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Optimality conditions for strict minimizers of higher-order in semi-infinite multi-objective optimization

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Abstract

This paper is devoted to the study of optimality conditions for strict minimizers of higher-order for a non-smooth semi-infinite multi-objective optimization problem. We propose a generalized Guignard constraint qualification and a generalized Abadie constraint qualification for this problem under which necessary optimality conditions are proved. Under the assumptions of generalized higher-order strong convexity for the functions appearing in the formulation of the non-smooth semi-infinite multi-objective optimization problem, three sufficient optimality conditions are derived.

MSC: Primary 90C34; 90C40; secondary 49J52

Keywords: optimality conditions; multi-objective optimization; semi-infinite optimization; strict minimizer of higher-order

1 Introduction

In recent years, there has been considerable interest in the so-called *semi-infinite multi-objective optimization* problems (SIMOPs), which is the simultaneous minimization of finitely many scalar objective functions subject to an infinitely many constraints. SIMOPs have been investigated intensively by many researchers from several different perspectives. For example, the pseudo-Lipschitz property and the semicontinuity of the efficient solution map under some types of perturbation with respect to a parameter have been discussed in [1–5]. The density of the set of all stable convex semi-infinite vector optimization problems has been established in [6]. However, the work on optimality conditions for SIMOPs is limited. Here we should mention that the authors in [7] have examined the optimality conditions and duality relations in SIMOPs involving differentiable functions, whose constraints are required to depend continuously on an index t belonging to a compact set T. For non-smooth semi-infinite multi-objective optimization problems, work has been done to obtain necessary optimality conditions for weakly efficient solutions and sufficient optimality conditions for efficient solutions by presenting several kinds of constraint qualifications and imposing assumptions of generalized convexity (see [8]), and to establish necessary and sufficient conditions for (weakly) efficient solutions of SIMOPs by applying some advanced tools of variational analysis and generalized differentiation and

proposing the concepts of (strictly) generalized convex functions defined by using the limiting subdifferential of locally Lipschitz functions (see [9]). It is worth noticing that all of the above mentioned literature studies only weakly efficient solutions or efficient solutions of SIMOPs.

On the other hand, a continuing interest in the theory of multi-objective optimization is to define and characterize its solutions. Besides the weak efficiency and efficiency mentioned above, a meaningful solution concept called a strict efficient solution of higher-order (also called a strict minimizer of higher-order) was recently extended by Jiménez in [10] from the strict minimizer of higher-order in scalar optimization given by Auslender in [11] and Ward in [12]. Recently, Bhatia [13] established necessary and sufficient optimality conditions for strict efficiency of higher-order in multi-objective optimization under the *basic regularity condition* and *generalized higher-order strong convexity assumption*, respectively.

In this paper, we introduce the notion of a semi-strict minimizer of higher-order for a semi-infinite multi-objective optimization problem, which includes arbitrary many (possibly infinite) inequality constraints. For the purpose of investigating this new solution concept, we found that the notion of convexity that appears to be most appropriate in the development of sufficient optimality conditions is the strong convexity of higher-order [14].

The rest of this paper is organized as follows. In Section 2, some basic notations and results of non-smooth and convex analysis are reviewed, and the concept of a semi-strict minimizer for a semi-infinite multi-objective optimization problem is presented. In Section 3, we introduce the generalized Guignard constraint qualification and Abadie constraint qualification for SIMOPs. Necessary optimality conditions of Karush-Kuhn-Tuchker type are derived under these two constraint qualifications. Finally, in Section 4, three sufficient optimality conditions for SIMOPs are obtained under the assumption of some generalized strong convexity of higher-order.

2 Notations and preliminaries

Throughout the paper, we let $\mathbb{R}^n$ be the *n*-dimensional Euclidean space endowed with the Euclidean norm $\|\cdot\|$, X be a convex subset of $\mathbb{R}^n$, and $m \geq 1$ be a positive integer. Let W be a subset of $\mathbb{R}^n$. We use cl W, co W, and cone W to denote the closure of W, the convex hull of W, and the conic hull of W (*i.e.*, the smallest convex cone containing W), respectively.

Definition 2.1 (see [15–17]) Let W be a nonempty subset of $\mathbb{R}^n$. The *tangent cone* to W at $\bar{x} \in \text{cl } W$ is the set defined by

$$T(W; \bar{x}) := \left\{ h \in \mathbb{R}^n : h = \lim_{n \to \infty} t_n (x^n - \bar{x}) \text{ such that } x^n \in W, \right.$$

$$\left. \lim_{n \to \infty} x^n = \bar{x} \text{ and } t_n > 0 \text{ for all } n = 1, 2, \ldots \right\}.$$

Recall that a function $\varphi : X \to \mathbb{R}$ is *Lipschitz* at $\bar{x} \in X$ if there exists a positive constant K such that

$$\left| \varphi(x) - \varphi(\bar{x}) \right| \leq K \|x - \bar{x}\| \quad \text{for all } x \in X,$$

where K is called the rank of φ at $\bar{x}$. φ is said to be Lipschitz on X if φ is Lipschitz at each $x \in X$. Suppose that φ is Lipschitz at $\bar{x} \in X$, then *Clarke's generalized directional derivative*

of φ at $\bar{x} \in X$ in the direction $v \in \mathbb{R}^n$, denoted by $\varphi^0(\bar{x}, v)$, is defined as

$$\varphi^0(\bar{x}, v) = \limsup_{(x,t)\to(\bar{x},0^+)} \frac{\varphi(x+tv) - \varphi(x)}{t}.$$

Clarke's generalized gradient of φ at $\bar{x} \in X$, denoted by $\partial\varphi(\bar{x})$, is defined as

$$\partial\varphi(\bar{x}) = \left\{ \xi \in \mathbb{R}^n : \varphi^0(\bar{x}, v) \geq \langle \xi, v \rangle \text{ for all } v \in \mathbb{R}^n \right\}.$$

It is well known that $\partial\varphi(\bar{x})$ is a nonempty convex compact set in $\mathbb{R}^n$.

