp(-2dT+dt_1)(2\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_2^2h^2\exp(-2d(T-t_1)), \\ x_0 = h\exp(dt_1) + x_0\exp(-\delta t_1) - p_1x_0\exp(-dT)(\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - p_2h\exp(-d(T-t_1)). \end{cases} \tag{2.3}$$

By the second equation of system (2.3) we obtain

$$x_0 = \frac{h(1-p_2\exp(-dT))\exp(dt_1)}{1-\exp(-\delta t_1) + p_1\exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T))} \triangleq \bar{x}_0.$$

Let

$$\begin{aligned} f(x_0,t_1) &= p_1x_0\exp\bigl(-(d+\delta)T\bigr) + bp_1x_0\exp(-dT)\bigl(\exp(-\delta t_1) \\ &\quad - \exp(-\delta T)\bigr) + bp_2h\exp\bigl(-d(T-t_1)\bigr) \\ &\quad - cp_1^2x_0^2\exp(-2dT-\delta t_1)\bigl(\exp(-\delta t_1)-\exp(-\delta T)\bigr) \\ &\quad - cp_1p_2hx_0\exp(-2dT+dt_1)\bigl(2\exp(-\delta t_1) \\ &\quad - \exp(-\delta T)\bigr) - cp_2^2h^2\exp\bigl(-2d(T-t_1)\bigr) - x_0. \end{aligned}$$

Thus,

$$\begin{aligned} f(\bar{x}_0,t_1)|_{t_1=0} &= \frac{h(\exp(dT)-p_2)}{p_1(1-\exp(-\delta T))}\bigl[p_1\exp\bigl(-(d+\delta)T\bigr)-1\bigr] + bh \\ &\quad - \frac{ch^2(\exp(dT)-p_2)[\exp(dT)+p_2-p_2\exp(-\delta T)]}{\exp(2dT)(1-\exp(-\delta T))} \\ &\quad - cp_2^2h^2\exp(-2dT) \end{aligned}$$

and

$$\begin{aligned} f(\bar{x}_0,t_1)|_{t_1=T} &= \frac{h(\exp(dT)-p_2)}{1-\exp(-\delta T)}\bigl[p_1(1-cp_2h)\exp\bigl(-(d+\delta)T\bigr)-1\bigr] \\ &\quad + bp_2h - cp_2^2h^2. \end{aligned}$$

If

$$\bigl(f(\bar{x}_0,t_1)|_{t_1=0}\bigr)\bigl(f(\bar{x}_0,t_1)|_{t_1=T}\bigr) < 0, \tag{2.4}$$

then there exists $\bar{t}_1 \in (0,T)$ such that

$$f(\bar{x}_0,t_1)|_{t_1=\bar{t}_1} = 0.$$

Hence, there exists a period-$T$ solution of system (1.2) where the fixed-time birth pulse occurs at $t = nT$, $n = 1,2,3,\ldots$, whereas the state feedback pulse occurs at $t = (n-1)T + \bar{t}_1$. The initial point is $(x_0^*, y_0^*)$ with

$$\begin{cases} x_0^* = \frac{h(1-p_2\exp(-dT))\exp(d\bar{t}_1)}{1-\exp(-\delta\bar{t}_1)+p_1\exp(-dT)(\exp(-\delta\bar{t}_1)-\exp(-\delta T))}, \\ y_0^* = h\exp(d\bar{t}_1) + x_0^*\exp(-\delta\bar{t}_1) - x_0^*. \end{cases} \tag{2.5}$$

It is easy to see that

$$\begin{aligned} y_0^* &= h\exp(d\bar{t}_1) + x_0^*\exp(-\delta\bar{t}_1) - x_0^* \\ &= h\exp(d\bar{t}_1)\left(1 - \frac{(1-\exp(-\delta\bar{t}_1))(1-p_2\exp(-dT))}{1-\exp(-\delta\bar{t}_1)+p_1\exp(-dT)(\exp(-\delta\bar{t}_1)-\exp(-\delta T))}\right) \\ &> h\exp(d\bar{t}_1)\left(1 - \frac{1-\exp(-\delta\bar{t}_1)}{1-\exp(-\delta\bar{t}_1)+p_1\exp(-dT)(\exp(-\delta\bar{t}_1)-\exp(-\delta T))}\right) \\ &> 0. \end{aligned}$$

**Theorem 2.1** *Assume that condition* (2.4) *holds. Then there exists a period-$T$ solution of system* (1.2) *where the initial point* $(x_0^*, y_0^*)$ *is as in* (2.5).

A special period-$2T$ solution that is subject to spraying pesticide once, and birth pulse two times per period $2T$ is investigated in the following. Set the initial point of system (1.2) as $A_0(x_0, y_0)$. The trajectory originating from the initial point $A_0$ reaches the point $A_1(x_{11}, y_{11})$ at $t = T$ and jumps to the point $A_1^+(x_{11}^+, y_{11}^+)$ due to the effect of birth pulse. Suppose that the trajectory reaches the line $y = h$ at the point $A_2(x_{21}, y_{21})$ for $t = T + t_1$, where $0 < t_1 < T$, and jumps to the point $A_2^+(x_{21}^+, y_{21}^+)$. The trajectory reaches the point $A_3(x_{31}, y_{31})$ at $t = 2T$ and jumps to the point $A_3^+(x_{31}^+, y_{31}^+)$. Further, suppose that $x_{31}^+ = x_0$, $y_{31}^+ = y_0$. Then there exists a period-$2T$ solution.

By system (1.2) we obtain

$$\begin{cases} x_{11} = x_0 \exp(-(d+\delta)T), \\ y_{11} = (x_0 + y_0)\exp(-dT) - x_0 \exp(-(d+\delta)T), \end{cases}$$

$$\begin{cases} x_{11}^+ = x_{11} + (b - c(x_{11} + y_{11}))y_{11}, \\ y_{11}^+ = y_{11}, \end{cases}$$

$$\begin{cases} x_{21} = x_{11}^+ \exp(-(d+\delta)t_1), \\ y_{21} = (x_{11}^+ + y_{11}^+)\exp(-dt_1) - x_{11}^+ \exp(-(d+\delta)t_1), \end{cases}$$

$$\begin{cases} x_{21}^+ = p_1 x_{21}, \\ y_{21}^+ = p_2 y_{21}, \end{cases}$$

and

$$\begin{cases} x_{31} = x_{21}^+ \exp(-(d+\delta)(T - t_1)), \\ y_{31} = (x_{21}^+ + y_{21}^+)\exp(-d(T - t_1)) - x_{21}^+ \exp(-(d+\delta)(T - t_1)), \end{cases}$$

$$\begin{cases} x_{31}^+ = x_{31} + (b - c(x_{31} + y_{31}))y_{31}, \\ y_{31}^+ = y_{31}. \end{cases}$$

Thus,

$$\begin{cases} x_{11}^+ = x_0 \exp(-(d+\delta)T) + b(x_0 + y_0)\exp(-dT) - bx_0 \exp(-(d+\delta)T) \\ \qquad - c(x_0 + y_0)^2 \exp(-2dT) + cx_0(x_0 + y_0)\exp(-2d - \delta T), \\ y_{11}^+ = (x_0 + y_0)\exp(-dT) - x_0 \exp(-(d+\delta)T), \end{cases} \tag{2.6}$$

$$y_{21} = \left(x_{11}^+ + y_{11}^+\right)\exp(-dt_1) - x_{11}^+ \exp\left(-(d+\delta)t_1\right) = h, \tag{2.7}$$

$$\begin{cases} x_{31}^+ = p_1 x_{11}^+ \exp(-(d+\delta)T) + bp_1 x_{11}^+ \exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T)) \\ \qquad + bp_2 h \exp(-d(T - t_1)) - cp_1^2 (x_{11}^+)^2 \exp(-2dT - \delta t_1)(\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_1 p_2 h x_{11}^+ \exp(-2dT + dt_1)(2\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_2^2 h^2 \exp(-2d(T - t_1)), \\ y_{31}^+ = p_1 x_{11}^+ \exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T)) \\ \qquad + p_2 h \exp(-d(T - t_1)). \end{cases} \tag{2.8}$$

