

## RESEARCH

## Open Access



# The asymptotic property for nonlinear fourth-order Schrödinger equation with gain or loss

Cuihua Guo\*

\*Correspondence:  
gchzjq@sxu.edu.cn  
School of Mathematical Science,  
Shanxi University, Taiyuan, Shanxi  
030006, P.R. China

**Abstract**

We study the Cauchy problem of the nonlinear fourth-order Schrödinger equation with gain or loss:  $iu_t + \Delta^2 u + \lambda|u|^\alpha u + i\varepsilon a(t)|u|^\beta u = 0$ ,  $x \in R^n$ ,  $t \in R$ , where  $2 \leq \alpha \leq \frac{8}{n-4}$  and  $2 \leq \beta \leq \frac{8}{n-4}$ ,  $\varepsilon$  is a real number,  $a(t)$  is a real function, and  $n > 4$ . We study the asymptotic properties of its local and global solutions as  $\varepsilon \rightarrow 0$ .

**Keywords:** nonlinear fourth-order Schrödinger equation with gain or loss; Fourier restriction norm method; Cauchy problem

**1 Introduction**

In this paper we study the following nonlinear fourth-order Schrödinger equation with gain or loss:

$$\begin{cases} iu_t + \Delta^2 u + \lambda|u|^\alpha u + i\varepsilon a(t)|u|^\beta u = 0, & x \in R^n, t \in R, \\ u(x, 0) = u_0(x), & x \in R^n, \end{cases} \quad (1.1)$$

where  $u(x, t)$  are complex-valued function. We have  $2 \leq \alpha \leq \frac{8}{n-4}$  and  $2 \leq \beta \leq \frac{8}{n-4}$ ,  $\varepsilon$  is a real number,  $a(t)$  is a real function, and  $n > 4$ .

For the case  $\varepsilon = 0$ , the above equation is the nonlinear fourth-order Schrödinger equation,

$$\begin{cases} iu_t + \Delta^2 u + \lambda|u|^\alpha u = 0, & x \in R^n, t \in R, \\ u(x, 0) = u_0(x), & x \in R^n. \end{cases} \quad (1.2)$$

For (1.2), in [1] we have obtained the local well-posedness result in the space  $C([-T, T], H^2(R^n))$  if  $n > 4$  and  $2 \leq \alpha \leq \frac{8}{n-4}$ . We also get the global well-posedness result in the space  $C(R, H^2(R^n))$  if  $n > 4$  and  $\lambda > 0$ ,  $2 \leq \alpha \leq \frac{8}{n-4}$  or  $\lambda < 0$ ,  $2 \leq \alpha \leq \frac{8}{n}$ . For the energy-critical case, in [2] and [3], Pausader Benoit gives the global well-posedness and scattering for  $n \geq 5$  and radial initial data. In [4], Miao *et al.* study the defocusing case and obtain the global existence for  $n \geq 9$ . In [5], Zhang and Zheng obtain the global solution and scattering for  $n = 8$ . Pausader Benoit also discusses the mass-critical case in [6].

For the case  $\varepsilon \neq 0$ ,  $a(t)$  is the gain (loss) if  $a(t) < 0$  ( $a(t) > 0$ ). In [7], the authors discuss the Schrödinger equation with gain. They have obtained the result: The value of  $a(t)$  will

determine whether or not the solution will blow up. Feng *et al.* study the Schrödinger equation with gain/loss in [8] and [9]. They, respectively, give the limit behavior of solution as  $\varepsilon \rightarrow 0$  and the global solution and blow-up result. As far as we know, there are fewer results about the fourth-order Schrödinger equation with gain. In this paper, we will discuss the local well-posedness and the global well-posedness of (1.1); especially, we will discuss the asymptotic behavior of the solution as  $\varepsilon \rightarrow 0$ .

## 2 The preliminary estimates

First, we denote by  $U(t)$  ( $t \in \mathbb{R}$ ) the fundamental solution operator of the fourth-order Schrödinger equation [10], *i.e.*,

$$U(t)\varphi(x) = F^{-1}(e^{-it\xi^4} \hat{\varphi}(\xi)) \quad \text{for } \varphi \in S'(R),$$

where  $\hat{\varphi}$  denotes the Fourier transformation of  $\varphi$ , and  $F^{-1}$  represents the inverse Fourier transformation.

Thus the equivalent integral equations [11] of (1.1) and (1.2) are, respectively,

$$u_\varepsilon(t) = U(t)u_0 + i\lambda \int_0^t U(t-s)(|u_\varepsilon|^\alpha u_\varepsilon)(s) ds - \varepsilon \int_0^t U(t-s)a(s)(|u_\varepsilon|^\beta u_\varepsilon)(s) ds \quad (2.1)$$

and

$$u(t) = U(t)u_0 + i\lambda \int_0^t U(t-s)(|u|^\alpha u)(s) ds. \quad (2.2)$$

Second, we introduce the following notations. For any given  $T > 0$ , we define the space  $L^q(0, T; W^{2,r}(R^n))$  with the norm

$$\|u\|_{L^q(0,T;W^{2,r})} := \left( \int_0^T \|u(\cdot, t)\|_{W^{2,r}(R^n)}^q dt \right)^{\frac{1}{q}}.$$

For two integers  $8 \leq q \leq \infty$  and  $2 \leq r < \infty$ , we say that  $(q, r)$  is an admissible pair if the following condition is satisfied:

$$\frac{2}{q} = \frac{n}{4} \left( 1 - \frac{2}{r} \right).$$

For simplicity, in this paper, we will use  $C$  to denote various constants which may be different from line to line.

We have the following Strichartz estimate (see [1]): For any admissible pair  $(q, r)$

$$\|U(t)\varphi(x)\|_{L^q(0,t;L^r)} \leq C\|\varphi\|_{L^2} \quad (2.3)$$

and

$$\left\| \int_0^t U(t-s)f(x, s) ds \right\|_{L^q(0,t;L^r)} \leq C\|f\|_{L^{q'}(0,t;L^{r'})}, \quad (2.4)$$

where  $(\gamma, \rho)$  is an arbitrary admissible pair, and  $'$  represents the conjugate number.

From Theorem 4.5 of [1], we have the following results.

