Guo et al. *Advances in Difference Equations* 2013, 2013:294
http://www.advancesindifferenceequations.com/content/2013/1/294

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A certain class of completely monotonic sequences

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Abstract
In this article, we present some necessary conditions, a sufficient condition and a necessary and sufficient condition for sequences to be completely monotonic. One counterexample is also presented.
MSC: Primary 40A05; secondary 26A45; 26A48; 39A60

Keywords: necessary condition; sufficient condition; necessary and sufficient condition; difference equation; moment sequence; completely monotonic sequence; completely monotonic function; bounded variation; Stieltjes integral

1 Introduction and the main results

We first recall some definitions and basic results on or related to completely monotonic sequences and completely monotonic functions.

Definition 1 [1] A sequence $\{\mu_n\}_{n=0}^{\infty}$ is called a moment sequence if there exists a function $\alpha(t)$ of bounded variation on the interval $[0,1]$ such that

$$\mu_n = \int_0^1 t^n \, d\alpha(t), \quad n \in \mathbb{N}_0. \tag{1}$$

Here, in Definition 1 and throughout the paper,

$$\mathbb{N}_0 := \{0\} \cup \mathbb{N},$$

and $\mathbb{N}$ is the set of all positive integers.

Definition 2 [1] A sequence $\{\mu_n\}_{n=0}^{\infty}$ is called completely monotonic if

$$(-1)^k \Delta^k \mu_n \geqq 0, \quad n, k \in \mathbb{N}_0, \tag{2}$$

where

$$\Delta^0 \mu_n = \mu_n \tag{3}$$

and

$$\Delta^{k+1} \mu_n = \Delta^k \mu_{n+1} - \Delta^k \mu_n. \tag{4}$$

Such a sequence is called *totally monotone* in [2].

From Definition 2, using mathematical induction, we can prove, for a completely monotonic sequence $\{\mu_n\}_{n=0}^{\infty}$, that the sequence $\{(-1)^m \Delta^m \mu_n\}_{n=0}^{\infty}$ is non-increasing for any fixed $m \in \mathbb{N}_0$, and that the sequence $\{(-1)^m \Delta^m \mu_n\}_{m=0}^{\infty}$ is non-increasing for any fixed $n \in \mathbb{N}_0$. The difference equation (4) plays an important role in the proofs of these properties and our main results of this paper.

In [3], the authors showed that for a completely monotonic sequence $\{\mu_n\}_{n=0}^{\infty}$, we always have

$$(-1)^k \Delta^k \mu_n > 0, \quad n, k \in \mathbb{N}_0, \tag{5}$$

unless $\mu_n = c$, a constant for all $n \in \mathbb{N}$.

Let

$$\lambda_{k,m} := \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \mu_m, \quad k, m \in \mathbb{N}_0. \tag{6}$$

It was shown (see [1]) as follows.

Theorem 1 *A sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is a moment sequence if and only if there exists a constant* L *such that*

$$\sum_{m=0}^{k} |\lambda_{k,m}| < L, \quad k \in \mathbb{N}_0, \tag{7}$$

where in (7), $\lambda_{k,m}$ *is defined by* (6).

For completely monotonic sequences, the following is the well-known Hausdorff's theorem (see [1]).

Theorem 2 *A sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic if and only if there exists a non-decreasing and bounded function* $\alpha(t)$ *on* $[0,1]$ *such that*

$$\mu_n = \int_0^1 t^n \, d\alpha(t), \quad n \in \mathbb{N}_0. \tag{8}$$

From this theorem, we know (see [1]) that a completely monotonic sequence is a moment sequence and is as follows.

Theorem 3 *A necessary and sufficient condition that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *should be a moment sequence is that it should be the difference of two completely monotonic sequences.*

We also recall the following definition.

Definition 3 [1] A function f is said to be completely monotonic on an interval I if f is continuous on I has derivatives of all orders on I^o (the interior of I) and for all $n \in \mathbb{N}_0$,

$$(-1)^n f^{(n)}(x) \geqq 0, \quad x \in I^o. \tag{9}$$

Some mathematicians use the terminology *completely monotone* instead of completely monotonic. The class of all completely monotonic functions on the interval I is denoted by $\mathrm{CM}(I)$.