If $x_{31}^+ = x_0$, $y_{31}^+ = y_0$, then the evolution of the dynamics repeats itself. For this to hold, from systems (2.6), (2.8), and Eq. (2.7) we get

$$\begin{cases} x_0 = p_1 x_{11}^+ \exp(-(d+\delta)T) + bp_1 x_{11}^+ \exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T)) \\ \qquad + bp_2 h \exp(-d(T-t_1)) - cp_1^2 (x_{11}^+)^2 \exp(-2dT - \delta t_1)(\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_1 p_2 h x_{11}^+ \exp(-2dT + dt_1)(2\exp(-\delta t_1) \\ \qquad - \exp(-\delta T)) - cp_2^2 h^2 \exp(-2d(T-t_1)), \\ x_0 = x_0 \exp(-\delta T) + h \exp(d(T+t_1)) + x_{11}^+ \exp(dT - \delta t_1) - x_{11}^+ \exp(dT) \\ \qquad - p_1 x_{11}^+ \exp(-dT)(\exp(-\delta t_1) - \exp(-\delta T)) - p_2 h \exp(-d(T-t_1)), \\ y_0 = -x_0 + x_0 \exp(-\delta T) + h \exp(d(T+t_1)) + x_{11}^+ \exp(dT - \delta t_1) \\ \qquad - x_{11}^+ \exp(dT). \end{cases} \tag{2.9}$$

Suppose that $(x_0, y_0) = (\bar{x}_0, \bar{y}_0)$ is a solution of (2.9). Then there exists a period-$2T$ solution of system (1.2) where the birth pulse occurs at the moments $t = nT$, whereas the pesticide is sprayed at $t = (2n-1)T + t_1$. The initial point is $(\bar{x}_0, \bar{y}_0)$. Then we obtain the following result.

**Theorem 2.2** *If the initial point* $(x_0, y_0) = (\bar{x}_0, \bar{y}_0)$, *where* $(\bar{x}_0, \bar{y}_0)$ *is the solution of* (2.9), *then there exists a period*-$2T$ *solution of system* (1.2).

## 3 The stability of the trivial solution

Now, we discuss the stability of the trivial solution of system (1.2).

Let $N(t) = x(t) + y(t)$. Then system (1.2) is equivalent to

$$\begin{cases} \left.\begin{aligned} N'(t) &= -dN, \\ y'(t) &= \delta N - (\delta + d)y, \end{aligned}\right\} & t \neq nT, y(t) < h, \\ \left.\begin{aligned} \Delta N(t) &= (b - cN)y, \\ \Delta y(t) &= 0, \end{aligned}\right\} & t = nT, \\ \left.\begin{aligned} \Delta N(t) &= -N + p_1(N-y) + p_2 y, \\ \Delta y(t) &= -(1-p_2)y, \end{aligned}\right\} & y(t) = h. \end{cases} \tag{3.1}$$

Set $p = \max\{p_1, p_2\}$. Then $\Delta N(t) = -N + p_1(N-y) + p_2 y < -N + pN$.

Set the initial point of system (3.1) as $A_0(N_0, y_0)$. In the following, four cases are considered for $N(t)$.

(H1) The trajectory originating from the initial point $A_0$ does not reach the line $y(t) = h$ for $0 < t \leq T$.

(H2) The trajectory originating from the initial point $A_0$ reaches the line $y(t) = h$ once at time $T$ for $0 < t \leq T$.

(H3) The trajectory originating from the initial point $A_0$ reaches the line $y(t) = h$ once at time $t_1$ for $0 < t < T$ where $0 < t_1 < T$.

(H4) The trajectory originating from the initial point $A_0$ reaches the line $y(t) = h$ $k$ times for $0 < t < T$.

(H1) It follows from system (3.1) that if $y(t) < h$ for $0 < t \leq T$, then

$$\begin{aligned} N(T^+) &= N(T) + \bigl(b - cN(T)\bigr)y(T) \leq N(T) + \bigl(b - cN(T)\bigr)N(T) \\ &= (1+b)N_0 \exp(-dT) - cN_0^2 \exp(-2dT) \triangleq f_1(N_0). \end{aligned}$$

(H2) Suppose that the trajectory originating from the initial point $A_0$ reaches the line $y(t) = h$ at the point $A_{21}(N_{21}, y_{21})$ for $t = T$. Then birth pulse occurs, and the pesticide is spayed. The trajectory jumps to the point $A_{21}^+(N_{21}^+, y_{21}^+)$.

From system (3.1) we obtain

$$N_{21}^+ = N_{21} + (b - cN_{21})y_{21} \le (1 + b)N_{21} = (1 + b)N_0 \exp(-dT) \triangleq f_2(N_0).$$

(H3) Suppose that the trajectory originating from the initial point $A_0$ reaches the line $y = h$ at the point $A_{31}(N_{31}, y_{31})$ at time $t = t_1$, where $0 < t_1 < T$, $N_{31} = N_0 \exp(-dt_1)$, and $y_{31} = h$. Then the pesticide is spayed, and the trajectory jumps to the point $A_{31}^+(N_{31}^+, y_{31}^+)$. The trajectory reaches the point $A_{32}(N_{32}, y_{32})$ at $t = T$ and jumps to $A_{32}^+(N_{32}^+, y_{32}^+)$ due to the effect of birth pulse.

From system (3.1) we have

$$N_{31}^+ = p_1(N_{31} - y_{31}) + p_2 y_{31} \le pN_{31} = pN_0 \exp(-dt_1).$$

So

$$N_{32} = N_{31}^+ \exp\bigl(-d(T - t_1)\bigr) \le pN_0 \exp(-dt_1) \exp\bigl(-d(T - t_1)\bigr) = pN_0 \exp(-dT),$$
$$N_{32}^+ = N_{32} + (b - cN_{32})y_{32} \le (1 + b)N_{32} = (1 + b)pN_0 \exp(-dT) \triangleq f_3(N_0).$$

(H4) Suppose that the trajectory originating from the initial point $A_0$ reaches the line $y = h$ at the point $A_n(N_n, y_n)$ at time $t = t_n$, where $0 < t_n < T$, $0 < n \le k$, and jumps to the point $A_n^+(N_n^+, y_n^+)$. The trajectory reaches the point $A_{k+1}(N_{k+1}, y_{k+1})$ at $t = T$ and jumps to the point $A_{k+1}^+(N_{k+1}^+, y_{k+1}^+)$ due to the effect of birth pulse.

Similarly to the discussion of case (H3), we have

$$N_{k+1} \le p^k N_0 \exp(-dT),$$
$$N_{k+1}^+ = N_{k+1} + (b - cN_{k+1})y_{k+1} \le (1 + b)N_{k+1} \le (1 + b)p^k N_0 \exp(-dT) \triangleq f_4(N_0).$$

Hence, for $0 < b < \exp(dT) - 1$,

$$0 < f_1'(0) < 1, \qquad 0 < f_2'(0) < 1, \qquad 0 < f_3'(0) < 1, \qquad 0 < f_4'(0) < 1.$$

Therefore, $N(t)$ tends to zero with $t$ increasing for $0 < b < \exp(dT) - 1$. So $N(t) < \frac{(\delta+d)h}{2\delta}$ over a limited period of time for any initial point $(N_0, y_0)$. It is seen from (3.1) that

$$\frac{dy}{dt} = \delta N - (\delta + d)y < 0$$

if $N < \frac{(\delta+d)h}{2\delta}$, $y > \frac{h}{2}$, which means that the trajectory of system (3.1) will enter region $\{(N, y) | 0 < N < \frac{(\delta+d)h}{2\delta}, 0 < y < h\}$ over a limited period of time for any initial point $(N_0, y_0)$. Then the trajectory of system (3.1) no longer reaches the line $y = h$. Suppose the number of mature pests is small (less than the threshold level $h$). It follows from system (3.1) that

$$\begin{cases} \left.\begin{aligned} &N'(t) = -dN, \\ &y'(t) = \delta N - (\delta + d)y, \end{aligned}\right\} & t \neq nT, \\ \left.\begin{aligned} &\Delta N(t) = (b - cN)y, \\ &\Delta y(t) = 0, \end{aligned}\right\} & t = nT. \end{cases} \tag{3.2}$$