Definition 2.2 (see [18]) Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz at $\bar{x} \in \mathbb{R}^n$. It is said that φ admits a *strict derivative* at $\bar{x}$, an element of $\mathbb{R}^n$, denoted by $D_s\varphi(\bar{x})$, provided that, for each $x \in \mathbb{R}^n$, the following holds:

$$\lim_{(x',t)\to(\bar{x},0)} \frac{\varphi(x'+tx) - \varphi(x')}{t} = \langle D_s\varphi(\bar{x}), x \rangle.$$

If φ admits a strict derivative at $\bar{x}$, then φ is called *strictly differentiable* at $\bar{x}$.

Lemma 2.1 (see [19]) *Let φ, φ_1, and φ_2 be Lipschitz from X to $\mathbb{R}$, and $\bar{x} \in X$. Then the following properties hold*:
   (a) $\varphi^0(\bar{x}, v) = \max\{\langle \xi, v \rangle : \xi \in \partial\varphi(\bar{x})\}$, *for all $v \in \mathbb{R}^n$*.
   (b) $\partial(\lambda\varphi(\bar{x})) = \lambda\partial\varphi(\bar{x})$, *for all $\lambda \in \mathbb{R}$*.
   (c) $\partial(\varphi_1 + \varphi_2)(\bar{x}) \subset \partial\varphi_1(\bar{x}) + \partial\varphi_2(\bar{x})$.

Now, we recall the definition of the *strong convexity of order m* for a Lipschitz function.

Definition 2.3 (see [13, 14]) Let $\varphi : X \to \mathbb{R}$ be Lipschitz at $\bar{x} \in X$.
   (a) φ is said to be *strongly convex of order m* at $\bar{x}$ if there exists a constant $c > 0$ such that for each $x \in X$ and $\xi \in \partial\varphi(\bar{x})$

$$\varphi(x) - \varphi(\bar{x}) \geq \langle \xi, x - \bar{x} \rangle + c\|x - \bar{x}\|^m.$$

   (b) φ is said to be *strongly quasiconvex* of order m at $\bar{x}$ if there exists a constant $c > 0$ such that, for each $x \in X$ and $\xi \in \partial\varphi(\bar{x})$,

$$\varphi(x) \leq \varphi(\bar{x}) \quad \Rightarrow \quad \langle \xi, x - \bar{x} \rangle + c\|x - \bar{x}\|^m \leq 0.$$

Based upon the above definition of a strongly convex function of order m, we define the following generalized strong convexities of order m for a Lipschitz function.

Definition 2.4 Let $\varphi : X \to \mathbb{R}$ be Lipschitz at $\bar{x} \in X$.
   (a) φ is *strictly strong convex* of order m at $\bar{x}$ if there exists a constant $c > 0$ such that, for each $x \in X$ with $x \neq \bar{x}$ and $\xi \in \partial\varphi(\bar{x})$,

$$\varphi(x) - \varphi(\bar{x}) > \langle \xi, x - \bar{x} \rangle + c\|x - \bar{x}\|^m.$$

(b) φ is *strictly strong quasiconvex* of order m at $\bar{x}$ if there exists a constant $c > 0$ such that, for each $x \in X$ with $x \neq \bar{x}$ and any $\xi \in \partial\varphi(\bar{x})$,

$$\varphi(x) \leq \varphi(\bar{x}) \quad \Rightarrow \quad \langle \xi, x - \bar{x} \rangle + c\|x - \bar{x}\|^m < 0.$$

The next Lemma gives a basic property of generalized higher-order strong convexities, which will be used in Section 4.

Proposition 2.1 *Let* $\varphi_i : X \to \mathbb{R}$ *be Lipschitz at* $\bar{x} \in X$, $i = 0, 1, 2, \ldots, s$. *Suppose that* φ_0 *is a strictly strong convex function of order m and* $\varphi_1, \varphi_2, \ldots, \varphi_s$ *are strongly convex functions of order m at* $\bar{x}$. *If* $\lambda_0 > 0$ *and* $\lambda_i \geq 0$ *for* $i = 1, 2, \ldots, s$, *then* $\sum_{i=0}^{s} \lambda_i \varphi_i$ *is strictly strong convex of order m at* $\bar{x}$.