**Proposition 2.1** (subcritical case) *Assume that  $n > 4$ ,  $a \in L^\infty(0, \infty)$ ,  $2 \leq \alpha < \frac{8}{n-4}$ , and  $2 \leq \beta < \frac{8}{n-4}$ ,  $(\gamma_1, \rho_1) = (\alpha + 2, \frac{2n(\alpha+2)}{n(\alpha+2)-8})$ ,  $(\gamma_2, \rho_2) = (\frac{8(\beta+2)}{n\beta}, \beta + 2)$ . For any  $u_0 \in H^2(\mathbb{R}^n)$ , there exists  $\delta$  such that the Cauchy problem (1.1) has a unique solution  $u_\varepsilon$  in the space  $L^\infty(0, \delta; H^2(\mathbb{R}^n)) \cap L^{\gamma_1}(0, \delta; W^{2, \rho_1}(\mathbb{R}^n)) \cap L^{\gamma_2}(0, \delta; W^{2, \rho_2}(\mathbb{R}^n))$ . Moreover,*

$$\|u_\varepsilon\|_{L^\infty(0, \delta; H^2) \cap L^{\gamma_1}(0, \delta; W^{2, \rho_1}) \cap L^{\gamma_2}(0, \delta; W^{2, \rho_2})} \leq 2\|u_0\|_{H^2}.$$

**Proposition 2.2** (critical case) *Assume that  $n > 4$ ,  $a \in L^\infty(0, \infty)$ ,  $\alpha = \frac{8}{n-4}$ ,  $2 \leq \beta < \frac{8}{n-4}$ ,  $(\gamma^*, \rho^*) = (\frac{2n}{n-4}, \frac{2n^2}{n^2-4n+16})$ ,  $(\gamma_2, \rho_2) = (\frac{8(\beta+2)}{n\beta}, \beta + 2)$ . For any  $u_0 \in H^2(\mathbb{R}^n)$ , there exists  $\delta$  such that the Cauchy problem (1.1) has a unique solution  $u_\varepsilon$  in the space  $L^\infty(0, \delta; H^2(\mathbb{R}^n)) \cap L^{\gamma^*}(0, \delta; W^{2, \rho^*}(\mathbb{R}^n)) \cap L^{\gamma_2}(0, \delta; W^{2, \rho_2}(\mathbb{R}^n))$ . Moreover,*

$$\|u_\varepsilon\|_{L^\infty(0, \delta; H^2) \cap L^{\gamma^*}(0, \delta; W^{2, \rho^*}) \cap L^{\gamma_2}(0, \delta; W^{2, \rho_2})} \leq 3\|U(t)u_0\|_{L^{\gamma^*}(0, \delta; W^{2, \rho^*})}.$$

### 3 Main results

**Lemma 3.1** *Let  $n, \alpha, \beta, (\gamma_1, \rho_1), (\gamma_2, \rho_2)$  be as in Proposition 2.1. Assume that  $u$  is the solution of (1.2), defined on a maximal time interval  $[0, T^*)$ ,  $0 < l < T^*$ , and  $u_\varepsilon$  exists on  $[0, l]$ . If  $\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(0, l; H^2) \cap L^{\gamma_1}(0, l; W^{2, \rho_1}) \cap L^{\gamma_2}(0, l; W^{2, \rho_2})} < +\infty$ , then we have  $u_\varepsilon \rightarrow u$  in  $L^q(0, l; W^{2, r}(\mathbb{R}^n))$  as  $\varepsilon \rightarrow 0$ , where  $(q, r)$  is arbitrary admissible pair.*

*Proof* First, we prove

$$\|u_\varepsilon - u\|_{L^q(0, l; L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

From (2.1) and (2.2), using Strichartz estimates, we have

$$\begin{aligned} & \|u_\varepsilon - u\|_{L^q(0, l; L^r)} \\ & \leq \|J(t)\|_{L^q(0, l; L^r)} + \|K(t)\|_{L^q(0, l; L^r)} \\ & \leq C\| |u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u \|_{L^{\gamma'}(0, l; L^{\rho'})} + C\varepsilon \|a\|_{L^\infty(0, l)} \| |u_\varepsilon|^\beta u_\varepsilon \|_{L^{\gamma'_2}(0, l; L^{\rho'_2})}, \end{aligned} \tag{3.1}$$

where  $J(t) = i\lambda \int_0^t U(t-s)(|u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u)(s) ds$ ,  $K(t) = -\varepsilon \int_0^t U(t-s)a(s)(|u_\varepsilon|^\beta u_\varepsilon)(s) ds$ ,  $(\gamma, \rho) = (\frac{8(\alpha+2)}{n\alpha}, \alpha + 2)$ .

Since  $\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(0, l; H^2) \cap L^{\gamma_1}(0, l; W^{2, \rho_1}) \cap L^{\gamma_2}(0, l; W^{2, \rho_2})} < +\infty$ , there exist  $N_1, \varepsilon_0$  such that

$$\|u_\varepsilon\|_{L^\infty(0, l; H^2) \cap L^{\gamma_1}(0, l; W^{2, \rho_1}) \cap L^{\gamma_2}(0, l; W^{2, \rho_2})} \leq N_1 \quad \text{for all } \varepsilon < \varepsilon_0.$$

Let  $N_2 = \|u\|_{L^\infty(0, l; H^2)}$ , it is obvious that  $N_2 < +\infty$ . Using the Hölder inequality and the Sobolev embedding [12], we have

$$\begin{aligned} & \| |u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u \|_{L^{\gamma'}(0, l; L^{\rho'})} \leq C(\|u_\varepsilon\|_{L^\alpha(0, l; L^{\alpha+2})}^\alpha + \|u\|_{L^\alpha(0, l; L^{\alpha+2})}^\alpha) \|u_\varepsilon - u\|_{L^\gamma(0, l; L^\rho)} \\ & \leq C(\|u_\varepsilon\|_{L^\infty(0, l; H^2)}^\alpha + \|u\|_{L^\infty(0, l; H^2)}^\alpha) \|u_\varepsilon - u\|_{L^\gamma(0, l; L^\rho)} \\ & \leq C(N_1^\alpha + N_2^\alpha) \|u_\varepsilon - u\|_{L^\gamma(0, l; L^\rho)}, \end{aligned} \tag{3.2}$$

where  $a = \frac{4\alpha(\alpha+2)}{8-(n-4)\alpha}$ .

Similarly, we have

$$\begin{aligned} \| |u_\varepsilon|^\beta u_\varepsilon \|_{L^{\gamma'_2}(0,t;L^{\rho'_2})} &\leq \| u_\varepsilon \|_{L^b(0,t;L^{\beta+2})}^\beta \| u_\varepsilon \|_{L^{\gamma_2}(0,t;L^{\rho_2})} \\ &\leq \| u_\varepsilon \|_{L^\infty(0,t;H^2)}^\beta \| u_\varepsilon \|_{L^{\gamma_2}(0,t;L^{\rho_2})} \leq N_1^{\beta+1}, \end{aligned} \tag{3.3}$$

where  $b = \frac{4\beta(\beta+2)}{8-(n-4)\beta}$ .