The trajectory originating from the initial point $A_0(N_0, y_0)$ reaches the point $A_1(N_1, y_1)$ at time $t = T$ and jumps to the point $A_1^+(N_1^+, y_1^+)$ due to the effect of birth pulse. Similarly to the above analysis, we get

$$\begin{cases} N_1^+ = N_1(T) + (b - cN_1(T))y_1(T) \leq N_1(T) + (b - cN_1(T))N_1(T) \\ \qquad = (1 + b)N_0 \exp(-dT) - cN_0^2 \exp(-2dT), \\ y_1^+ = N_0 \exp(-dT) - (N_0 - y_0) \exp(-(\delta + d)T). \end{cases}$$

Then the following map is obtained:

$$\begin{cases} N_{n+1} = (1 + b)N_n \exp(-dT) - cN_n^2 \exp(-2dT), \\ y_{n+1} = (\exp(-dT) - \exp(-(\delta + d)T))N_n + y_n \exp(-(\delta + d)T). \end{cases} \tag{3.3}$$

There exists a fixed point $\bar{A}(0,0)$ of map (3.3). The associated characteristic polynomial of the fixed point $\bar{A}$ of map (3.3) is given by

$$\begin{vmatrix} \lambda - (1 + b)\exp(-dT) & 0 \\ -\exp(-dT) + \exp\bigl(-(\delta + d)T\bigr) & \lambda - \exp\bigl(-(\delta + d)T\bigr) \end{vmatrix}.$$

Therefore,

$$\lambda_1 = (1 + b)\exp(-dT), \qquad \lambda_2 = \exp\bigl(-(\delta + d)T\bigr).$$

Note that $0 < \lambda_1 < 1$ and $0 < \lambda_2 < 1$ for $0 < b < \exp(dT) - 1$. Then it follows that the fixed point $\bar{A}(0,0)$ of map (3.3) is locally asymptotically stable. Hence, the trivial solution of system (1.2) is locally asymptotically stable for $0 < b < \exp(dT) - 1$, which is given in the following result.

**Theorem 3.1** *The trivial solution of system* (1.2) *is locally asymptotically stable for* $0 < b < \exp(dT) - 1$.

## 4 Permanence

In the following, we discuss the permanence of system (1.2) by means of system (3.1) and assume that $c > 0$. We set the initial point of system (3.1) as $A_0(N_0, y_0)$ where $0 < y_0 < h$. Two cases are considered.

(E1) The trajectory originating from the initial point $A_0$ does not reach the line $y(t) = h$ for $0 < t \leq T$.

(E2) The trajectory originating from the initial point $A_0$ reaches the line $y(t) = h$ at time $t_1$ where $0 < t_1 \leq T$.

(E1) It follows from system (3.1) that

$$\begin{aligned} N(T^+) &= N(T) + \big(b - cN(T)\big)y(T) \\ &= N_0 \exp(-dT) + bN_0 \exp(-dT) - b(N_0 - y_0)\exp\big(-(\delta + d)T\big) \\ &\quad - cN_0^2 \exp(-2dT) + cN_0 \exp(-dT)(N_0 - y_0)\exp\big(-(\delta + d)T\big) \\ &> N_0\big[(1 + b)\exp(-dT) - b\exp\big(-(\delta + d)T\big) - cN_0 \exp(-2dT)\big]. \end{aligned}$$

Suppose $(1 + b)\exp(-dT) - b\exp(-(\delta + d)T) > 1$. Then there exists $\epsilon_0 > 0$ such that $(1 + b)\exp(-dT) - b\exp(-(\delta + d)T) > 1 + \epsilon_0$. Thus, $N(T^+) > (1 + \epsilon_0)N_0$ if

$$0 < N_0 < \frac{\exp(dT)}{c}\big[1 + b - b\exp(-\delta T) - (1 + \epsilon_0)\exp(dT)\big] \triangleq D_1. \tag{4.1}$$

Assume that

$$\delta D_1 - (\delta + d)h < 0. \tag{4.2}$$

Then there exists $\epsilon > 0$ small enough such that $\delta D_1 - (\delta + d)(1 - \epsilon)h < 0$. From (4.2) we obtain

$$\frac{dy}{dt} = \delta N - (\delta + d)y < \delta D_1 - (\delta + d)(1 - \epsilon)h < 0 \quad \text{for } N < D_1, y > (1 - \epsilon)h.$$

Thus, $y(t) < h$ if $N(t) < D_1$ for all $t > 0$. If $N(t) < D_1$ for $0 < t \le nT$, then

$$N(nT^+) \ge (1 + \epsilon_0)N\big((n - 1)T^+\big) \ge (1 + \epsilon_0)^n N_0. \tag{4.3}$$

It follows from (4.3) that there exists $n_0 > 0$ such that $N(n_0T^+) > D_1$. In this case, there exists $t_2 > 0$, where $n_0T < t_2 < (n_0 + 1)T$, such that $y(t_2) = h$ and $N(t_2) \ge D_1$. Then the pesticide is sprayed. Let $\bar{p} = \min\{p_1, p_2\}$. Then

$$N(t_2^+) = p_1x(t_2) + p_2y(t_2) \ge \bar{p}N(t_2).$$

We need to consider the following two cases.

(1) $N(t_2^+) \ge D_1$.

(2) $N(t_2^+) < D_1$.

(1) If $N(t_2^+) \ge D_1$, then, similarly to the above analysis, there exists $t_2 < t_3 < (n_0 + 1)T$ such that $y(t_3) = h$ and $N(t_3^+) \ge \bar{p}N(t_3)$. For $N(t_3^+) \ge D_1$, we may continue the same argument. For $N(t_3^+) < D_1$, by the first equation of system (3.1) we find

$$N(t) < N(t_3^+) < D_1 \quad \text{for } t_3 < t < (n_0 + 1)T.$$

Then $y(t) < h$ for $t_3 < t < (n_0 + 1)T$. Thus,

$$N\big((n_0 + 1)T\big) = N(t_3^+)\exp\big(-d\big((n_0 + 1)T - t_3\big)\big) \ge \bar{p}N(t_3)\exp(-dT).$$

Since $y(t_3) = h$, from the above discussion we get

$$N(t_3) > D_1.$$

Then $N((n_0+1)T) \geq \bar{p}D_1\exp(-dT)$. It is easy to see that the birth pulse $\Delta N((n_0+1)T) > 0$; then $N((n_0+1)T^+) \geq N((n_0+1)T)$. It is possible that $N((n_0+1)T^+) \geq D_1$. It is well known that this case coincides with the case $N(t_2^+) \geq D_1$, and therefore we omit it.

From the above analysis we get

$$N(t) \geq N\big((n_0+1)T\big) \geq \bar{p}D_1\exp(-dT) \quad \text{for } n_0T < t \leq (n_0+1)T.$$

If $N((n_0+1)T^+) < D_1$, then we have

$$\begin{aligned} N\big((n_0+2)T\big) &= N\big((n_0+1)T^+\big)\exp(-dT) \geq N\big((n_0+1)T\big)\exp(-dT) \\ &\geq \bar{p}D_1\exp(-2dT) \triangleq m_1. \end{aligned} \tag{4.4}$$

Therefore, $N(t) \geq m_1$ for $n_0T < t < (n_0+2)T$. Since $N((n_0+1)T^+) < D_1$, we obtain $N((n_0+2)T^+) \geq N((n_0+1)T^+)$. For $N((n_0+2)T^+) \geq D_1$, this case coincides with $N(t_2^+) \geq D_1$. For $N((n_0+2)T^+) < D_1$, we obtain

$$N\big((n_0+3)T\big) = N\big((n_0+2)T^+\big)\exp(-dT) \geq N\big((n_0+1)T^+\big)\exp(-dT) \geq m_1.$$

Hence,

$$N(t) > N\big((n_0+3)T\big) \geq m_1 \quad \text{for } (n_0+2)T < t \leq (n_0+3)T.$$

Continuing the same argument, we get $N(t) \geq m_1$ for $t > n_0T$.