Proof It is evident that the function $\sum_{i=0}^{s} \lambda_i \varphi_i$ is Lipschitz at $\bar{x}$. Thus, we get

$$\partial \left(\sum_{i=0}^{s} \lambda_i \varphi_i \right)(\bar{x}) \neq \emptyset.$$

Taking $\xi \in \partial(\sum_{i=0}^{s} \lambda_i \varphi_i)(\bar{x})$. It follows from Lemma 2.1 that

$$\partial \left(\sum_{i=0}^{s} \lambda_i \varphi_i \right)(\bar{x}) \subset \sum_{i=0}^{s} \partial(\lambda_i \varphi_i)(\bar{x}) = \sum_{i=0}^{s} \lambda_i \partial \varphi_i(\bar{x}).$$

This means that there exist $\xi_i \in \partial\varphi_i(\bar{x})$, $i = 0, 1, \ldots, s$, such that

$$\xi = \sum_{i=0}^{s} \lambda_i \xi_i.$$

Since φ_0 is strictly strong convex of order m at $\bar{x}$ and φ_i, $i = 1, 2, \ldots, s$, is strongly convex of order m at $\bar{x}$, we derive that there exist $c_i > 0$, $i = 0, 1, \ldots, s$, such that, for all $x \in \mathbb{R}^n$,

$$\begin{cases} \varphi_0(x) - \varphi_0(\bar{x}) > \langle \xi_0, x - \bar{x} \rangle + c_0 \|x - \bar{x}\|^m, \\ \varphi_i(x) - \varphi_i(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle + c_i \|x - \bar{x}\|^m \quad \text{for all } i = 1, 2, \ldots, s, \end{cases}$$

which implies that

$$\sum_{i=0}^{s} \lambda_i \big(\varphi_i(x) - \varphi_i(\bar{x}) \big) > \sum_{i=0}^{s} \lambda_i \langle \xi_i, x - \bar{x} \rangle + \left(\sum_{i=0}^{s} \lambda_i c_i \right) \|x - \bar{x}\|^m.$$

Therefore, we get

$$\left(\sum_{i=0}^{s} \lambda_i \varphi_i \right)(x) - \left(\sum_{i=0}^{s} \lambda_i \varphi_i \right)(\bar{x}) > \langle \xi, x - \bar{x} \rangle + c\|x - \bar{x}\|^m,$$

where $c = \sum_{i=0}^{s} \lambda_i c_i$. This completes the proof of the proposition. $\square$

Consider the following *semi-infinite multi-objective optimization* problem:

(P)   Minimize $f(x) = (f_1(x), f_2(x), \ldots, f_p(x))$

subject to $g_t(x) \leq 0$ for $t \in T$,

$x \in X$,

where f_i, $i \in P = \{1, 2, \ldots, p\}$, and g_t, $t \in T$ are Lipschitz from X to $\mathbb{R}$, and the index set T is arbitrary, not necessarily finite (but nonempty). The feasible set of (P) is denoted by Ω,

$$\Omega := \{x \in X : g_t(x) \leq 0, \forall t \in T\}.$$

For a given $\bar{x} \in \Omega$, set

$$\hat{T}(\bar{x}) := \{t \in T : g_t(\bar{x}) = 0\}.$$

In the sequel, we use the following notations. For $x, y \in \mathbb{R}^n$.

 (i) $f(x) < f(y) \Leftrightarrow f_i(x) < f_i(y)$ for every $i \in P$;
 (ii) $f(x) \not< f(y)$ is the negation of $f(x) < f(y)$;
 (iii) $f(x) \leq f(y) \Leftrightarrow f_i(x) \leq f_i(y)$ for every $i \in P$, but there is at least one $i_0 \in P$ such that
    $f_{i_0}(x) < f_{i_0}(y)$;
 (iv) $f(x) \not\leq f(y)$ is the negation of $f(x) \leq f(y)$.

Definition 2.5 (see [10]) A point $\bar{x} \in \Omega$ is said to be a strict minimizer of order m for (P) if there exists $c = (c_1, c_2, \ldots, c_p) \in \mathbb{R}^p$ with $c_i > 0$, $i \in P$, such that

$$f(x) \not< f(\bar{x}) + c\|x - \bar{x}\|^m \quad \text{for all } x \in \Omega.$$

Definition 2.6 A point $\bar{x} \in \Omega$ is said to be a semi-strict minimizer of order m for (P) if there exists $c = (c_1, c_2, \ldots, c_p) \in \mathbb{R}^p$ with $c_i > 0$, $i \in P$, such that

$$f(x) \not\leq f(\bar{x}) + c\|x - \bar{x}\|^m \quad \text{for all } x \in \Omega.$$

Remark 2.1 It is obvious that if $\bar{x} \in \Omega$ is a semi-strict minimizer of order m for (P), then $\bar{x} \in \Omega$ is a strict minimizer of order m for (P).

Example 2.1 The functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, defined by

$$f_1(x) = \begin{cases} x^4 + e^x & \text{if } x \geq 0, \\ x^4 + 1 & \text{if } x < 0, \end{cases} \qquad f_2(x) = \begin{cases} x^4 + x & \text{if } x \geq 0, \\ x^4 - x & \text{if } x < 0, \end{cases}$$

are Lipschitz at $\bar{x} = 0$. It is easy to verify that $\bar{x}$ is a semi-strict minimizer of order 4 with $c = (1, 1)$ for the following optimization problem:

(P$_0$)   Minimize $f(x) = (f_1(x), f_2(x))$

subject to $x \in \mathbb{R}$.

Motivated by the notion of *a linearizing cone* at a point to the feasible set of a differentiable multi-objective optimization problem, which was introduced by Maeda in [15], we give the definition of a linearizing cone for the semi-infinite multi-objective optimization problem (P). We first need to define a set.