Let  $N_3 = C \| a \|_{L^\infty} N_1^{\beta+1}$ . Substituting (3.2) and (3.3) into (3.1), we have

$$\| u_\varepsilon - u \|_{L^q(0,t;L^r)} \leq C (N_1^\alpha + N_2^\alpha) \| u_\varepsilon - u \|_{L^{\gamma}(0,t;L^\rho)} + \varepsilon N_3. \tag{3.4}$$

In the following we will prove that  $\| u_\varepsilon - u \|_{L^{\gamma}(0,t;L^\rho)} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

Noting that  $N_1, N_2 < \infty$ , we can divide the time interval  $[0, l]$  into subintervals  $[t_i, t_{i+1}]$ ,  $i = 0, 1, \dots, J - 1$ , where  $t_0 = 0, t_{J-1} = l$  such that in each part  $C(\| u_\varepsilon \|_{L^a(t_i, t_{i+1}; L^{\alpha+2})}^\alpha + \| u \|_{L^a(t_i, t_{i+1}; L^{\alpha+2})}^\alpha) = \frac{1}{2}$ .

On  $[t_0, t_1]$ , since  $u_\varepsilon(t_0) = u(t_0) = u_0$ , we have

$$\| u_\varepsilon - u \|_{L^{\gamma}(t_0, t_1; L^\rho)} \leq \frac{1}{2} \| u_\varepsilon - u \|_{L^{\gamma}(t_0, t_1; L^\rho)} + \varepsilon N_3,$$

which means

$$\| u_\varepsilon - u \|_{L^{\gamma}(t_0, t_1; L^\rho)} \leq 2\varepsilon N_3.$$

By (3.4), we have

$$\| u_\varepsilon - u \|_{L^\infty(t_0, t_1; L^2)} \leq 2\varepsilon N_3.$$

On  $[t_1, t_2]$ , we have

$$\begin{aligned} \| u_\varepsilon - u \|_{L^{\gamma}(t_1, t_2; L^\rho)} &\leq \| u_\varepsilon(t_1) - u(t_1) \|_{L^2} + \frac{1}{2} \| u_\varepsilon - u \|_{L^{\gamma}(t_1, t_2; L^\rho)} + \varepsilon N_3 \\ &\leq 3\varepsilon N_3 + \frac{1}{2} \| u_\varepsilon - u \|_{L^{\gamma}(t_1, t_2; L^\rho)}, \end{aligned}$$

from which we can obtain

$$\| u_\varepsilon - u \|_{L^{\gamma}(t_1, t_2; L^\rho)} \leq 6\varepsilon N_3.$$

Especially, we have  $\| u_\varepsilon - u \|_{L^\infty(t_1, t_2; L^2)} \leq 6\varepsilon N_3$ .

By induction, we have

$$\begin{aligned} \| u_\varepsilon - u \|_{L^{\gamma}(t_i, t_{i+1}; L^\rho)} &\leq 2(2^{i+1} - 1)\varepsilon N_3, \\ \| u_\varepsilon - u \|_{L^\infty(t_i, t_{i+1}; L^2)} &\leq 2(2^{i+1} - 1)\varepsilon N_3, \quad \text{for } i = 0, 1, \dots, J - 1. \end{aligned}$$

So we have

$$\| u_\varepsilon - u \|_{L^{\gamma}(0,t;L^\rho)} \leq \sum_{i=0}^{J-1} 2(2^{i+1} - 1)\varepsilon N_3 = [4(2^J - 1) - 2J]\varepsilon N_3 \rightarrow 0.$$

Furthermore, we have

$$\|u_\varepsilon - u\|_{L^q(0,t;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Second, we prove

$$\|\nabla u_\varepsilon - \nabla u\|_{L^q(0,t;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

From (2.1) and (2.2), we have

$$\nabla(u_\varepsilon - u) = i\lambda \int_0^t U(t-s)\nabla(|u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u)(s) ds - \varepsilon \int_0^t U(t-s)a(s)\nabla(|u_\varepsilon|^\beta u_\varepsilon)(s) ds.$$

Let  $g_1(u) = |u|^\alpha u$ ,  $g_2(u) = |u|^\beta u$ . Then, using Strichartz estimates, we have

$$\begin{aligned} & \|\nabla(u_\varepsilon - u)\|_{L^q(0,t;L^r)} \\ & \leq C \left\| \int_0^t U(t-s)\nabla(g_1(u_\varepsilon) - g_1(u))(s) ds \right\|_{L^q(0,t;L^r)} \\ & \quad + C\varepsilon \left\| \int_0^t U(t-s)a(s)\nabla g_2(u_\varepsilon)(s) ds \right\|_{L^q(0,t;L^r)} \\ & \leq C \|\nabla(g_1(u_\varepsilon) - g_1(u))\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} + C\varepsilon \|a\|_{L^\infty(0,t)} \|\nabla g_2(u_\varepsilon)\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})} \\ & \leq C \|g_1'(u_\varepsilon)\nabla(u_\varepsilon - u)\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} + C \|(g_1'(u_\varepsilon) - g_1'(u))\nabla u\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ & \quad + C\varepsilon \|a\|_{L^\infty(0,t)} \|\nabla g_2(u_\varepsilon)\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})}. \end{aligned} \tag{3.5}$$

Using the Hölder inequality, the Sobolev embedding, and the Young inequality, we obtain

$$\begin{aligned} & \|g_1'(u_\varepsilon)\nabla(u_\varepsilon - u)\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ & \leq C \|u_\varepsilon\|_{L^{\gamma_1}(0,t;L^c)}^\alpha \|\nabla(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})} \\ & \leq C \|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha \|\nabla(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})} \quad \left(c = \frac{\rho_1\alpha}{\rho_1 - 2}\right), \end{aligned} \tag{3.6}$$

$$\|\nabla g_2(u_\varepsilon)\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})} \leq C \|u_\varepsilon\|_{L^\infty(0,t;H^2)}^\beta \|u_\varepsilon\|_{L^{\gamma_2}(0,t;W^{2,\rho_2})}, \tag{3.7}$$

and

$$\begin{aligned} & \|(g_1'(u_\varepsilon) - g_1'(u))\nabla u\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ & \leq C (\|u_\varepsilon\|_{L^{\gamma_1}(0,t;L^{d_1})}^{\alpha-1} + \|u\|_{L^{\gamma_1}(0,t;L^{d_1})}^{\alpha-1}) \|u_\varepsilon - u\|_{L^{\gamma_1}(0,t;L^{e_1})} \|\nabla u\|_{L^{\gamma_1}(0,t;L^{e_1})} \\ & \leq C (\|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^{\alpha-1} + \|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^{\alpha-1}) \|\nabla(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})} \|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})} \\ & \leq C (\|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha + \|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha) \|\nabla(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})}, \end{aligned} \tag{3.8}$$

where  $d_1 = \frac{2n(\alpha+2)(\alpha-1)}{24-(n-4)(\alpha+2)}$ ,  $e_1 = \frac{2n(\alpha+2)}{(n-2)(\alpha+2)-8}$ .