(2) If $N(t_2^+) < D_1$, then we find

$$\begin{aligned} N\big((n_0+1)T\big) &= N\big(t_2^+\big)\exp\big(-d\big((n_0+1)T - t_2\big)\big) \geq \bar{p}N(t_2)\exp(-dT) \\ &\geq \bar{p}D_1\exp(-dT). \end{aligned}$$

From the first and fifth equations of system (3.1) we have

$$N(t) \geq N\big((n_0+1)T\big) \geq \bar{p}D_1\exp(-dT) > m_1 \quad \text{for } n_0T < t \leq (n_0+1)T.$$

It is easy to see that $N((n_0+1)T^+) \geq N((n_0+1)T)$. If $N((n_0+1)T^+) \geq D_1$, it is well known that the case coincides with case (1). For $N((n_0+1)T^+) < D_1$, we obtain

$$\begin{aligned} N\big((n_0+2)T^+\big) \geq N\big((n_0+2)T\big) &= N\big((n_0+1)T^+\big)\exp(-dT) \\ &\geq N\big((n_0+1)T\big)\exp(-dT) \geq \bar{p}D_1\exp(-2dT). \end{aligned}$$

From the first and fifth equations of system (3.1) we get

$$N(t) \geq N\big((n_0+2)T\big) \geq \bar{p}D_1\exp(-2dT) = m_1 \text{for } (n_0+1)T < t \leq (n_0+2)T.$$

Since $N((n_0+1)T^+) < D_1$, we obtain $N((n_0+2)T^+) \geq N((n_0+1)T^+)$. For $N((n_0+2)T^+) \geq D_1$, this case coincides with case (1). For $N((n_0+2)T^+) < D_1$, we obtain

$$\begin{aligned} N\big((n_0+3)T\big) &= N\big((n_0+2)T^+\big)\exp(-dT) \geq N\big((n_0+1)T^+\big)\exp(-dT) \\ &\geq \bar{p}D_1\exp(-2dT) = m_1. \end{aligned}$$

Hence,

$$N(t) > N\big((n_0+3)T\big) \geq m_1 \quad \text{for } (n_0+2)T < t \leq (n_0+3)T.$$

Continuing the same argument, we get $N(t) \geq m_1$ for $t > n_0T$.

(E2) In this case, we have

$$y(t_1) = h \quad \text{for } 0 < t_1 \leq T.$$

By case (E1) we obtain

$$N(t_1) > D_1.$$

Similarly to the discussion of case (1), that is, $N(t_1^+) > D_1$, it is easy to see that $N(t) > m_1$ for large enough $t > 0$.

In conclusion, for any initial value $N_0 > 0$, there exists $t_4 > 0$ such that $N(t) \geq m_1$ for $t > t_4$.

If $y(t_5) = h$ for $0 < t_5 < T$, then

$$\Delta N(t_5) = -N(t_5) + p_1\big(N(t_5) - y(t_5)\big) + p_2y(t_5) \leq -N(t_5) + pN(t_5).$$

Thus, $N(t_5^+) \leq pN(t_5) < N(t_5)$. Then the number of the total pests is decreasing when the pesticide is spayed. Therefore, $N(T) \leq N_0\exp(-dT)$.

In view of the birth pulse $\Delta N = (b - cN)N > 0$, it is easy to see that $b - cN(T) \geq b - cN_0\exp(-dT) > 0$, that is, $N_0 < \frac{b\exp(dT)}{c}$. From the first and fifth equations of system (3.1) we get

$$N(t) \leq N_0 < \frac{b\exp(dT)}{c} \quad \text{for } 0 < t \leq T.$$

It is easy to see that

$$\begin{aligned} N\big(T^+\big) &= N(T) + \big(b - cN(T)\big)y(T) \leq N(T) + \big(b - cN(T)\big)N(T) \\ &\leq -cN^2(T) + (1+b)N(T) \leq \frac{(1+b)^2}{4c}. \end{aligned}$$

It is well known that

$$N(2T) \leq N\big(T^+\big)\exp(-dT) \leq \frac{(1+b)^2}{4c}\exp(-dT).$$

In view of the birth pulse $\Delta N(2T) = (b - cN(2T))N(2T) > 0$, we get

$$N(2T) < \frac{b}{c}.$$

For this to hold, the following condition is satisfied;

$$\frac{(1+b)^2}{4c}\exp(-dT) < \frac{b}{c}, \quad \text{that is,} \quad (1+b)^2 \le 4b\exp(dT).$$

From the first and fifth equations of system (3.1) we obtain

$$N(t) \le N(T^+) \le \frac{(1+b)^2}{4c} \quad \text{for } T < t \le 2T.$$

It is easy to see that

$$\begin{aligned} N(2T^+) &= N(2T) + \big(b - cN(2T)\big)y(2T) \le N(2T) + \big(b - cN(2T)\big)N(2T) \\ &\le -cN^2(2T) + (1+b)N(2T) \le \frac{(1+b)^2}{4c}. \end{aligned}$$

From the first and fifth equations of system (3.1) we get

$$N(t) \le N(2T^+) \le \frac{(1+b)^2}{4c} \quad \text{for } 2T < t \le 3T$$

and

$$N(3T) < N(2T^+)\exp(-dT) \le \frac{(1+b)^2}{4c}\exp(-dT) \le \frac{b}{c}.$$

It is well known that

$$\begin{aligned} N(3T^+) &= N(3T) + \big(b - cN(3T)\big)y(3T) \le N(3T) + \big(b - cN(3T)\big)N(3T) \\ &\le -cN^2(3T) + (1+b)N(3T) \le \frac{(1+b)^2}{4c}. \end{aligned}$$

Continuing the same argument, we obtain

$$N(t) \le \frac{(1+b)^2}{4c} \quad \text{for } nT < t \le (n+1)T.$$

Let

$$M_1 = \max\left\{\frac{b\exp(dT)}{c}, \frac{(1+b)^2}{4c}\right\}. \tag{4.5}$$

Thus, for any initial value $0 < N_0 < \frac{b\exp(dT)}{c}$ and $t > 0$,

$$N(t) < M_1.$$

Therefore, for large enough $t > 0$,

$$m_1 \leq N(t) \leq M_1. \tag{4.6}$$

Now we consider the persistence of the mature pest populations. It follows from (4.6) that there exists $t_6 > 0$ such that $N(t) \geq m_1$ for $t > t_6$. Thus, from system (3.1) we have

$$\frac{dy}{dt} = \delta N - (\delta + d)y \geq \delta m_1 - (\delta + d)y \quad \text{for } t > t_6.$$

In the following, we consider three cases.

(1) $p_2 h < \frac{\delta m_1}{\delta+d} < h$.

It is easy to see that for large enough $t > 0$ and $y < \frac{\delta m_1}{\delta+d}$,

$$\frac{dy}{dt} \geq \delta m_1 - (\delta + d)y > 0.$$

Hence, for large enough $t > 0$, $p_2 h \leq y(t) \leq h$.

(2) $\frac{\delta m_1}{\delta+d} \leq p_2 h$.

It is well known that for large enough $t > 0$, $\frac{\delta m_1}{\delta+d} \leq y(t) \leq h$.

(3) $\frac{\delta m_1}{\delta+d} \geq h$.