Let $\bar{x} \in \Omega$ be a semi-strict minimizer of order m for (P), and define

$$Q^i(\bar{x}) := \left\{ x \in \mathbb{R}^n : f_k(x) \leq f_k(\bar{x}) + c_k \|x - \bar{x}\|^m, k \in P \text{ and } k \neq i \right\} \cap \Omega.$$

It is obvious that $\bar{x} \in Q^i(\bar{x})$, $i \in P$.

Definition 2.7 Let $\bar{x} \in \Omega$. The *linearizing cone* at $\bar{x}$ is the set defined by

$$C(\bar{x}) = \left\{ z \in \mathbb{R}^n : \langle \xi, z \rangle \leq 0 \text{ for all } \xi \in \partial f_i(\bar{x}), i \in P \text{ and} \right.$$
$$\left. \langle \zeta, z \rangle \leq 0 \text{ for all } \zeta \in \partial g_t(\bar{x}), t \in \hat{T}(\bar{x}) \right\}.$$

3 Necessary conditions

In this section, we shall examine necessary optimality conditions for a semi-strict (strict) minimizer of order m for (P). we begin with presenting two constraint qualifications, which are the non-smooth, semi-infinite version of the generalized Guignard constraint qualification and generalized Abadie constraint qualification presented in [15] and [16].

Definition 3.1 The problem (P) satisfies the *generalized Guignard constraint qualification* at a given point $\bar{x} \in \Omega$ which is a semi-strict minimizer of order m for (P) if the following holds: $C(\bar{x}) \subseteq \bigcap_{i=1}^p \mathrm{cl}[\mathrm{co}(T(Q^i; \bar{x}))]$, where $Q^i := Q^i(\bar{x})$.

Definition 3.2 The problem (P) satisfies the *generalized Abadie constraint qualification* at a given point $\bar{x} \in \Omega$ which is a semi-strict minimizer of order m for (P) if the following holds: $C(\bar{x}) \subseteq \bigcap_{i=1}^p T(Q^i; \bar{x})$.

Next, we recall the *generalized Motzkin theorem* discussed in [20].

Lemma 3.1 (see [20]) *Let A be a compact set in $\mathbb{R}^n$, B an arbitrary set in $\mathbb{R}^n$. Suppose that the set* cone *B is closed. Then either the system*

$$\begin{cases} \langle a, z \rangle < 0 & \text{for all } a \in A, \\ \langle b, z \rangle \leq 0 & \text{for all } b \in B, \end{cases}$$

has a solution $z \in \mathbb{R}^n$, or there exist integers μ and ν, with $0 \leq \nu \leq n+1$, such that there exist μ points $a^i \in A$ ($i = 1, 2, \ldots, \mu$), ν points $b^m \in B$ ($m = 1, 2, \ldots, \nu$), μ nonnegative numbers u_i, with $u_i > 0$ for at least one $i \in \{1, 2, \ldots \mu\}$, and ν positive numbers v_m for $m \in \{1, 2, \ldots \nu\}$, such that

$$\sum_{i=1}^{\mu} u_i a^i + \sum_{m=1}^{\nu} v_m b^m = 0,$$

but never both.

The following lemma is a generalization of the classical Tucker theorem of the alternative. We use this lemma in the proof of our necessary efficiency result. The proof of the lemma is similar to that of Lemma 3.6 in [17], hence it is omitted.

Lemma 3.2 *Let $A^i \subset \mathbb{R}^n$, $i \in P$, be compact convex sets, B an arbitrary set in $\mathbb{R}^n$. Suppose that, for each $i \in P$, the set* $\mathrm{cone}(B \cup [\bigcup_{j \in P, j \neq i} A^j])$ *is closed. Then either the system*

$$
\begin{cases}
\langle a, z \rangle < 0 & \text{for at least one } i \in P \text{ and for all } a \in A^i, \\
\langle a, z \rangle \leq 0 & \text{for all } a \in \bigcup_{i=1}^{p} A^i, \\
\langle b, z \rangle \leq 0 & \text{for all } b \in B,
\end{cases}
$$

has a solution $z \in \mathbb{R}^n$, or there exist $u \in U \equiv \{u \in \mathbb{R}^p : u > 0, \sum_{i=1}^{p} u_i = 1\}$, $a^i \in A^i$ for $i \in P$, and integer v with $0 \leq v \leq n + 1$, such that there exist v points $b^m \in B$, and v positive numbers v_m, $m \in \{1, 2, \ldots, v\}$, such that

$$
\sum_{i=1}^{p} u_i a^i + \sum_{m=1}^{v} v_m b^m = 0,
$$

but never both.

Lemma 3.3 *Let $\bar{x}$ be a semi-strict minimizer of order m for (P), let $f_i(x)$, $i \in P$, be Lipschitz at $\bar{x}$ of rank K_i, for all $t \in T$, let the functions $g_t(x)$ be Lipschitz at $\bar{x}$. If the generalized Guignard constraint qualification holds at $\bar{x}$ and $f_i(x)$, $i \in P$, are strictly differentiable at $\bar{x}$ or the generalized Abadie constraint qualification holds at $\bar{x}$, then the system*

$$
\begin{cases}
\langle \xi, z \rangle < 0 & \text{for at least one } i \in P \text{ and for all } \xi \in \partial f_i(\bar{x}), \\
\langle \xi, z \rangle \leq 0 & \text{for all } \xi \in \bigcup_{i=1}^{p} \partial f_i(\bar{x}), \\
\langle \zeta, z \rangle \leq 0 & \text{for all } \zeta \in \partial g_t(\bar{x}), t \in \hat{T}(\bar{x}),
\end{cases} \tag{3.1}
$$

has no solution $z \in \mathbb{R}^n$.

