Dong et al. *Journal of Inequalities and Applications* 2013, **2013**:269
http://www.journalofinequalitiesandapplications.com/content/2013/1/269

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Rate of convergence of Mann, Ishikawa and Noor iterations for continuous functions on an arbitrary interval

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Abstract
In this paper, we relax the control condition of convergence of SP-iteration presented by Phuengrattana and Suantai (J. Comput. Appl. Math. 235:3006-3014, 2011). We compare the rate of convergence of Mann, Ishikawa and Noor iterations from another point of view and come to a different conclusion. Finally, we provide a numerical example for Ishikawa and Noor iterations, which supports our theoretical results.
MSC: 47H05; 47H07; 47H10

Keywords: continuous functions; convergence theorem; fixed point; nondecreasing functions; rate of convergence

1 Introduction

Let E be a closed interval on the real line and let $f : E \to E$ be a continuous function. A point $p \in E$ is a fixed point of f if $f(p) = p$. We denote by $F(f)$ the set of fixed points of f. It is known that if E is also bounded, then $F(f)$ is nonempty.

Mann iteration (see [1]) is defined by $u_1 \in E$ and

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \tag{1.1}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$. Ishikawa iteration (see [2]) is defined by $s_1 \in E$ and

$$\begin{cases} t_n = (1 - \beta_n)s_n + \beta_n f(s_n), \\ s_{n+1} = (1 - \alpha_n)s_n + \alpha_n f(t_n) \end{cases} \tag{1.2}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$. Noor iteration (see [3]) is defined by $w_1 \in E$ and

$$\begin{cases} r_n = (1 - \gamma_n)w_n + \gamma_n f(w_n), \\ q_n = (1 - \beta_n)w_n + \beta_n f(r_n), \\ w_{n+1} = (1 - \alpha_n)w_n + \alpha_n f(q_n) \end{cases} \tag{1.3}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$. Clearly, Mann and Ishikawa iterations are special cases of Noor iteration, and Mann iteration is a special case of Ishikawa iteration.

In 1974, Rhoades [4] proved the convergence of Mann iteration for a class of continuous and nondecreasing functions on a closed unit interval, and then he [5] extended convergence results to Ishikawa iterations. Further, Borwein and Borwein [6] proved the convergence of Mann iteration of continuous functions on a bounded closed interval. Recently, Qing and Qihou [7] extended results in [6] to an arbitrary interval and to Ishikawa iteration and presented a necessary and sufficient condition for the convergence of Ishikawa iteration of continuous functions on an arbitrary interval.

Very recently, Phuengrattana and Suantai [8] introduced SP-iteration as follows:

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n f(x_n), \\ y_n = (1-\beta_n)z_n + \beta_n f(z_n), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n f(y_n) \end{cases} \tag{1.4}$$

for all $n \geq 1$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$, and it will be denoted by $\mathrm{SP}(x_1,\alpha_n,\beta_n,\gamma_n,f)$. They presented a necessary and sufficient condition for the convergence of SP-iteration (1.4) of continuous functions on an arbitrary interval. They also compared the convergence speed of Mann, Ishikawa, Noor iterations and SP-iteration and concluded that SP-iteration is better than the others.

Inspired by the above work, in this paper, we compare the rate of convergence of Mann, Ishikawa and Noor iterations under the same computation cost and come to a different conclusion with Phuengrattana and Suantai [8]. We also present a numerical example for Ishikawa and Noor iterations, which verifies our theoretical results.

2 Convergence theorem

In this section, we present a new necessary and sufficient condition for the convergence of SP-iteration (1.4), which relaxes the control condition presented by Phuengrattana and Suantai [8].

Phuengrattana and Suantai [8] proposed the following necessary and sufficient condition for the convergence of SP-iteration (1.4) of continuous functions on an arbitrary interval.