For large enough $t > 0$ and $y \leq h$,

$$\frac{dy}{dt} \geq \delta m_1 - (\delta + d)y > 0.$$

Thus, for large enough $t > 0$, $p_2 h \leq y(t) \leq h$.

In conclusion, for large enough $t > 0$, $m_2 \leq y(t) \leq h$, where

$$m_2 = \min\left\{p_2 h, \frac{\delta m_1}{\delta + d}\right\}. \tag{4.7}$$

Then we obtain the following result.

**Theorem 4.1** *Assume that condition* (4.2) *holds*, $c > 0$, $(1 + b)^2 \leq 4b\exp(dT)$, *and* $(1 + b)\exp(-dT) - b\exp(-(\delta + d)T) > 1$. *Then for any initial point* $A_0(x_0, y_0)$ *in system* (1.2) *such that* $0 < x_0 + y_0 < \frac{b\exp(\sigma T)}{c}$ *and* $0 < y_0 \leq h$, *there exists* $\bar{t} > 0$ *such that* $m_1 \leq x(t) + y(t) \leq M_1$ *and* $m_2 \leq y(t) \leq h$ *for* $t > \bar{t}$, *where* $m_1$, $M_1$, *and* $m_2$ *are given in* (4.4), (4.5), *and* (4.7), *respectively.*

The particular case $c = 0$ is investigated as follows. Set the initial point of system (1.2) as $\bar{A}_1(x_1, y_1)$ with $0 < y_1 < h$. We consider two cases.

- (B1) The trajectory originating from the initial point $A_1$ does not reach the line $y(t) = h$ for $0 < t \leq T$.
- (B2) The trajectory originating from the initial point $A_1$ reaches the line $y(t) = h$ at time $t_7$ where $0 < t_7 \leq T$.

(B1) It follows from system (1.2) that

$$\begin{aligned} x(T^+) &= x_1 \exp\bigl(-(\delta + d)T\bigr) + b(x_1 + y_1)\exp(-dT) - bx_1\exp\bigl(-(\delta + d)T\bigr) \\ &> x_1 \exp\bigl(-(\delta + d)T\bigr) + bx_1\exp(-dT) - bx_1\exp\bigl(-(\delta + d)T\bigr). \end{aligned}$$

It is well known that $x(T^+) > x_0$ if

$$1 - b + b\exp(\delta T) - \exp\big((\delta + d)T\big) > 0. \tag{4.8}$$

From system (1.2) we have

$$\frac{dy}{dt} = \delta x - dy < 0 \quad \text{for } x < \frac{dp_2 h}{\delta}, y > p_2 h.$$

Thus,

$$y(t) < h \quad \text{if } x(t) < \frac{dp_2 h}{\delta} \text{ for all } t > 0.$$

By (4.8) it is easy to see that there exists $\epsilon_1 > 0$ such that

$$1 - b + b\exp(\delta T) - \exp\big((\delta + d)T\big) > \epsilon_1,$$

that is,

$$\exp\big(-(\delta + d)T\big) + b\exp(-dT) - b\exp\big(-(\delta + d)T\big) > 1 + \epsilon_1.$$

Hence,

$$x\big(T^+\big) > x_1(1 + \epsilon_1).$$

If $x(t) < \frac{dp_2 h}{\delta}$ for $0 < t \leq nT$, then

$$x\big(nT^+\big) \geq (1 + \epsilon_1)x\big((n-1)T^+\big) \geq (1 + \epsilon_1)^n x_1. \tag{4.9}$$

It follows from (4.9) that there exists $n_0 > 0$ such that $x(nT^+) > \frac{dp_2 h}{\delta}$. Similarly to the case $c > 0$, there exists $t_8 > 0$ such that, for $t > t_8$,

$$x(t) > \frac{dp_2 h}{\delta}\exp\big(-2(d + \delta)T\big) \triangleq \bar{m}_1. \tag{4.10}$$

(B2) In this case, we get

$$y(t_7) = h \quad \text{for } 0 < t_7 \leq T.$$

By case (B1) we obtain

$$x(t_7) > \frac{dp_2 h}{\delta}.$$

Similarly to the discussion of case (B1), it is easy to see that $x(t) > \bar{m}_1$ for large enough $t > 0$.

By system (1.2) we obtain

$$\frac{dy}{dt} = \delta x - dy \geq \delta\bar{m}_1 - dy \quad \text{for large enough } t > 0.$$

In the following, we discuss three cases.

(1) $p_2 h < \frac{\delta \bar{m}_1}{d} < h$.

It is easy to see that

$$\frac{dy}{dt} \geq \delta \bar{m}_1 - dy > 0 \quad \text{for } y < \frac{\delta \bar{m}_1}{d}.$$

Hence, for large enough $t > 0$, $p_2 h \leq y(t) \leq h$.

(2) $\frac{\delta \bar{m}_1}{d} < p_2 h$.

It is well known that for large enough $t > 0$, $\frac{\delta \bar{m}_1}{d} \leq y(t) \leq h$.

(3) $\frac{\delta \bar{m}_1}{d} \geq h$.

For large enough $t > 0$ and $y \leq h$,

$$\frac{dy}{dt} \geq \delta \bar{m}_1 - dy > 0.$$

Thus, for large enough $t > 0$, $p_2 h \leq y(t) \leq h$.

In conclusion, for large enough $t > 0$, $\bar{m}_2 \leq y(t) \leq h$, where

$$\bar{m}_2 = \min \left\{ p_2 h, \frac{\delta \bar{m}_1}{d} \right\}. \tag{4.11}$$

Then we obtain the following result.

**Theorem 4.2** *Assume that condition* (4.8) *holds and* $c = 0$. *Then for any solution of system* (1.2), *there exists* $t^* > 0$ *such that* $\bar{m}_1 \leq x(t)$ *and* $\bar{m}_2 \leq y(t) \leq h$ *for* $t > t^*$, *where* $\bar{m}_1$ *and* $\bar{m}_2$ *are given in* (4.10) *and* (4.11), *respectively.*

## 5 Numerical simulation

Now consider the following example:

$$\begin{cases} \left.\begin{aligned} x'(t) &= -0.1x - 0.2x, \\ y'(t) &= 0.2x - 0.1y, \end{aligned}\right\} & t \neq nT, y(t) < h, \\ \left.\begin{aligned} \Delta x(t) &= (b - c(x+y))y, \\ \Delta y(t) &= 0, \end{aligned}\right\} & t = nT, \\ \left.\begin{aligned} \Delta x(t) &= -(1-p_1)x, \\ \Delta y(t) &= -(1-p_2)y, \end{aligned}\right\} & y(t) = h. \end{cases} \tag{5.1}$$

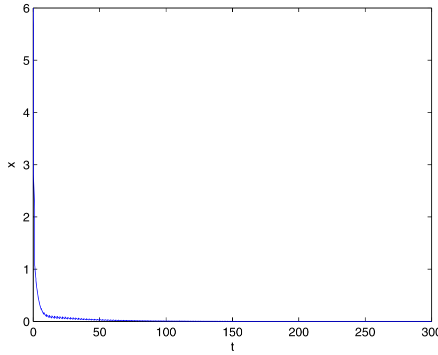
In our case, $d = 0.1$, $\delta = 0.2$, $T = 2$, $c = 0.2$, $b_0 = \exp(dT) - 1 = \exp(0.1 \times 2) - 1 \approx 0.2215$.

### 5.1 Species extinction

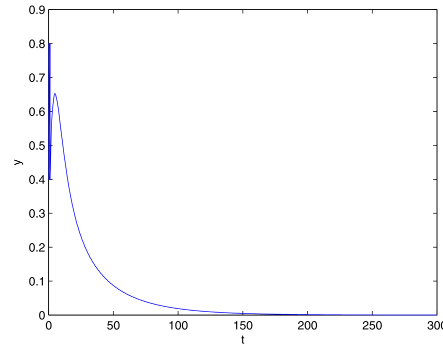
Set $b = 0.2$, $h = 0.8$, $p_1 = 0.4$, $p_2 = 0.5$, where $b \in (0, b_0)$. The phase portrait of system (5.1) with the initial point $(6, 0.6)$ is shown in Figure 1. It is seen that the solution tends to the trivial solution with $t$ increasing, which means that the trivial solution is asymptotically stable.