Proof Suppose to the contrary that (3.1) has a solution z. Then $z \neq 0$ and $z \in C(\bar{x})$. Without loss of generality, we can assume that

$$
\langle \xi, z \rangle < 0 \quad \text{for all } \xi \in \partial f_1(\bar{x}),
$$

$$
\langle \xi, z \rangle \leq 0 \quad \text{for all } \xi \in \bigcup_{i=2}^{p} \partial f_i(\bar{x}).
$$

By our generalized Guignard constraint qualification assumption, $z \in \mathrm{cl}[\mathrm{co}(T(Q^1; \bar{x}))]$, and hence there exists a sequence $\{z^m\}_{m=1}^{\infty} \subset \mathrm{co}(T(Q^1; \bar{x}))$ such that

$$
\lim_{m \to \infty} z^m = z. \tag{3.2}
$$

For each z^m, $m = 1, 2, \ldots$, there exist numbers L_m and $\lambda_{ml} \geq 0$, and $z^{ml} \in T(\Omega^1; \bar{x})$, $z^{ml} \neq 0$, $l = 1, 2, \ldots, L_m$, such that

$$
\sum_{l=1}^{L_m} \lambda_{ml} = 1, \qquad \sum_{l=1}^{L_m} \lambda_{ml} z^{ml} = z^m. \tag{3.3}
$$

Since, for each $m = 1, 2, \ldots$ and $l = 1, 2, \ldots, L_m$, $z^{ml} \in T(Q^1; \bar{x})$, there exist sequences $\{x^{mln}\}_{n=1}^{\infty} \subset Q^1$ and $\{t_{mln}\}_{n=1}^{\infty} \subset \mathbb{R}$, with $t_{mln} > 0$ for all n, such that

$$\lim_{n \to \infty} x^{mln} = \bar{x}, \qquad \lim_{n \to \infty} t_{mln}(x^{mln} - \bar{x}) = z^{ml}. \tag{3.4}$$

Noticing that $x^{mln} \in Q^1$ for all n, we have $x^{mln} \in \Omega$ and $f_i(x^{mln}) \le f_i(\bar{x}) + c_i \|x^{mln} - \bar{x}\|^m$ for $i = 2, 3, \ldots, p$.

Since $\bar{x}$ is a semi-strict minimizer of order m for (P), for all n we may assume

$$f_1(x^{mln}) \ge f_1(\bar{x}) + c_1 \|x^{mln} - \bar{x}\|^m. \tag{3.5}$$

Since $z^{ml} \ne 0$, for each $m = 1, 2, \ldots$ and $l = 1, 2, \ldots, L_m$, we must have $t_{mln} \to +\infty$ as $n \to +\infty$, and hence $x^{mln} - \frac{1}{t_{mln}} z^{ml} \to \bar{x}$ as $n \to +\infty$. Therefore,

$$
\begin{aligned}
f_1^0(\bar{x}; z^{ml}) &= \limsup_{\substack{y \to \bar{x} \\ t \downarrow 0}} \frac{f_1(y + t z^{ml}) - f_1(y)}{t} \\
&\ge \limsup_{n \to \infty} \frac{f_1(x^{mln} - \frac{1}{t_{mln}} z^{ml} + \frac{1}{t_{mln}} z^{ml}) - f_1(x^{mln} - \frac{1}{t_{mln}} z^{ml})}{\frac{1}{t_{mln}}} \\
&\ge \limsup_{n \to \infty} \frac{f_1(x^{mln}) - f_1(\bar{x}) + f_1(\bar{x}) - f_1(x^{mln} - \frac{1}{t_{mln}} z^{ml})}{\frac{1}{t_{mln}}} \\
&\ge \limsup_{n \to \infty} \frac{f_1(x^{mln}) - f_1(\bar{x})}{\frac{1}{t_{mln}}} - \limsup_{n \to \infty} \frac{|f_1(\bar{x}) - f_1(x^{mln} - \frac{1}{t_{mln}} z^{ml})|}{\frac{1}{t_{mln}}} \\
&\ge \limsup_{n \to \infty} \frac{c_1 \|x^{mln} - \bar{x}\|^m}{\frac{1}{t_{mln}}} - \limsup_{n \to \infty} \frac{|f_1(\bar{x}) - f_1(x^{mln} - \frac{1}{t_{mln}} z^{ml})|}{\frac{1}{t_{mln}}} \\
&\qquad \text{(by (3.5))} \\
&\ge -\limsup_{n \to \infty} \frac{|f_1(\bar{x}) - f_1(x^{mln} - \frac{1}{t_{mln}} z^{ml})|}{\frac{1}{t_{mln}}} \\
&\ge -\limsup_{n \to \infty} t_{mln} K_1 \left| \bar{x} - x^{mln} + \frac{1}{t_{mln}} z^{ml} \right| \\
&\qquad \text{(by the Lipschitz continuity of } f_1) \\
&= -\limsup_{n \to \infty} K_1 \left| t_{mln}(\bar{x} - x^{mln}) + z^{ml} \right| \\
&= 0 \quad \text{(by (3.4))}. \tag{3.6}
\end{aligned}
$$