The completely monotonic functions and completely monotonic sequences have remarkable applications in probability and statistics [4–10], physics [11, 12], numerical and asymptotic analysis [2], *etc.*

For the completely monotonic functions on the interval $[0,\infty)$, Widder proved (see [1]).

Theorem 4 *A function f on the interval $[0,\infty)$ is completely monotonic if and only if there exists a bounded and non-decreasing function $\alpha(t)$ on $[0,\infty)$ such that*

$$f(x) = \int_0^\infty e^{-xt}\,d\alpha(t). \tag{10}$$

There is rich literature on completely monotonic functions. For more recent works, see, for example, [13–26].

There exists a close relationship between completely monotonic functions and completely monotonic sequences. For example, Widder [27] showed the following.

Theorem 5 *Suppose that $f \in \mathrm{CM}[a,\infty)$, then for any $\delta \geqq 0$, the sequence $\{f(a+n\delta)\}_{n=0}^\infty$ is completely monotonic.*

This result was generalized in [28] as follows.

Theorem 6 *Suppose that $f \in \mathrm{CM}[a,\infty)$. If the sequence $\{\Delta x_k\}_{k=0}^\infty$ is completely monotonic and $x_0 \geqq a$, then the sequence $\{f(x_k)\}_{k=0}^\infty$ is also completely monotonic.*

For the meaning of Δx_k, $k \in \mathbb{N}_0$ in Theorem 6, see (3) and (4).

Suppose that $f \in \mathrm{CM}[0,\infty)$. By Theorem 5, we know that $\{f(n)\}_{n=0}^\infty$ is completely monotonic.

The following result was obtained in [16].

Theorem 7 *Suppose that the sequence $\{\mu_n\}_{n=0}^\infty$ is completely monotonic, then for any $\varepsilon \in (0,1)$, there exists a continuous interpolating function $f(x)$ on the interval $[0,\infty)$ such that $f|_{[0,\varepsilon]}$ and $f|_{[\varepsilon,\infty)}$ are both completely monotonic and*

$$f(n) = \mu_n, \quad n \in \mathbb{N}_0.$$

From this result or Theorem 2, we can get the following.

Theorem 8 *Suppose that the sequence $\{\mu_n\}_{n=0}^\infty$ is completely monotonic. Then there exists a completely monotonic interpolating function $g(x)$ on the interval $[1,\infty)$ such that*

$$g(n) = \mu_n, \quad n \in \mathbb{N}.$$

It should be noted that (see [1, Chapter IV]) under the condition of Theorem 8, we cannot guarantee that there exists a completely monotonic interpolating function $g(x)$ on the interval $[0, \infty)$ such that

$$g(n) = \mu_n, \quad n \in \mathbb{N}_0.$$

In this article, we shall further investigate the properties of the completely monotonic sequences. We shall give some necessary conditions, a sufficient condition and a necessary and sufficient condition for sequences to be completely monotonic. More precisely we have the following results.

Theorem 9 *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic. Then, for any* $m \in \mathbb{N}_0$, *the series*

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_{m+1}$$

converges and

$$\mu_m \geqq \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_{m+1}.$$

Corollary 1 *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic. Then for* $m, k \in \mathbb{N}_0$,

$$\mu_m = (-1)^{k+1} \Delta^{k+1} \mu_m + \sum_{i=0}^{k} (-1)^i \Delta^i \mu_{m+1}. \tag{11}$$

Remark 1 Although from the complete monotonicity of the sequence $\{\mu_n\}_{n=0}^{\infty}$, we can deduce that for any $m \in \mathbb{N}_0$, the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_{m+1}$$

converges, it cannot guarantee the convergence of the series

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_0.$$

For example, let

$$\mu_n = \frac{1}{n+1}, \quad n \in \mathbb{N}_0.$$

Since the function

$$f(x) = \frac{1}{x+1}$$

is completely monotonic on the interval $[0, \infty)$, by Theorem 5, we see that the sequence

$$\{\mu_n\}_{n=0}^{\infty} := \{f(n)\}_{n=0}^{\infty} = \left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$$

is completely monotonic. This conclusion can also be obtained by setting

$$\alpha(t) = t$$

in Theorem 2.