Substituting (3.6)-(3.8) into (3.5), we have

$$\|\nabla(u_\varepsilon - u)\|_{L^q(0,t;L^r)} \leq C(\|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha + \|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha) \|\nabla(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})} + \varepsilon N_3.$$

Similar to the proof in the first step, we have

$$\|\nabla(u_\varepsilon - u)\|_{L^q(0,t;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

At last, we prove

$$\|D^2u_\varepsilon - D^2u\|_{L^q(0,t;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

By simple computing, we have

$$D^2u_\varepsilon - D^2u = i(K_1 + K_2 + K_3), \tag{3.9}$$

where  $K_1 = \lambda \int_0^t U(t-s)A_1(u_\varepsilon, u)(s) ds$ ,  $K_2 = \lambda \int_0^t U(t-s)A_2(u_\varepsilon, u)(s) ds$ ,  $K_3 = -\varepsilon \int_0^t U(t-s)a(s)A_3(u_\varepsilon)(s) ds$ . The arrays  $A_1(u_\varepsilon, u) = g'_1(u_\varepsilon)D^2(u_\varepsilon - u) + g''_1(u_\varepsilon)D(u_\varepsilon - u) \times Du$ ,  $A_2(u_\varepsilon, u) = Du \times [g''_1(u_\varepsilon)Du_\varepsilon - g''_1(u)Du] + [g'_1(u_\varepsilon) - g'_1(u)]D^2u$ ,  $A_3(u_\varepsilon) = g''_2(u_\varepsilon)Du_\varepsilon \times Du_\varepsilon + g'_2(u_\varepsilon)D^2u_\varepsilon$ .

By the Hölder inequality and the Sobolev embedding, we have

$$\|g'_1(u_\varepsilon)D^2(u_\varepsilon - u)\|_{L^{\gamma'_1}(0,t;L^{\rho'_1})} \leq \|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha \|D^2(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})} \tag{3.10}$$

and

$$\begin{aligned} & \|g''_1(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du_\varepsilon\|_{L^{\gamma'_1}(0,t;L^{\rho'_1})} \\ & \leq \|u_\varepsilon\|_{L^{\gamma_1}(0,t;L^{d_1})}^{\alpha-1} \|D(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})} \|Du_\varepsilon\|_{L^{\gamma_1}(0,t;L^{\rho_1})} \\ & \leq \|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha \|D^2(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})}. \end{aligned} \tag{3.11}$$

Thus we have from (3.10) and (3.11)

$$\begin{aligned} \|K_1\|_{L^q(0,t;L^r)} & \leq \|g'_1(u_\varepsilon)D^2(u_\varepsilon - u)\|_{L^{\gamma'_1}(0,t;L^{\rho'_1})} + \|g''_1(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du_\varepsilon\|_{L^{\gamma'_1}(0,t;L^{\rho'_1})} \\ & \leq C\|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha \|D^2(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})}. \end{aligned} \tag{3.12}$$

Similar to the proof of (3.11), we obtain

$$\begin{aligned} & \|g''_1(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du\|_{L^{\gamma'_1}(0,t;L^{\rho'_1})} \\ & \leq (\|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha + \|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha) \|D^2(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})}. \end{aligned} \tag{3.13}$$

Noting that  $\alpha \geq 2$ , we have

$$\begin{aligned} & \|(g''_1(u_\varepsilon) - g''_1(u))Du^\perp \times Du\|_{L^{\gamma'_1}(0,t;L^{\rho'_1})} \\ & \leq C(|u_\varepsilon|^{\alpha-2} + |u|^{\alpha-2})(u_\varepsilon - u)Du^\perp \times Du\|_{L^{\gamma'_1}(0,t;L^{\rho'_1})} \end{aligned}$$

$$\begin{aligned} &\leq C(\|u_\varepsilon\|_{L^{\gamma_1}(0,t;L^c)}^{\alpha-2} + \|u\|_{L^{\gamma_1}(0,t;L^c)}^{\alpha-2})\|u_\varepsilon - u\|_{L^{\gamma_1}(0,t;L^{d_2})} \|Du\|_{L^{\gamma_1}(0,t;L^{e_2})}^2 \\ &\leq (\|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha + \|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha) \|D^2(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})}, \end{aligned} \tag{3.14}$$

where  $e_2 = \frac{2n(\alpha+2)}{(n-2)(\alpha+2)-8}$ ,  $\frac{1}{\rho_1'} = \frac{(\rho_1-2)(\alpha-2)}{\rho_1\alpha} + \frac{1}{d_2} + \frac{2}{e_2}$ .

Similarly, using the Hölder inequality and the Sobolev embedding, we obtain

$$\begin{aligned} &\|(g_1'(u_\varepsilon) - g_1'(u))D^2u\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ &\leq C(\|u_\varepsilon\|_{L^{\gamma_1}(0,t;L^c)}^{\alpha-1} + \|u\|_{L^{\gamma_1}(0,t;L^c)}^{\alpha-1})\|u_\varepsilon - u\|_{L^{\gamma_1}(0,t;L^c)} \|D^2u\|_{L^{\gamma_1}(0,t;L^{\rho_1})} \\ &\leq C(\|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha + \|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha) \|D^2(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})}. \end{aligned} \tag{3.15}$$

Thus we have from (3.13) and (3.15)

$$\begin{aligned} \|K_2\|_{L^q(0,t;L^r)} &\leq \|A_2(u_\varepsilon, u)\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ &\leq \|g_1''(u_\varepsilon)D(u_\varepsilon - u)^\perp \times Du\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ &\quad + \|(g_1''(u_\varepsilon) - g_1''(u))Du^\perp \times Du\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ &\quad + \|(g_1'(u_\varepsilon) - g_1'(u))D^2u\|_{L^{\gamma_1'}(0,t;L^{\rho_1'})} \\ &\leq C(\|u_\varepsilon\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha + \|u\|_{L^{\gamma_1}(0,t;W^{2,\rho_1})}^\alpha) \|D^2(u_\varepsilon - u)\|_{L^{\gamma_1}(0,t;L^{\rho_1})}. \end{aligned} \tag{3.16}$$