### 5.2 Species persistence

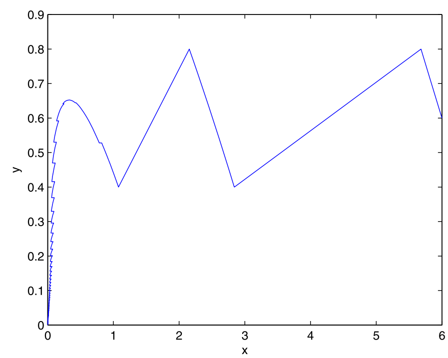
Firstly, set $c = 0.2$. The phase portrait of system (5.1) with $h = 4$, $b = 3$, $p_1 = 0.4$, $p_2 = 0.5$, and the initial points $(0.001, 0.001)$ and $(0.001, 0.1)$ are shown in Figure 2. It is seen that




(a) Time series of $x$.

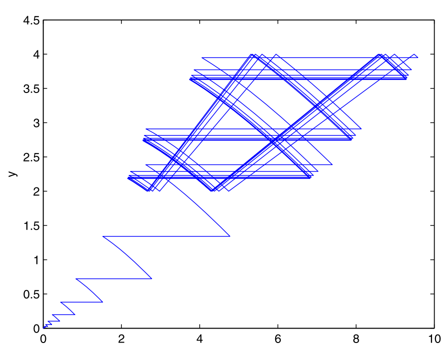



(b) Time series of $y$.

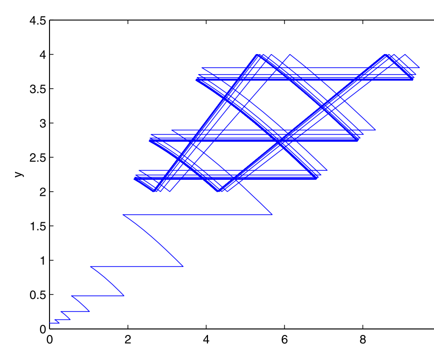



(c) Phase portrait.

**Figure 1 The stable trivial solution of system (5.1).**




(a) The solution with the initial point (0.001, 0.001).

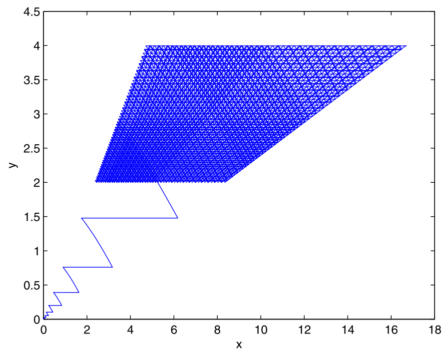



(b) The solution with the initial point (0.001, 0.1).

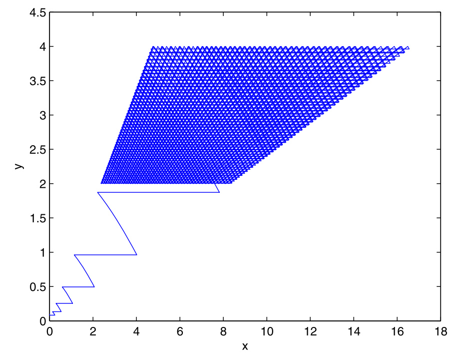
**Figure 2 The solutions of system (5.1) with $c = 0.2$.**

the trajectory of system (5.1) enters a quadrilateral and stays in it forever, which verifies Theorem 4.1.

Set $c = 0$. The phase portrait of system (5.1) with $h = 4$, $b = 3$, $p_1 = 0.4$, $p_2 = 0.5$, and the initial points $(0.001, 0.001)$ and $(0.001, 0.1)$ are shown in Figure 3. It is seen that the trajectory of system (5.1) enters a quadrilateral and stays in it forever, which verifies Theorem 4.2.

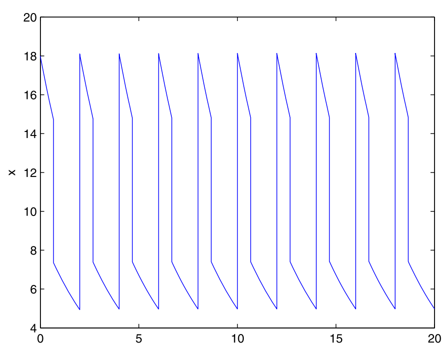



(a) The solution with the initial point (0.001, 0.001).

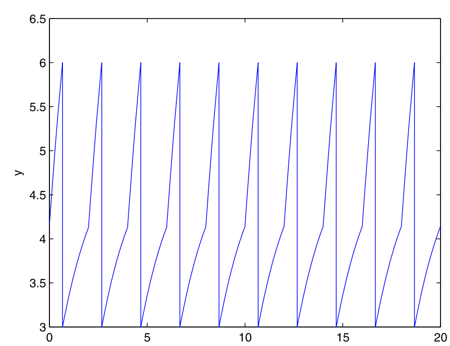



(b) The solution with the initial point (0.001, 0.1).

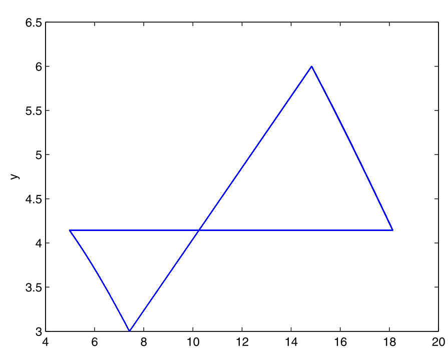
**Figure 3 The solutions of system (5.1) with $c = 0$.**




(a) Time series of $x$.

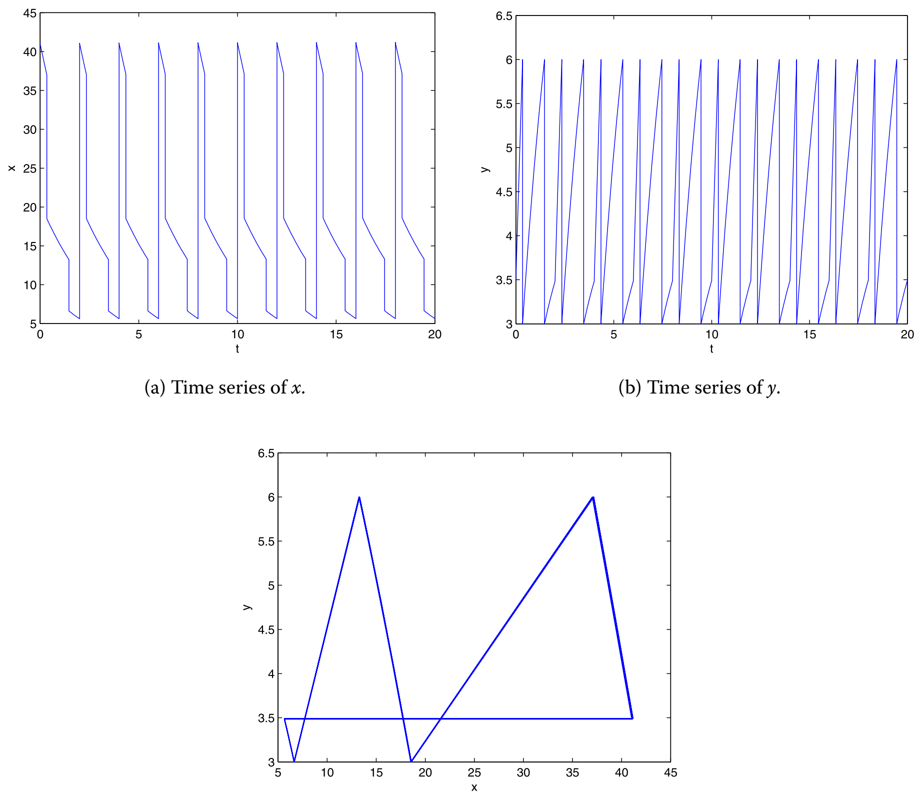



(b) Time series of $y$.