We can verify that

$$\Delta^j \mu_0 = \frac{(-1)^j}{j+1}.$$

Hence,

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_0 = \sum_{j=0}^{\infty} \frac{1}{j+1}$$

is the famous harmonic series, which is divergent.

Theorem 10 *Suppose that the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic. Then for any* $k, m \in \mathbb{N}_0$,

$$(-1)^k \Delta^k \mu_m \geqq \sum_{j=k}^{\infty} (-1)^j \Delta^j \mu_{m+1}. \qquad (12)$$

Theorem 11 *Suppose that the sequence* $\{\mu_n\}_{n=1}^{\infty}$ *is completely monotonic and that the series*

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1$$

converges. Let μ_0 *be such that*

$$\mu_0 \geqq \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1.$$

Then the sequence $\{\mu_n\}_{n=0}^{\infty}$ *is completely monotonic.*

Theorem 12 *A necessary and sufficient condition for the sequence* $\{\mu_n\}_{n=0}^{\infty}$ *to be completely monotonic is that the sequence* $\{\mu_n\}_{n=1}^{\infty}$ *is completely monotonic, the series*

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1$$

converges and

$$\mu_0 \geqq \sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_1.$$

2 Proofs of the main results

Now, we are in a position to prove the main results.

Proof of Theorem 9 Since $\{\mu_n\}_{n=0}^{\infty}$ is completely monotonic, by Theorem 2, there exists a non-decreasing and bounded function $\alpha(t)$ on the interval $[0,1]$ such that

$$\mu_n = \int_0^1 t^n \, d\alpha(t), \quad n \in \mathbb{N}_0. \tag{13}$$

From (3), (4) and (13), we can prove that

$$(-1)^i \Delta^i \mu_n = \int_0^1 (1-t)^i t^n \, d\alpha(t), \quad i, n \in \mathbb{N}_0. \tag{14}$$

Now, for $k \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{i=0}^{k-1} (-1)^i \Delta^i \mu_{m+1} &= \sum_{i=0}^{k-1} \int_0^1 (1-t)^i t^{m+1} \, d\alpha(t) \\
&= \int_0^1 t^{m+1} \sum_{i=0}^{k-1} (1-t)^i \, d\alpha(t) \\
&= \int_0^1 t^m \bigl(1 - (1-t)^k\bigr) \, d\alpha(t) \\
&= \int_0^1 t^m \, d\alpha(t) - \int_0^1 (1-t)^k t^m \, d\alpha(t) \\
&= \mu_m - (-1)^k \Delta^k \mu_m, \quad m \in \mathbb{N}_0.
\end{aligned}$$

Hence, for $k \in \mathbb{N}$,

$$\mu_m = (-1)^k \Delta^k \mu_m + \sum_{i=0}^{k-1} (-1)^i \Delta^i \mu_{m+1}, \quad m \in \mathbb{N}_0. \tag{15}$$

Since

$$(-1)^i \Delta^i \mu_n \geqq 0, \quad i, n \in \mathbb{N}_0, \tag{16}$$

from (15), we get, for $k \geqq 1$,

$$\mu_m \geqq \sum_{i=0}^{k-1} (-1)^i \Delta^i \mu_{m+1}, \quad m \in \mathbb{N}_0. \tag{17}$$

From (16), we also know that

$$\sum_{j=0}^{\infty} (-1)^j \Delta^j \mu_{m+1}, \quad m \in \mathbb{N}_0$$

is a positive series. Then by (17), we obtain that

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_{m+1}, \quad m\in\mathbb{N}_0$$

converges and that

$$\mu_m \geqq \sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_{m+1}, \quad m\in\mathbb{N}_0. \tag{18}$$

The proof of Theorem 9 is thus completed. □

Proof of Corollary 1 This corollary can be obtained from (15). □

Proof of Theorem 10 Let m be a fixed non-negative integer.