Because $f_1(x)$ is strictly differentiable at $\bar{x}$, we know $f_1^0(x; \nu) = \langle D_s f_1(\bar{x}), \nu \rangle$. By Proposition 2.2.4 in [18], we also know that $\partial f_1(\bar{x}) = \{D_s f_1(\bar{x})\}$. In view of (3.3) we obtain $\langle D_s f_1(\bar{x}), z^m \rangle = f_1^0(\bar{x}; z^m) \ge 0$, which further gives us $\langle D_s f_1(\bar{x}), z \rangle = f_1^0(\bar{x}; z) \ge 0$ because of (3.2), contradicting the assumption that z is a solution of the system (3.1). Therefore, under the assumption that the generalized Guignard constraint qualification holds and $f_i(x)$, $i \in P$, are strictly differentiable at $\bar{x}$, (3.1) has no solution $z \in \mathbb{R}^n$.

Now let us show that (3.1) has no solution $z \in \mathbb{R}^n$ under the generalized Abadie constraint qualification. Suppose to the contrary that (3.1) has a solution z. Then $z \ne 0$ and

$z \in C(\bar{x})$. By the generalized Abadie constraint qualification, without loss of generality we may assume that $z \in T(Q^1; \bar{x})$, and hence there exist sequences $\{x^n\}_{n=1}^{\infty} \subset Q^1$ and $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$, with $t_n > 0$ for all n, such that

$$\lim_{n \to \infty} x^n = \bar{x}, \qquad \lim_{n \to \infty} t_n (x^n - \bar{x}) = z.$$

Replacing t_{mln} by t_n, x^{mln} by x^n, and z^{ml} by z in (3.6), we arrive at $f_1^0(\bar{x}; z) \geq 0$. By Proposition 2.1.2 in [18], we know that there is a $\xi \in \partial f_1(\bar{x})$ such that $\langle \xi, z \rangle = f_1^0(\bar{x}; z) \geq 0$, contradicting the assumption that z is a solution of the system (3.1). Therefore, (3.1) has no solution $z \in \mathbb{R}^n$. □

Now we are ready to prove the following necessary optimality condition for (P).

Theorem 3.1 *Let $\bar{x} \in \Omega$ and let the functions $f_i(x)$ for $i \in P$ and $g_t(x)$ for $t \in T$ be Lipschitz at $\bar{x}$. If $\bar{x}$ is a semi-strict minimizer of order m for* (P), *if the generalized Guignard constraint qualification holds at $\bar{x}$ and $f_i(x)$ for each $i \in P$ is strictly differentiable at $\bar{x}$ (or the generalized Abadie constraint qualification holds at $\bar{x}$), and if for each $i_0 \in P$, the set* cone$(\{\zeta \in \partial g_t(\bar{x}) : t \in \hat{T}(\bar{x})\} \cup \{\xi \in \partial f_i(\bar{x}) : i \in P, i \neq i_0\})$ *is closed, then there exist $u^* \in U \equiv \{u \in \mathbb{R}^p : u > 0, \sum_{i=1}^p u_i = 1\}$, integers ν^*, with $0 \leq \nu^* \leq n + 1$, such that there exist $t^m \in \hat{T}(\bar{x})$ and $\nu_m^* > 0$, $m \in \{1, 2, \ldots, \nu^*\}$, with the property that*

$$0 \in \sum_{i=1}^p u_i^* \partial f_i(\bar{x}) + \sum_{m=1}^{\nu^*} \nu_m^* \partial g_{t^m}(\bar{x}). \tag{3.7}$$

Proof In Lemma 3.2, set

$$A^i = \partial f_i(\bar{x}), \quad i \in P,$$

$$B = \bigcup_{t \in \hat{T}(\bar{x})} \partial g_t(\bar{x}).$$

By Proposition 2.1.2 in [18], we know that A^i is a compact convex set. According to Lemma 3.3, the system (3.1) has no solution and, therefore, by Lemma 3.2, there exist $u^* \in U$, $a^i \in \partial f_i(x)$ for $i \in P$, integers ν^* with $0 \leq \nu^* \leq n + 1$, such that there exist ν^* points $t^m \in \hat{T}(\bar{x})$ and ν^* positive numbers $\nu_m^* > 0$, $m \in \{1, 2, \ldots, \nu^*\}$, such that (3.7) holds. □

Remark 3.1 If we modify the definition of $Q^i(\bar{x})$ as follows:

$$Q^i(\bar{x}) := \{x \in \mathbb{R}^n : f_k(x) < f_k(\bar{x}) + c_k \|x - \bar{x}\|, k \in P \text{ and } k \neq i\} \cap \Omega,$$

and define the generalized Guignard constraint qualification and generalized Abadie constraint qualification accordingly, we can prove a similar necessary result for $\bar{x}$ to be a strict minimizer of order m for (P).

4 Sufficient conditions

In this section we discuss sufficient optimality results under various generalized higher-order strong convexity (introduced in Section 2) hypotheses imposed on the involved functions.