Similar to the proof of (3.3), we obtain

$$\begin{aligned} \|g_2'(u_\varepsilon)D^2u_\varepsilon\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})} &\leq \|u_\varepsilon\|_{L^b(0,t;L^{\beta+2})}^\beta \|D^2u_\varepsilon\|_{L^{\gamma_2}(0,t;L^{\rho_2})} \leq N_1^{\beta+1} \\ &\leq \|u_\varepsilon\|_{L^\infty(0,t;H^2)}^\beta \|u_\varepsilon\|_{L^{\gamma_2}(0,t;W^{2,\rho_2})} \leq N_1^{\beta+1} \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \|g_2''(u_\varepsilon)Du_\varepsilon^\perp \times Du_\varepsilon\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})} &\leq \|u_\varepsilon\|_{L^b(0,t;L^{\beta+2})}^{\beta-1} \|Du_\varepsilon\|_{L^{\gamma_2}(0,t;L^{\rho_2})}^2 \\ &\leq \|u_\varepsilon\|_{L^{\gamma_2}(0,t;W^{2,\rho_2})}^{\beta+1} \leq N_1^{\beta+1}. \end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we immediately obtain

$$\begin{aligned} \|K_3\|_{L^q(0,t;L^r)} &\leq \varepsilon \|a\|_{L^\infty(0,t)} \|A_3(u_\varepsilon)\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})} \\ &\leq \varepsilon \|a\|_{L^\infty(0,t)} [\|g_2'(u_\varepsilon)D^2u_\varepsilon\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})} + \|g_2''(u_\varepsilon)Du_\varepsilon^\perp \times Du_\varepsilon\|_{L^{\gamma_2'}(0,t;L^{\rho_2'})}] \\ &\leq \varepsilon N_3. \end{aligned} \tag{3.19}$$

Taking, respectively,  $(q, r) = (\gamma, \rho)$  and  $(q, r) = (\gamma_1, \rho_1)$  in (3.9), (3.12), (3.16), and (3.19), similar to the method of the first step, we can obtain

$$\|D^2u_\varepsilon - D^2u\|_{L^q(0,t;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad \square$$

Noting that if  $\alpha = \frac{8}{n-4}$ ,  $a$  in (3.2) will be meaningless. So we will need the following lemma for the critical case.

**Lemma 3.2** *Let  $n, \alpha, \beta, (\gamma^*, \rho^*), (\gamma_2, \rho_2)$  be as in Proposition 2.2. Assume that  $u$  is the solution of (1.2), defined on a maximal time interval  $[0, T^*)$ ,  $0 < l < T^*$ , and  $u_\varepsilon$  exists on  $[0, l]$ . If  $\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(0,l;H^2) \cap L^{\gamma^*}(0,l;W^{2,\rho^*}) \cap L^{\gamma_2}(0,l;W^{2,\rho_2})} < +\infty$ , then we have  $u_\varepsilon \rightarrow u$  in  $L^q(0, l; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , where  $(q, r)$  is arbitrary admissible pair.*

*Proof* Using the Hölder inequality and a Sobolev embedding, we have

$$\begin{aligned} & \| |u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u \|_{L^{\gamma^* \prime}(0,l;L^{\rho^* \prime})} \\ & \leq C \left( \|u_\varepsilon\|_{L^{\gamma^*}(0,l;L^{\frac{\rho^* \alpha}{\rho^* - 2}})}^\alpha + \|u\|_{L^{\gamma^*}(0,l;L^{\frac{\rho^* \alpha}{\rho^* - 2}})}^\alpha \right) \|u_\varepsilon - u\|_{L^{\gamma^*}(0,l;L^{\rho^*})} \\ & \leq C \left( \|u_\varepsilon\|_{L^{\gamma^*}(0,l;W^{2,\rho^*})}^\alpha + \|u\|_{L^{\gamma^*}(0,l;W^{2,\rho^*})}^\alpha \right) \|u_\varepsilon - u\|_{L^{\gamma^*}(0,l;L^{\rho^*})}. \end{aligned} \tag{3.20}$$

From (2.1) and (2.2), using Strichartz estimates, we have

$$\begin{aligned} \|u_\varepsilon - u\|_{L^q(0,l;L^r)} & \leq C \| |u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u \|_{L^{\gamma^* \prime}(0,l;L^{\rho^* \prime})} + C\varepsilon \|a\|_{L^\infty(0,l)} \| |u_\varepsilon|^\beta u_\varepsilon \|_{L^{\gamma_2 \prime}(0,l;L^{\rho_2 \prime})} \\ & \leq C \left( \|u_\varepsilon\|_{L^{\gamma^*}(0,l;W^{2,\rho^*})}^\alpha + \|u\|_{L^{\gamma^*}(0,l;W^{2,\rho^*})}^\alpha \right) \|u_\varepsilon - u\|_{L^{\gamma^*}(0,l;L^{\rho^*})} \\ & \quad + C\varepsilon \|a\|_{L^\infty(0,l)} \| |u_\varepsilon|^\beta u_\varepsilon \|_{L^{\gamma_2 \prime}(0,l;L^{\rho_2 \prime})}; \end{aligned} \tag{3.21}$$

similarly as in Lemma 3.1, we can obtain

$$\|u_\varepsilon - u\|_{L^q(0,l;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Noting that for  $(\gamma_1, \rho_1)$  in Lemma 3.1 in the case  $\alpha = \frac{8}{n-4}$ ,  $2 \leq \beta < \frac{8}{n-4}$ , we have

$$(\gamma_1, \rho_1) = (\gamma^*, \rho^*),$$

thus obviously

$$\|\nabla(u_\varepsilon - u)\|_{L^q(0,l;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

and

$$\|D^2 u_\varepsilon - D^2 u\|_{L^q(0,l;L^r)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

for all admissible pairs  $(q, r)$ . □

**Remark 3.1** For the critical case  $2 \leq \alpha < \frac{8}{n-4}$ ,  $\beta = \frac{8}{n-4}$ , we only take the working space as  $L^\infty(0, \delta; H^2(R^n)) \cap L^{\gamma_1}(0, \delta; W^{2,\rho_1}(R^n)) \cap L^{\gamma^*}(0, \delta; W^{2,\rho^*}(R^n))$ .

For the case  $\alpha = \beta = \frac{8}{n-4}$ , we take the working space as  $L^\infty(0, \delta; H^2(R^n)) \cap L^{\gamma^*}(0, \delta; W^{2,\rho^*}(R^n))$ .

**Theorem 3.1** *Assume that  $n > 4$ ,  $a \in L^\infty(0, \infty)$ ,  $2 \leq \alpha \leq \frac{8}{n-4}$ , and  $2 \leq \beta \leq \frac{8}{n-4}$ . Assume that  $u$  is the solution of (1.2) with initial value  $u_0 \in H^2(R^n)$ , defined on a maximal time interval  $[0, T^*)$ . Then we have:*



- (1) For any given  $0 < T < T^*$ , there is a solution  $u_\varepsilon$  on  $[0, T]$ .
- (2)  $u_\varepsilon \rightarrow u$  in  $L^q(0, T; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , where  $(q, r)$  is an arbitrary admissible pair.