(c) Phase portrait.

**Figure 4 The period-$T$ solution of system (5.1) with $h = 6$, $b = 5$, $p_1 = p_2 = 0.5$, and the initial point (18, 4.154).**




(a) Time series of $x$.

(b) Time series of $y$.

(c) Phase portrait.

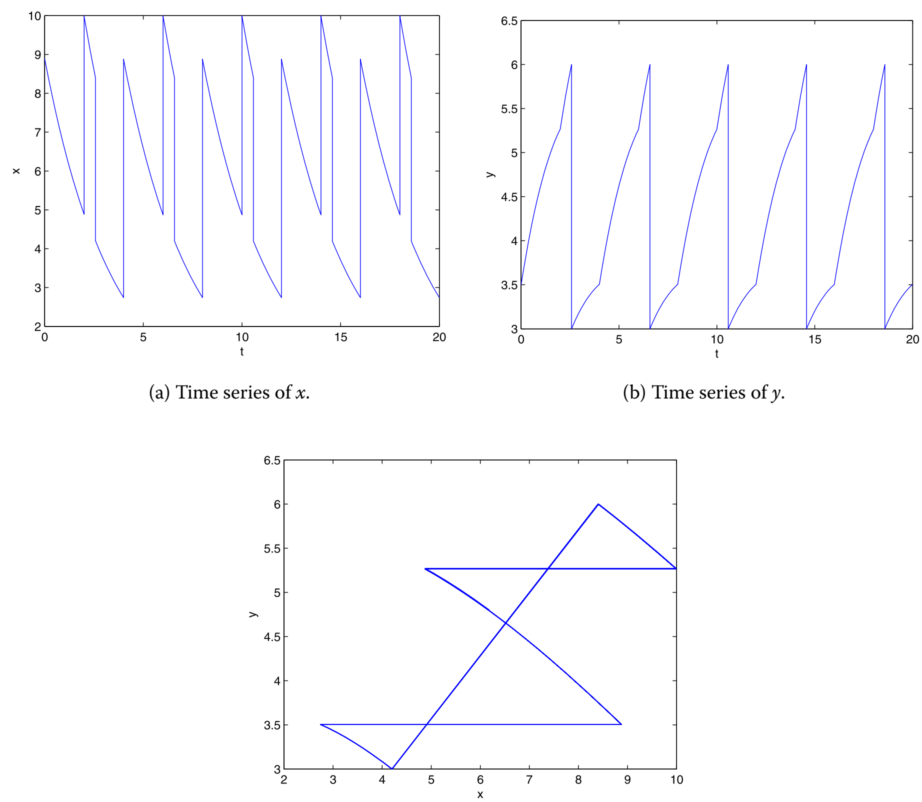
**Figure 5 The period-$T$ solution of system (5.1) with $h = 6$, $b = 12$, $p_1 = p_2 = 0.5$, and the initial point (41, 3.5).**

### 5.3 The existence of periodic solutions

Set $h = 6$, $b = 5$, $p_1 = p_2 = 0.5$. The phase portrait and time series of $x$ and $y$ of system (5.1) with the initial point (18, 4.154) are shown in Figure 4, where $t_1 = 0.646$. It is seen that $y(t)$ reaches the threshold value $y(t) = 6$ at $t = 2n - 2 + t_1$, where $n = 1, 2, \ldots$ . The pulse treatment and birth pulse occur at $t = 2n - 2 + t_1$ and $t = 2n$, respectively. Then system (5.1) has a period-$T$ solution, which verifies Theorem 2.1.

Set $h = 6$, $b = 12$, $p_1 = p_2 = 0.5$. The phase portrait and time series of $x$ and $y$ of system (5.1) with the initial point (41, 3.5) are shown in Figure 5, where $t_1 \approx 0.34$, $t_2 \approx 1.45$. It is seen that $y(t)$ reaches the threshold value $y(t) = 6$ at $t = 2n - 2 + t_1$ and $t = 2n - 2 + t_2$, respectively, where $n = 1, 2, \ldots$ . The pulse treatment occurs at $t = 2n - 2 + t_1$ and $t = 2n - 2 + t_2$, respectively. The birth pulse occurs at $t = 2n$. Then system (5.1) has a period-$T$ solution.

Set $h = 6$, $b = 3$, $p_1 = p_2 = 0.5$. The phase portrait and time series of $x$ and $y$ of system (5.1) with the initial point (8.9, 3.5) are shown in Figure 6, where $t_1 \approx 2.6$. It is seen that $y(t)$ reaches the threshold value $y(t) = 6$ at $t = 2n - 1 + t_1$, where $n = 1, 2, \ldots$ . The pulse treatment and birth pulse occur at $t = 2n - 1 + t_1$ and $t = 2n$, respectively. Then system (5.1) has a period-$2T$ solution.




(a) Time series of $x$.

(b) Time series of $y$.

(c) Phase portrait.

**Figure 6 The period-2$T$ solution of system (5.1) with $h = 6$, $b = 3$, $p_1 = p_2 = 0.5$, and the initial point (8.9, 3.5).**

## 5.4 The superiority of the mixed control strategy

Set $d = 0.1$, $\delta = 0.2$, $T = 2$, $c = 0.2$, $h = 6$, $b = 3$, $p_1 = 0.7$, $p_2 = 0.75$, and the initial point $(12, 0.1)$ in system (1.2). The system of the time-fixed pulse control strategy

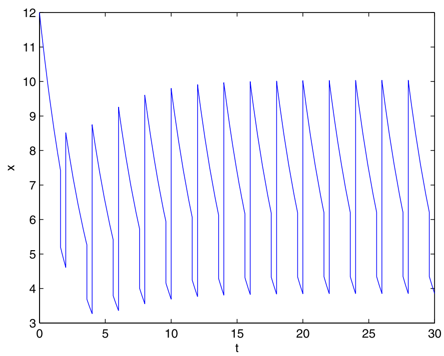
$$\begin{cases} \left.\begin{aligned} x'(t) &= -0.1x - 0.2x, \\ y'(t) &= 0.2x - 0.1y, \end{aligned}\right\} & t \neq n, n = 1, 2, 3, \ldots, \\ \left.\begin{aligned} \Delta x(t) &= -(1 - p_1)x, \\ \Delta y(t) &= -(1 - p_2)y, \end{aligned}\right\} & t = 2n - 1, \\ \left.\begin{aligned} \Delta x(t) &= (b - c(x + y))y, \\ \Delta y(t) &= 0, \end{aligned}\right\} & t = 2n, \end{cases} \tag{5.2}$$

and the state feedback control strategy

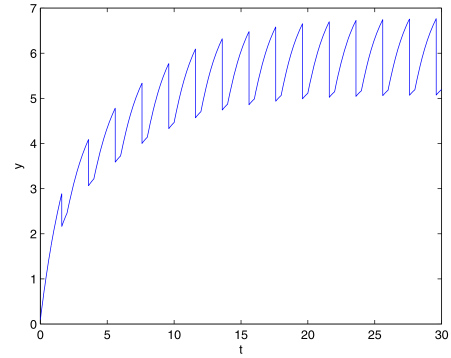
$$\begin{cases} x(t^+) = p_1 x, \\ y(t^+) = p_2 y, \end{cases} \qquad y(t) = 6,$$

are shown in Figure 7 and Figure 8, respectively, where $0 \leq t \leq 30$.