Proposition 2.1 *Let E be a closed interval on the real line and let $f: E \to E$ be a continuous function. For $x_1 \in E$, let SP-iteration $\{x_n\}_{n=1}^{\infty}$ be defined by* (1.4), *where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in* $[0,1]$ *satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$,
(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$ *and*
(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Next proposition reveals that it is of interest to relax the control conditions on$\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ in Proposition 2.1.

Proposition 2.2 [8] *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\alpha_n^*\}_{n=1}^{\infty}$, $\{\beta_n^*\}_{n=1}^{\infty}$, $\{\gamma_n^*\}_{n=1}^{\infty}$ be sequences in $[0,1)$ such that $\alpha_n < \alpha_n^*$, $\beta_n < \beta_n^*$ and $\gamma_n < \gamma_n^*$ for all $n \geq 1$. Let $\{x_n\}_{n=1}^{\infty}$ and $\{x_n^*\}_{n=1}^{\infty}$ be defined by $\mathrm{SP}(x_1, \alpha_n, \beta_n, \gamma_n, f)$ and $\mathrm{SP}(x_1^*, \alpha_n^*, \beta_n^*, \gamma_n^*, f)$, respectively. If $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then $\{x_n^*\}_{n=1}^{\infty}$ converges to p. Moreover, $\{x_n^*\}_{n=1}^{\infty}$ is better than $\{x_n\}_{n=1}^{\infty}$, provided that $x_1^* = x_1 \in E$.*

Consider the following three-step Mann iteration:

$$\begin{cases} v_{3n-1} = (1 - \lambda_{3n-2})v_{3n-2} + \lambda_{3n-2} f(v_{3n-2}), \\ v_{3n} = (1 - \lambda_{3n-1})v_{3n-1} + \lambda_{3n-1} f(v_{3n-1}), \\ v_{3n+1} = (1 - \lambda_{3n})v_{3n} + \lambda_{3n} f(v_{3n}), \end{cases} \tag{2.1}$$

where $\{\lambda_{3n-i}\}_{n=1}^{\infty}$, $i = 0, 1, 2$, are sequences in $[0,1]$.

Remark 2.1 Let $\lambda_{3n-2} = \gamma_n$, $\lambda_{3n-1} = \beta_n$, $\lambda_{3n} = \alpha_n$ and $v_1 = x_1$, then (2.1) transforms into (1.4) with $x_n = v_{3n-2}$, $z_n = v_{3n-1}$, $y_n = v_{3n}$, $x_{n+1} = v_{3n+1}$. So, one-step SP-iteration is exactly three-step Mann iteration.

Theorem 2.1 *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous function. For $x_1 \in E$, let SP-iteration $\{x_n\}_{n=1}^{\infty}$ be defined by* (1.4), *where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying the conditions*:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \gamma_n = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Proof Let $\lambda_{3n-2} = \gamma_n$, $\lambda_{3n-1} = \beta_n$, $\lambda_{3n} = \alpha_n$ and $v_1 = x_1$, then $x_n = v_{3n-2}$, $z_n = v_{3n-1}$, $y_n = v_{3n}$, $x_{n+1} = v_{3n+1}$. We divide the proof into three steps.

Step 1. By conditions (i)-(ii), it is obvious that $\{\lambda_n\}_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. From Proposition 2.1, it follows that $\{v_n\}_{n=1}^{\infty}$ is bounded if and only if $\{v_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Step 2. Since $\{x_n\}_{n=1}^{\infty}$ is a subsequence of $\{v_n\}_{n=1}^{\infty}$, so $\{x_n\}_{n=1}^{\infty}$ is bounded if $\{v_n\}_{n=1}^{\infty}$ is bounded. On the other hand, assume that $\{x_n\}_{n=1}^{\infty}$ is bounded, then $\{x_n\}_{n=1}^{\infty}$ belongs to a bounded closed interval. By the continuity of f, we have that $\{f(x_n)\}_{n=1}^{\infty}$ belongs to another bounded closed interval, and thus $\{f(x_n)\}_{n=1}^{\infty}$ is bounded. Since $z_n = (1 - \gamma_n)x_n + \gamma_n f(x_n)$, so $\{z_n\}_{n=1}^{\infty}$ is bounded, and thus $\{f(z_n)\}_{n=1}^{\infty}$ is bounded. Similarly, since $y_n = (1 - \beta_n)z_n + \gamma_n f(z_n)$, we have $\{y_n\}_{n=1}^{\infty}$ and $\{f(y_n)\}_{n=1}^{\infty}$ are bounded. Since $v_{3n-1} = z_n$, $v_{3n} = y_n$, $v_{3n+1} = x_{n+1}$ for all $n \geq 1$, we obtain that $\{v_n\}_{n=1}^{\infty}$ is bounded. So, $\{x_n\}_{n=1}^{\infty}$ is bounded if and only if $\{v_n\}_{n=1}^{\infty}$ is bounded.