*Proof* (1) The case  $2 \leq \alpha < \frac{8}{n-4}$  and  $2 \leq \beta < \frac{8}{n-4}$ .

From Proposition 2.1, we find that there exists  $u_\varepsilon$  on  $[0, \delta]$  such that

$$\|u_\varepsilon\|_{L^\infty(0,\delta;H^2) \cap L^{\gamma_1}(0,\delta;W^{2,\rho_1}) \cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} \leq 2\|u_0\|_{H^2}.$$

So for small  $\varepsilon$ , we have

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(0,\delta;H^2) \cap L^{\gamma_1}(0,\delta;W^{2,\rho_1}) \cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} < +\infty.$$

Using Lemma 3.1, we have  $u_\varepsilon \rightarrow u$  in  $L^q(0, \delta; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , for any arbitrary admissible pair  $(q, r)$ .

Especially, we have  $\|u_\varepsilon(\delta)\|_{H^2} \leq 2\|u_0\|_{H^2}$ . Again using Proposition 2.1, there exists  $u_\varepsilon$  on  $[\delta, 2\delta]$  such that

$$\|u_\varepsilon\|_{L^\infty(\delta,2\delta;H^2) \cap L^{\gamma_1}(\delta,2\delta;W^{2,\rho_1}) \cap L^{\gamma_2}(\delta,2\delta;W^{2,\rho_2})} \leq 2\|u_\varepsilon(\delta)\|_{H^2} \leq 2\|u_0\|_{H^2}.$$

By a continuation extension method, we obtain the solution  $u_\varepsilon$  on  $[0, T]$  ( $0 < T < T^*$ ) such that

$$\|u_\varepsilon\|_{L^\infty(0,T;H^2) \cap L^{\gamma_1}(0,T;W^{2,\rho_1}) \cap L^{\gamma_2}(0,T;W^{2,\rho_2})} \leq 2\|u_0\|_{H^2}.$$

So

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(0,T;H^2) \cap L^{\gamma_1}(0,T;W^{2,\rho_1}) \cap L^{\gamma_2}(0,T;W^{2,\rho_2})} < +\infty,$$

using Lemma 3.1, we immediately have  $u_\varepsilon \rightarrow u$  in  $L^q(0, T; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , for any arbitrary admissible pair  $(q, r)$ .

- (2) *Case 1:*  $\alpha = \frac{8}{n-4}$ ,  $2 \leq \beta < \frac{8}{n-4}$ .

From Proposition 2.2, we find that there exists  $u_\varepsilon$  on  $[0, \delta]$  such that

$$\|u_\varepsilon\|_{L^\infty(0,\delta;H^2) \cap L^{\gamma^*}(0,\delta;W^{2,\rho^*}) \cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} \leq 3\|U(t)u_0\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}.$$

So for small  $\varepsilon$ , we have

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(0,\delta;H^2) \cap L^{\gamma^*}(0,\delta;W^{2,\rho^*}) \cap L^{\gamma_2}(0,\delta;W^{2,\rho_2})} < +\infty.$$

Using Lemma 3.2, we have  $u_\varepsilon \rightarrow u$  in  $L^q(0, \delta; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , for any arbitrary admissible pair  $(q, r)$ .

Noting that

$$\|U(t)u_\varepsilon(\delta)\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})} \leq C\|u_\varepsilon(\delta)\|_{H^2} \leq 3C\|U(t)u_0\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})},$$

so, again using Proposition 2.2, there exists  $u_\varepsilon$  on  $[\delta, 2\delta]$  such that

$$\begin{aligned} \|u_\varepsilon\|_{L^\infty(\delta,2\delta;H^2)\cap L^{\gamma^*}(\delta,2\delta;W^{2,\rho^*})\cap L^{\gamma_2}(\delta,2\delta;W^{2,\rho_2})} &\leq 3C\|U(t)u_\varepsilon(\delta)\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})} \\ &\leq (3C)^2\|U(t)u_0\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}. \end{aligned}$$

By continuation extension method, we obtain the solution  $u_\varepsilon$  on  $[0, T]$  ( $0 < T < T^*$ ) such that

$$\|u_\varepsilon\|_{L^\infty(0,T;H^2)\cap L^{\gamma^*}(0,T;W^{2,\rho^*})\cap L^{\gamma_2}(0,T;W^{2,\rho_2})} \leq C(T)\|U(t)u_0\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})}.$$

So

$$\limsup_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(0,T;H^2)\cap L^{\gamma^*}(0,T;W^{2,\rho^*})\cap L^{\gamma_2}(0,T;W^{2,\rho_2})} \leq C(T)\|U(t)u_0\|_{L^{\gamma^*}(0,\delta;W^{2,\rho^*})} < +\infty,$$

using Lemma 3.2, we immediately have  $u_\varepsilon \rightarrow u$  in  $L^q(0, T; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , for any arbitrary admissible pair  $(q, r)$ .

Case 2:  $\beta = \frac{8}{n-4}$ ,  $2 \leq \alpha < \frac{8}{n-4}$  or  $\alpha = \beta = \frac{8}{n-4}$ .

See Remark 3.1, the proof is similar; here we omit it. □

**Lemma 3.3** *Assume that  $u$  is the global solution of (1.2) with the initial value  $u_0 \in H^2(R^n)$  and  $u \in L^q_{loc}(0, \infty; W^{2,r}(R^n))$ . Then we have:*

- (1) *The solution  $u_\varepsilon$  of (1.1) with the initial value  $u_0$  is global for sufficiently small  $\varepsilon$ .*
- (2)  *$u_\varepsilon \rightarrow u$  in  $L^q(0, \infty; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , where  $(q, r)$  is an arbitrary admissible pair.*

*Proof* (1) We will prove that  $u_\varepsilon$  is also global for small  $\varepsilon$  if  $u$  is global.

From Theorem 3.1, we can see

$$\|u_\varepsilon(T) - u(T)\|_{H^2} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

for all  $T < \infty$ .

Since  $u$  is global, for any  $\eta > 0$ , there exists sufficient large  $T$  such that

$$\|u\|_{L^{\gamma_1}(T,\infty;W^{2,\rho_1})} \leq \frac{\eta}{4},$$

$(\gamma_1, \rho_1)$  is the same as in Theorem 3.1.

Case 1:  $2 \leq \alpha < \frac{8}{n-4}$ ,  $2 \leq \beta \leq \frac{8}{n-4}$ .