Theorem 4.1 (Sufficient optimality conditions I) *Let $\bar{x} \in \Omega$ and $\hat{T}(\bar{x}) \neq \emptyset$. Suppose that there exist scalars $\alpha_i \geq 0$, $i = 1, 2, \ldots, p$ with $\sum_{i=1}^{p} \alpha_i = 1$, and $\beta_t \geq 0$, $t \in \hat{T}(\bar{x})$ with $\beta_t \neq 0$ for finitely many indices t, such that*

$$0 \in \sum_{i=1}^{p} \alpha_i \partial f_i(\bar{x}) + \sum_{t \in \hat{T}(\bar{x})} \beta_t \partial g_t(\bar{x}). \tag{4.1}$$

If the functions f_i, $i = 1, 2, \ldots, p$, are strongly convex of order m at $\bar{x}$, and g_t for $t \in \hat{T}(\bar{x})$ and $\beta_t \neq 0$, are strongly quasiconvex of order m at $\bar{x}$, then $\bar{x}$ is a strict minimizer of order m for (P).

Proof Let $J(\bar{x}) := \{t \in \hat{T}(\bar{x}) : \beta_t \neq 0\}$. Because of (4.1), we derive that there exist $\xi_i \in \partial f_i(\bar{x})$ for $i \in \{1, 2, \ldots, p\}$ and $\zeta_t \in \partial g_t(\bar{x})$ for $t \in J(\bar{x})$ such that

$$\sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in J(\bar{x})} \beta_t \zeta_t = 0. \tag{4.2}$$

Since f_i for $i \in \{1, 2, \ldots, p\}$ are strongly convex of order m at $\bar{x}$, we see that there exist $\bar{c}_i > 0$ for $i \in \{1, 2, \ldots, p\}$ such that

$$f_i(x) - f_i(\bar{x}) \geq \langle \xi_i, x - \bar{x} \rangle + \bar{c}_i \|x - \bar{x}\|^m \quad \text{for all } x \in \Omega, \xi_i \in \partial f_i(\bar{x}), i = 1, 2, \ldots, p.$$

Noticing $\alpha_i \geq 0$ for $i \in \{1, 2, \ldots, p\}$, we have

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x})\big) \geq \left\langle \sum_{i=1}^{p} \alpha_i \xi_i, x - \bar{x} \right\rangle + \sum_{i=1}^{p} \alpha_i \bar{c}_i \|x - \bar{x}\|^m. \tag{4.3}$$

On the other hand, $g_t(x) \leq g_t(\bar{x}) = 0$ for $x \in \Omega$, $t \in J(\bar{x})$. By the strong quasiconvexity of order m at $\bar{x}$ for g_t with $t \in J(\bar{x})$, we see that there exist $\bar{c}_t > 0$ for $t \in J(\bar{x})$ such that, for $\zeta_t \in \partial g_t(\bar{x})$,

$$\langle \zeta_t, x - \bar{x} \rangle + \bar{c}_t \|x - \bar{x}\|^m \leq 0,$$

furthermore, it follows from $\beta_t \geq 0$ for $t \in J(\bar{x})$ that

$$\left\langle \sum_{t \in J(\bar{x})} \beta_t \zeta_t, x - \bar{x} \right\rangle + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t \|x - \bar{x}\|^m \leq 0. \tag{4.4}$$

Adding (4.3) to (4.4), we get

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x})\big)$$

$$\geq \left\langle \sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in J(\bar{x})} \beta_t \zeta_t, x - \bar{x} \right\rangle + \left(\sum_{i=1}^{p} \alpha_i \bar{c}_i + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t \right) \cdot \|x - \bar{x}\|^m. \tag{4.5}$$

It follows from (4.2) that

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x})\big) - \left(\sum_{i=1}^{p} \alpha_i \bar{c}_i + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t\right) \cdot \|x - \bar{x}\|^m \geq 0.$$

Let $\bar{c} = \sum_{i=1}^{p} \alpha_i \bar{c}_i + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t$ and $c_i = \alpha_i \bar{c}$. Noticing that $\sum_{i=1}^{p} \alpha_i = 1$, we obtain

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x})\big) \geq \sum_{i=1}^{p} \alpha_i c \|x - \bar{x}\|^m.$$

This implies

$$\big\langle \alpha, f(x) - f(\bar{x}) - c\|x - \bar{x}\|^m \big\rangle \geq 0,$$

where $c = (\bar{c}, \bar{c}, \ldots, \bar{c})$ with $\bar{c} > 0$. Since $\alpha_i \geq 0$ and $\sum_{i=1}^{p} \alpha_i = 1$, we further see that for all $x \in \Omega$

$$f(x) \not< f(\bar{x}) + c\|x - \bar{x}\|^m,$$

which implies that $\bar{x}$ is a strict minimizer of order m for (P). $\qquad\square$

Theorem 4.2 (Sufficient optimality conditions II) *Let $\bar{x} \in \Omega$ and $\hat{T}(\bar{x}) \neq \emptyset$. Suppose that there exist scalars $\alpha_i \geq 0$, $i = 1, 2, \ldots, p$ with $\sum_{i=1}^{p} \alpha_i = 1$, and $\beta_t \geq 0$, $t \in \hat{T}(\bar{x})$ with $\beta_t \neq 0$ for finitely many indices t, such that*

$$0 \in \sum_{i=1}^{p} \alpha_i \partial f_i(\bar{x}) + \sum_{t \in \hat{T}(\bar{x})} \beta_t \partial g_t(\bar{x}).$$

If the functions f_i, $i \in \{1, 2, \ldots, p : \alpha_i \neq 0\}$, are strongly convex of order m at $\bar{x}$ and at least one of them is strictly strong convex of order m at $\bar{x}$, and g_t for $t \in \hat{T}(\bar{x})$ and $\beta_t \neq 0$, are strongly quasiconvex of order m at $\bar{x}$, then $\bar{x}$ is a semi-strict minimizer of order m for (P).