From Theorem 9, we see that

$$\mu_m \geqq \sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_{m+1}, \tag{19}$$

which means that (12) is valid for $k = 0$.

Suppose that (12) is valid for $k = r$. Then

$$\begin{aligned}(-1)^{r+1}\Delta^{r+1}\mu_m &= (-1)^{r+1}\left(\Delta^r\mu_{m+1}-\Delta^r\mu_m\right)\\ &= (-1)^r\left(\Delta^r\mu_m-\Delta^r\mu_{m+1}\right)\\ &= (-1)^r\Delta^r\mu_m-(-1)^r\Delta^r\mu_{m+1}\\ &\geqq \sum_{j=r}^{\infty}(-1)^j\Delta^j\mu_{m+1}-(-1)^r\Delta^r\mu_{m+1}\\ &= \sum_{j=r+1}^{\infty}(-1)^j\Delta^j\mu_{m+1},\end{aligned} \tag{20}$$

which means that (12) is valid for $k = r + 1$. Therefore, by mathematical induction, (12) is valid for all $k \in \mathbb{N}_0$. The proof of Theorem 10 is completed. □

Proof of Theorem 11 By the definition of completely monotonic sequence, we only need to prove that

$$(-1)^k\Delta^k\mu_0 \geqq 0, \quad k\in\mathbb{N}_0. \tag{21}$$

We first prove that

$$(-1)^k\Delta^k\mu_0 \geqq \sum_{j=k}^{\infty}(-1)^j\Delta^j\mu_1, \quad k\in\mathbb{N}_0. \tag{22}$$

From the condition of Theorem 11, (22) is valid for $k = 0$.

Suppose that (22) is valid for $k = m$. Then we have

$$\begin{aligned}(-1)^{m+1}\Delta^{m+1}\mu_0 &= (-1)^{m+1}\left(\Delta^m\mu_1 - \Delta^m\mu_0\right)\\ &= (-1)^m\left(\Delta^m\mu_0 - \Delta^m\mu_1\right)\\ &= (-1)^m\Delta^m\mu_0 - (-1)^m\Delta^m\mu_1\\ &\geqq \sum_{j=m}^{\infty}(-1)^j\Delta^j\mu_1 - (-1)^m\Delta^m\mu_1\\ &= \sum_{j=m+1}^{\infty}(-1)^j\Delta^j\mu_1,\end{aligned} \tag{23}$$

which means that (22) is valid for $k = m+1$. Therefore, by mathematical induction, (22) is valid for all $k \in \mathbb{N}_0$.

Since

$$\sum_{j=0}^{\infty}(-1)^j\Delta^j\mu_1$$

is a convergent positive series, we know that

$$\sum_{j=k}^{\infty}(-1)^j\Delta^j\mu_1 \geqq 0, \quad k \in \mathbb{N}_0. \tag{24}$$

From (22) and (24), we obtain that

$$(-1)^k\Delta^k\mu_0 \geqq 0, \quad k \in \mathbb{N}_0.$$

The proof of Theorem 11 is completed. □

Proof of Theorem 12 By Definition 2 and by setting $m = 0$ in Theorem 9, we see that the condition is necessary. By Theorem 11, we know that the condition is sufficient. The proof of Theorem 12 is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to the writing of the present article. They also read and approved the final manuscript.

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Acknowledgements
The authors thank the editor and the referees, one of whom brought our attention to the reference [6], for their valuable suggestions to improve the quality of this paper. The present investigation was supported, in part, by the Natural Science Foundation of Henan Province of China under Grant 112300410022.

Received: 6 June 2013 Accepted: 3 September 2013 Published: 07 Nov 2013

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10.1186/1687-1847-2013-294
Cite this article as: Guo et al.: **A certain class of completely monotonic sequences.** *Advances in Difference Equations* 2013, 2013:294