From (2.2), (2.3)-(2.4), using a continuity argument we can obtain

$$\begin{aligned} &\|U(t)u(T)\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} \\ &\leq C\|u(T)\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} + C\left\|\int_T^t U(t-\tau)|u|^\alpha u(\tau) d\tau\right\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} \\ &\leq C\|u(t)\|_{L^{\gamma_1}(T,\infty;W^{2,\rho_1})} + C\| |u|^{\alpha+1} \|_{L^{\gamma_1}(T,\infty;W^{2,\rho_1})} \\ &\leq \frac{\eta}{2}. \end{aligned}$$

Thus we have

$$\begin{aligned} \|U(t)u_\varepsilon(T)\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} &\leq \|U(t)(u_\varepsilon(T) - u(T))\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} \\ &\quad + \|U(t)u(T)\|_{L^{\gamma_1}(0,\infty;W^{2,\rho_1})} \leq \eta. \end{aligned}$$

Obviously  $\|U(t)u_\varepsilon(T)\|_{L^q(0,\infty;W^{2,r})} \leq \eta$  for suitable  $T$  and any admissible pair  $(q, r)$ .

Furthermore we define the working space as follows:

$$\begin{aligned} X(0, t) &= L^{\frac{2(n+4)}{n}}(0, t; L^{\frac{2(n+4)}{n}}(R^n)) \cap L^{\frac{2(n+4)}{n-4}}(0, t; W^{2, \frac{2n(n+4)}{n^2+16}}(R^n)) \\ &\quad \cap L^{\frac{2(n+4)}{n-2}}(0, t; W^{2, \frac{2n(n+4)}{n^2+8}}(R^n)) \cap L^{\gamma_2}(0, t; W^{2,\rho_2}(R^n)) \cap L^\infty(0, t; H^2(R^n)), \end{aligned}$$

where  $(\gamma_2, \rho_2)$  is the same as in Theorem 3.1.

Using the Hölder inequality, the interpolation inequality [13], and the Sobolev embedding, we have

$$\begin{aligned} &\| |u_\varepsilon|^\alpha u_\varepsilon \|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})} \\ &\leq \|u_\varepsilon\|_{L^{\frac{(n+4)\alpha}{4}}(0,t;L^{\frac{(n+4)\alpha}{4}})}^\alpha \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})} \\ &\leq C \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}^{2-\frac{(n-4)\alpha}{4}} \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n-4}}(0,t;L^{\frac{2(n+4)}{n-4}})}^{\frac{n\alpha}{4}-2} \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})} \\ &\leq C \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}^{3-\frac{(n-4)\alpha}{4}} \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n-4}}(0,t;W^{2, \frac{2n(n+4)}{n^2+16}})}^{\frac{n\alpha}{4}-2} \\ &\leq C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1}. \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} &\|\nabla(|u_\varepsilon|^\alpha u_\varepsilon)\|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})} \leq C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1}; \\ &\| |u_\varepsilon|^{\alpha-1} D^2 u_\varepsilon \|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})} \leq C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1}. \end{aligned}$$

For the case  $4 < n < 8$ , we have

$$\begin{aligned} &\| |u_\varepsilon|^{\alpha-1} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})} \\ &\leq \| |u_\varepsilon|^{\alpha-1} \|_{L^{\frac{2(n+4)}{8-n}}(0,t;L^{\frac{2(n+4)}{8-n}})} \| \nabla u_\varepsilon \|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}^2 \\ &\leq C \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;L^{\frac{2(n+4)}{n}})}^{\frac{8-n}{4}-\frac{(n-4)(\alpha-1)}{4}} \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n-4}}(0,t;W^{2, \frac{2n(n+4)}{n^2+16}})}^{\frac{n(\alpha-1)}{4}-\frac{8-n}{4}} \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t;W^{2, \frac{2n(n+4)}{n^2+16}})}^2 \\ &\leq C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1}. \end{aligned}$$

For the case  $8 \leq n < 12$ , we have

$$\begin{aligned} &\| |u_\varepsilon|^{\alpha-1} \nabla u_\varepsilon \cdot \nabla u_\varepsilon \|_{L^{\frac{2(n+4)}{n+8}}(0,t;L^{\frac{2(n+4)}{n+8}})} \\ &\leq \| |u_\varepsilon|^{\alpha-1} \|_{L^{\frac{2(n+4)}{12-n}}(0,t;L^{\frac{2(n+4)}{12-n}})} \| \nabla u_\varepsilon \|_{L^{\frac{2(n+4)}{n-2}}(0,t;L^{\frac{2(n+4)}{n-2}})}^2 \end{aligned}$$

$$\begin{aligned} &\leq C \|u_\varepsilon\|_{L^{\frac{12-n}{4}, \frac{(n-4)(\alpha-1)}{4}}(0,t; L^{\frac{2(n+4)}{n}}(0,t; L^{\frac{2(n+4)}{n}}))} \|u_\varepsilon\|_{L^{\frac{n(\alpha-1)}{4}, \frac{12-n}{4}}(0,t; W^{2, \frac{2(n+4)}{n^2+8}})} \|u_\varepsilon\|_{L^{\frac{2(n+4)}{n}}(0,t; W^{2, \frac{2(n+4)}{n}})} \\ &\leq C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1}; \end{aligned}$$

thus we have

$$\|D^2(|u_\varepsilon|^\alpha u_\varepsilon)\|_{L^{\frac{2(n+4)}{n+8}}(0,t; L^{\frac{2(n+4)}{n+8}})} \leq C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1}.$$

Noting that  $(\frac{2(n+4)}{n+8}, \frac{2(n+4)}{n+8})$  is an admissible pair, using Strichartz estimates, we can obtain

$$\left\| \int_T^t U(t-\tau) |u_\varepsilon|^\alpha u_\varepsilon \right\|_{X(0,t)} \leq C \| |u_\varepsilon|^\alpha u_\varepsilon \|_{L^{\frac{2(n+4)}{n-8}}(0,t; W^{2, \frac{2(n+4)}{n-8}})} \leq C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1}.$$

Using (2.1), we have

$$\|u_\varepsilon(T)\|_{X(0,t)} \leq C \|U(t)u_\varepsilon(T)\|_{X(0,t)} + C \|u_\varepsilon\|_{X(0,t)}^{\alpha+1} + C \|u_\varepsilon\|_{X(0,t)}^{\beta+1}.$$

Using a continuity argument, we immediately have

$$\|u_\varepsilon(T)\|_{X(0,t)} \leq 3\eta \quad \text{for sufficiently small } \varepsilon,$$

which means that  $\|u_\varepsilon\|_{X(T,\infty)} \leq M$ , where  $M$  is a constant.