Step 3. Since $\{x_n\}_{n=1}^{\infty}$ is a subsequence of $\{v_n\}_{n=1}^{\infty}$, so $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f if $\{v_n\}_{n=1}^{\infty}$ converges to a fixed point of f. On the other hand, assume that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point p of f, then $\{x_n\}_{n=1}^{\infty}$ is bounded. From Step 2, it follows that $\{f(x_n)\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ and $\{f(z_n)\}_{n=1}^{\infty}$ are bounded. Using (1.4), we have $z_n - x_n = \gamma_n(f(x_n) - x_n)$ and $y_n - z_n = \beta_n(f(z_n) - z_n)$. By the condition (ii), we get $|z_n - x_n| \to 0$ and $|y_n - z_n| \to 0$, so $\{z_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge to p. Since $v_{3n-1} = z_n$, $v_{3n} = y_n$, $v_{3n+1} = x_{n+1}$ for all $n \geq 1$, $\{v_n\}_{n=1}^{\infty}$ converges to p. Therefore, $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f if and only if $\{v_n\}_{n=1}^{\infty}$ converges to a fixed point of f. □

From Step 1, Step 2 and Step 3, the result follows.

Remark 2.2 The comparison of Proposition 2.1 and Theorem 2.1 implies that the conditions on parameters $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are relaxed.

3 Rate of convergence

In this section, we compare the rate of convergence of Ishikawa and Noor iterations under the same computation cost.

In order to compare the rate of convergence, we use the following definition introduced by Rhoades [5].

Definition 3.1 Let E be a closed interval on the real line and let $f : E \to E$ be a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}_{n=1}^{\infty}$ is said to be better than $\{y_n\}_{n=1}^{\infty}$ if

$$|x_n - p| \le |y_n - p| \quad \text{for all } n \ge 1.$$

Phuengrattana and Suantai [8] obtained the following theorem on the relation of the convergence of Mann, Ishikawa and Noor iterations and SP-iteration.

Proposition 3.1 *Let E be a closed interval on the real line and $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. For $u_1 = s_1 = w_1 = x_1 \in E$, let $\{u_n\}_{n=1}^{\infty}$, $\{s_n\}_{n=1}^{\infty}$, $\{w_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$ be the sequences defined by* (1.1)-(1.4), *respectively. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in* $[0,1)$. *Then the following are satisfied*:

- (i) *Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if Mann iteration $\{u_n\}_{n=1}^{\infty}$ converges to p. Moreover, Ishikawa iteration is better than Mann iteration*;
- (ii) *Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to p. Moreover, Noor iteration is better than Ishikawa iteration*;
- (iii) *SP-iteration $\{x_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to p. Moreover, SP-iteration is better than Noor iteration.*

Remark 3.1 In above Proposition 3.1, Phuengrattana and Suantai [8] compared the rate of convergence of Mann, Ishikawa, Noor iterations and SP-iteration and drew the conclusion that SP-iteration is better than other iterations, Noor iteration is better than Ishikawa iteration and Ishikawa iteration is better than Mann iteration. However, we know from Remark 2.1 that one-step SP-iteration is three-step Mann iteration. Clearly, the computation cost of one-step Ishikawa iteration and one-step Noor iteration equals to that of two-step Mann iteration and three-step Mann iteration, respectively. So, it seems to be more reasonable to compare the rate of convergence of Mann, Ishikawa and Noor iterations under the same computation cost. In this sense, from Proposition 3.1(iii), Mann iteration is better than Ishikawa and Noor iterations.