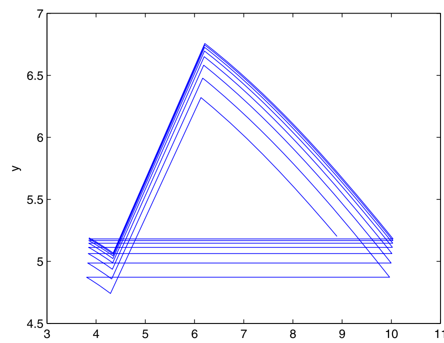
It is seen from Figure 7(b) that the total number of spraying pesticide is 15 in the case of time-fixed pulse control strategy. However, $y(t)$ is larger than the threshold value $h = 6$ at some points. It is seen from Figure 8(b) that the total number of spraying pesticide is




(a) Time series of $x$ with time-fixed pulse control strategy.




(b) Time series of $y$ with time-fixed pulse control strategy.



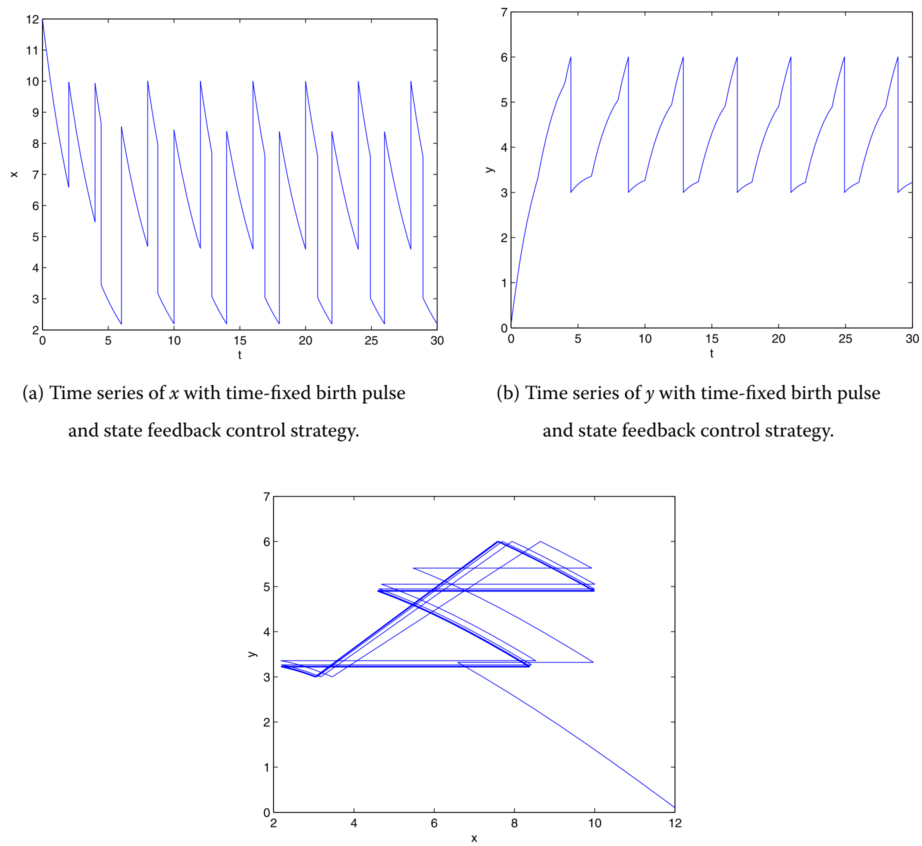

(c) Phase portrait with time-fixed pulse control strategy.

**Figure 7 The solution of system (5.2) with $h = 6$, $b = 3$, $p_1 = 0.7$, $p_2 = 0.75$, and the initial point $(12, 0.1)$.**

seven times in the case of state feedback control strategy, and $y(t)$ is controlled under the threshold value $h = 6$. In view of the total number of spraying pesticide and results of two control strategies, the state feedback control strategy is more effective than the time-fixed pulse control strategy.

## 6 Conclusion

In this paper, we study a stage structure population model with fixed-time birth pulse and state feedback control strategy. The stability of the trivial solution and the existence of periodic solutions are discussed. The sufficient conditions for the permanence of system (1.2) are obtained. It is shown that the maximum birth rate $b$ plays an important role in the population dynamics. The trivial solution of system (1.2) is asymptotically stable for $0 < b < \exp(dT) - 1$. For $b > \exp(dT) - 1$, the amount of mature pests can be controlled under the threshold value $y = h$. There exist many kinds of periodic solutions of system (1.2). The period-$T$ and period-$2T$ solutions are discussed.




(a) Time series of $x$ with time-fixed birth pulse and state feedback control strategy.

(b) Time series of $y$ with time-fixed birth pulse and state feedback control strategy.

(c) Phase portrait with time-fixed birth pulse and state feedback control strategy.

**Figure 8 The solution of system (5.1) with $h = 6$, $b = 3$, $p_1 = 0.7$, $p_2 = 0.75$, and the initial point $(12, 0.1)$.**

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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**References**

1. Hastings, A: Age-dependent predation is not a simple process. I. Continuous models. Theor. Popul. Biol. **23**, 347-362 (1983)
2. Aiello, WG, Freedman, HI: A time delay model of single-species growth with stage structure. Math. Biosci. **101**, 139-153 (1990)
3. Laksmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
4. Bainov, DD, Simeonov, PS: Impulsive Differential Equations: Periodic Solutions and Applications. Longman, New York (1993)
5. Zhang, H, Chen, L, Nieto, JJ: A delayed epidemic model with stage-structure and pulses for pest management strategy. Nonlinear Anal., Real World Appl. **9**, 1714-1726 (2008)
6. Jiang, G, Liu, Q, Peng, L: Impulsive ecological control of a stage-structured pest management system. Math. Biosci. Eng. **2**(2), 329-344 (2005)
7. Shi, R, Chen, L: Staged-structured Lotka-Volterra predator-prey models for pest management. Appl. Math. Comput. **203**, 258-265 (2008)
8. Caughley, G: Analysis of Vertebrate Population. Wiley, New York (1977)

9. Liu, Z, Chen, L: Periodic solution of a two-species competitive system with toxicant and birth pulse. Chaos Solitons Fractals **32**, 1703-1712 (2007)
10. Liu, B, Duan, Y, Luan, S: Dynamics of an SI epidemic model with external effects in a polluted environment. Nonlinear Anal., Real World Appl. **13**, 27-38 (2012)
11. Zhao, Z, Zhang, XQ, Chen, LS: The effect of pulsed harvesting policy on the inshore-offshore fishery model with the impulsive diffusion. Nonlinear Dyn. **63**, 537-545 (2011)
12. Zhang, H, Georgescu, P, Chen, LS: On the impulsive controllability and bifurcation of a predator-pest model of IPM. Biosystems **93**, 151-171 (2008)
13. Georgescu, P, Morosanu, G: Pest regulation by means of impulsive controls. Appl. Math. Comput. **190**, 790-803 (2007)
14. Nie, L, Peng, J, Teng, Z, Hua, L: Existence and stability of periodic solution of a Lotka-Volterra predator-prey model with state dependent impulsive effects. J. Comput. Appl. Math. **224**, 544-555 (2009)
15. Wu, X: Bifurcation of nontrivial periodic solutions for an impulsively controlled pest management model. Appl. Math. Comput. **202**, 675-687 (2008)
16. Ma, Z, Yang, J, Jiang, G: Impulsive control in a stage structure population model with birth pulses. Appl. Math. Comput. **217**, 3453-3460 (2010)
17. Jiang, G, Liu, S, Ling, L: Periodic solutions of a SIS epidemic model with fixed-time birth pulses and state feedback pulse treatments. Int. J. Comput. Math. **91**, 844-856 (2014)
18. Ling, L, Jiang, G, Long, T: The dynamics of an SIS epidemic model with fixed-time birth pulses and state feedback pulse treatments. Appl. Math. Model. (2015). doi:10.1016/j.apm.2015.01.022