Proof In the proof of Theorem 4.1, we derived that there exist $\xi_i \in \partial f_i(\bar{x})$ for $i \in \{1, 2, \ldots, p\}$ and $\zeta_t \in \partial g_t(\bar{x})$ for $t \in J(\bar{x})$, such that

$$\sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in J(\bar{x})} \beta_t \zeta_t = 0, \tag{4.6}$$

and that there exist $\bar{c}_t > 0$ for $t \in J(\bar{x})$, such that, for $\zeta_t \in \partial g_t(\bar{x})$, we have

$$\left\langle \sum_{t \in J(\bar{x})} \beta_t \zeta_t, x - \bar{x} \right\rangle + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t \|x - \bar{x}\|^m \leq 0. \tag{4.7}$$

Since the functions f_i for $i \in \{1, 2, \ldots, p\}$ are strongly convex of order m at $\bar{x}$ and there is at least one $i_0 \in \{1, 2, \ldots, p : \alpha_i \neq 0\}$ such that f_{i_0} is strictly strong convex of order m at $\bar{x}$,

using the same argument as in the proof of Theorem 4.1, we arrive at the conclusion that there exist $\bar{c}_i > 0$ for $i \in \{1, 2, \ldots, p\}$ such that

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x}) \big) > \left\langle \sum_{i=1}^{p} \alpha_i \xi_i, x - \bar{x} \right\rangle + \sum_{i=1}^{p} \alpha_i \bar{c}_i \| x - \bar{x} \|^m.$$

Adding the above inequality to (4.7), we obtain

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x}) \big)$$

$$> \left\langle \sum_{i=1}^{p} \alpha_i \xi_i + \sum_{t \in J(\bar{x})} \beta_t \zeta_t, x - \bar{x} \right\rangle + \left(\sum_{i=1}^{p} \alpha_i \bar{c}_i + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t \right) \| x - \bar{x} \|^m. \tag{4.8}$$

It follows from (4.6) that

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x}) \big) - \left(\sum_{i=1}^{p} \alpha_i \bar{c}_i + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t \right) \| x - \bar{x} \|^m > 0.$$

Let $\bar{c} = \sum_{i=1}^{p} \alpha_i \bar{c}_i + \sum_{t \in J(\bar{x})} \beta_t \bar{c}_t$ and $c_i = \alpha_i \bar{c}$. Noticing that $\sum_{i=1}^{p} \alpha_i = 1$, we get

$$\sum_{i=1}^{p} \alpha_i \big(f_i(x) - f_i(\bar{x}) \big) > \sum_{i=1}^{p} \alpha_i c \| x - \bar{x} \|^m.$$

Hence, with $c = (\bar{c}, \bar{c}, \ldots, \bar{c})$ we have

$$\big\langle \alpha, f(x) - f(\bar{x}) - c \| x - \bar{x} \|^m \big\rangle > 0.$$

Because $\alpha_i \geq 0$ for $i \in \{1, 2, \ldots, p\}$ and $\sum_{i=1}^{p} \alpha_i = 1$, we know that, for all $x \in \Omega$,

$$f(x) \nleq f(\bar{x}) + c \| x - \bar{x} \|^m,$$

which implies that $\bar{x}$ is a semi-strict minimizer of order m for (P). □

Theorem 4.3 (Sufficient optimality conditions III) *Let $\bar{x} \in \Omega$ and $\hat{T}(\bar{x}) \neq \emptyset$. Suppose that there exist scalars $\alpha_i \geq 0$, $i = 1, 2, \ldots, p$ with $\sum_{i=1}^{p} \alpha_i = 1$, and $\beta_t \geq 0$, $t \in \hat{T}(\bar{x})$ with $\beta_t \neq 0$ for finitely many indices t, such that*

$$0 \in \sum_{i=1}^{p} \alpha_i \partial f_i(\bar{x}) + \sum_{t \in \hat{T}(\bar{x})} \beta_t \partial g_t(\bar{x}).$$

If the functions f_i, $i = 1, 2, \ldots, p$, are strongly convex of order m at $\bar{x}$, and $\sum_{t \in \hat{T}(\bar{x})} \beta_t g_t$ is strictly strong quasiconvex of order m at $\bar{x}$, then $\bar{x}$ is a semi-strict minimizer of order m for (P).

Proof The proof is similar to that of Theorems 4.1 and 4.2. □

5 Conclusions

We have defined a strict minimizer of higher-order and a semi-strict minimizer of higher-order for a semi-infinite multi-objective optimization problem in this paper. We have presented a non-smooth semi-infinite version of the generealized Guignard and Abadie constraint qualifications. Under those constraint qualifications, utilizing the method in [15, 16] we have proved necessary optimality conditions for a semi-strict minimizer of higher-order and a strict minimizer of higher-order. Three sufficient optimality conditions have been proved under the assumption of strong convexities.

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