Furthermore, we have  $\|u_\varepsilon\|_{L^q(T,\infty; W^{2,r})} \leq M$ , for any admissible pair  $(q, r)$ . Thus  $u_\varepsilon$  is global.

Case 2:  $\alpha = \frac{8}{n-4}, 2 \leq \beta \leq \frac{8}{n-4}$ .

We need the following working space:

$$Y(0, t) = L^{\frac{2n}{n-4}}(0, t; W^{2, \frac{2n^2}{n^2-4n+16}}(R^n)) \cap L^{\gamma_2}(0, t; W^{2, \rho_2}(R^n)) \cap L^\infty(0, t; H^2(R^n)).$$

The process of proof is similar to the case 1, so here we omit the detailed proof.

(2) In the sequel, we prove  $u_\varepsilon \rightarrow u$  in  $L^q(0, \infty; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , for any admissible pair  $(q, r)$ .

Using (2.1) and (2.2), we have

$$\begin{aligned} &u_\varepsilon(T+t) - u(T+t) \\ &= U(t)(u_\varepsilon(T) - u(T)) + i \int_0^t U(t-\tau) (|u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u)(T+\tau) d\tau \\ &\quad + \varepsilon \int_0^t U(t-\tau) a(T+\tau) |u|^\beta u(T+\tau) d\tau \\ &= a(t) + b(t) + c(t); \\ &\|a(t)\|_{L^q(0,\infty; W^{2,r})} \leq C \|u_\varepsilon(T) - u(T)\|_{H^2} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0; \\ &\|b(t)\|_{L^q(0,\infty; W^{2,r})} \leq C \| |u_\varepsilon|^\alpha u_\varepsilon - |u|^\alpha u \|_{L^{\frac{2(n+4)}{n+8}}(0,\infty; W^{2, \frac{2(n+4)}{n+8}})} \\ &\leq C (\|u_\varepsilon\|_{X(0,\infty)}^\alpha + \|u\|_{X(0,\infty)}^\alpha) \|u_\varepsilon - u\|_{X(0,\infty)} \rightarrow 0; \end{aligned}$$

$$\|c(t)\|_{L^q(0,\infty;W^{2,r})} \leq C\varepsilon \|u_\varepsilon\|_{X(0,\infty)}^{\beta+1}.$$

Thus we have

$$\|u_\varepsilon(T+t) - u(T+t)\|_{L^q(0,\infty;W^{2,r})} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad \square$$

**Theorem 3.2** *Assume that  $n > 4$ ,  $a \in L^\infty(0, \infty)$ ,  $2 \leq \alpha \leq \frac{8}{n-4}$ , and  $2 \leq \beta \leq \frac{8}{n-4}$ . One of the following conditions holds:*

- (i)  $\lambda < 0$ ,
- (ii)  $\lambda > 0$ ,  $\|u_0\|_{H^2}$  is small.

Then we have

- (1) The solution  $u_\varepsilon$  of (1.1) is global for small  $\varepsilon$ .
- (2)  $u_\varepsilon \rightarrow u$  in  $L^q(0, \infty; W^{2,r}(R^n))$  as  $\varepsilon \rightarrow 0$ , where  $(q, r)$  is arbitrary admissible pair.

*Proof* Note that the solution  $u$  of (1.2) is global provided the conditions (i)  $\lambda < 0$  or (ii)  $\lambda > 0$ ,  $\|u_0\|_{H^2}$  is small hold. Combing Lemma 3.3, the proof of Theorem 3.2 immediately is complete. □

#### 4 Conclusions

The appearance of gain/loss does not affect the local well-posedness of the solution. Moreover, the solution  $u_\varepsilon$  will converge to  $u$  in the space  $L^q(0, T; W^{2,r}(R^n))$  as  $\varepsilon$  converges to 0. Furthermore, if (i)  $\lambda < 0$ , or (ii)  $\lambda > 0$ ,  $\|u_0\|_{H^2}$  is small, then we have found that the global solution  $u_\varepsilon$  will converge to  $u$  in the space  $L^q(0, \infty; W^{2,r}(R^n))$  as  $\varepsilon$  converges to 0.

#### Competing interests

The author declares that they have no competing interests.

#### Acknowledgements

This work is supported by Natural Science of the Shanxi province (No. 2013011003-2) and the Natural Science Foundation of China (Nos. 61473180, 11571209, 61503230).

Received: 22 May 2015 Accepted: 18 September 2015 Published online: 30 September 2015

#### References

1. Cui, SB, Guo, CH: Well-posedness of higher-order nonlinear Schrödinger equations in Sobolev spaces  $H^s(R^n)$  and applications. *Nonlinear Anal.* **67**(3), 687-707 (2007)
2. Pausader, B: The focusing energy-critical fourth-order Schrödinger equation with radial data. *Discrete Contin. Dyn. Syst.* **24**(4), 1275-1292 (2009)
3. Pausader, B: Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case. *Dyn. Partial Differ. Equ.* **4**(3), 197-225 (2007)
4. Miao, C, Xu, G, Zhao, L: Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equations of fourth order in dimensions  $n \geq 9$ . *J. Differ. Equ.* **251**(12), 3381-3402 (2011)
5. Zhang, J, Zheng, J: Energy critical fourth-order Schrödinger equations with subcritical perturbations. *Nonlinear Anal.* **73**(4), 1004-1014 (2010)
6. Pausader, B: The mass-critical fourth-order Schrödinger equation in high dimensions. *J. Hyperbolic Differ. Equ.* **7**(4), 651-705 (2010)
7. Allayarov, IM, Tsoy, EN: Dynamics of fronts in optical media with linear gain and nonlinear losses. *Phys. Lett. A* **377**(7), 550-554 (2013)
8. Feng, B, Zhao, D, Sun, C: The limit behavior of solutions for the nonlinear Schrödinger equation including nonlinear loss/gain with variable coefficient. *J. Math. Anal. Appl.* **405**(1), 240-251 (2013)
9. Feng, B, Zhao, D, Sun, C: On the Cauchy problem for the nonlinear Schrödinger equations with time-dependent linear loss/gain. *J. Math. Anal. Appl.* **416**(2), 901-923 (2014)
10. Ben-Artzi, M, Koch, H, Saut, J-C: Dispersion estimates for fourth order Schrödinger equations. *C. R. Math. Acad. Sci. Paris* **330**(2), 87-92 (2000)
11. Pazy, A: *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New York (1983)
12. Adams, RA, Fournier, JF: *Sobolev Spaces*. Academic Press, Singapore (2009)
13. Bergh, J, Löfström, T: *Interpolation Spaces*. Springer, New York (1976)