Next we compare the rate of convergence of Ishikawa and Noor iterations under the same computation cost. For purposes of comparison, we firstly define two iterations.

Three-step Ishikawa iteration (denoted by IshikawaIII iteration) is defined by $g_1 \in E$ and

$$\begin{cases} g_{n,1} = (1-\rho_{n,0})g_n + \rho_{n,0}f(g_n), \\ g_{n,2} = (1-\rho_{n,1})g_n + \rho_{n,1}f(g_{n,1}), \\ g_{n,3} = (1-\rho_{n,2})g_{n,2} + \rho_{n,2}f(g_{n,2}), \\ g_{n,4} = (1-\rho_{n,3})g_{n,2} + \rho_{n,3}f(g_{n,3}), \\ g_{n,5} = (1-\rho_{n,4})g_{n,4} + \rho_{n,4}f(g_{n,4}), \\ g_{n+1} = g_{n,6} = (1-\rho_{n,5})g_{n,4} + \rho_{n,5}f(g_{n,5}), \end{cases} \tag{3.1}$$

for all $n \geq 1$, where $\{\rho_{n,i}\}_{n=1}^{\infty}$, $i = 0,1,2,3,4,5$ are sequences in $[0,1]$. Two-step Noor iteration (denoted by NoorII iteration) is defined by $h_1 \in E$ and

$$\begin{cases} h_{n,1} = (1-\rho_{n,0})h_n + \rho_{n,0}f(h_n), \\ h_{n,2} = (1-\rho_{n,1})h_n + \rho_{n,1}f(h_{n,1}), \\ h_{n,3} = (1-\rho_{n,2})h_n + \rho_{n,2}f(h_{n,2}), \\ h_{n,4} = (1-\rho_{n,3})h_{n,3} + \rho_{n,3}f(h_{n,3}), \\ h_{n,5} = (1-\rho_{n,4})h_{n,3} + \rho_{n,4}f(h_{n,4}), \\ h_{n+1} = h_{n,6} = (1-\rho_{n,5})h_{n,3} + \rho_{n,5}f(h_{n,5}), \end{cases} \tag{3.2}$$

for all $n \geq 1$, where $\{\rho_{n,i}\}_{n=1}^{\infty}$, $i = 0,1,2,3,4,5$ are sequences in $[0,1]$. Since IshikawaIII and NoorII iterations are both six-step, their computation cost is same at every iteration.

Remark 3.2 It should be noted that IshikawaIII and NoorII iterations are not new iterations and we introduce them just for comparing the rate of convergence of Ishikawa and Noor iterations under the same computation cost.

Before proceeding with the main result, we present three lemmas (see Lemmas 3.2, 3.3 and 3.4 below). We first need to recall a lemma, which is used in the proof of Lemma 3.2.

Lemma 3.1 [8] *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing function. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$ be sequences in $[0,1)$. Let $\{s_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be defined by (1.2) and (1.3), respectively. Then the following hold:*

- (i) *if $f(s_1) < s_1$, then $f(s_n) < s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^{\infty}$ is nonincreasing;*
- (ii) *if $f(s_1) > s_1$, then $f(s_n) > s_n$ for all $n \geq 1$ and $\{s_n\}_{n=1}^{\infty}$ is nondecreasing;*
- (iii) *if $f(w_1) < w_1$, then $f(w_n) < w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^{\infty}$ is nonincreasing;*
- (iv) *if $f(w_1) > w_1$, then $f(w_n) > w_n$ for all $n \geq 1$ and $\{w_n\}_{n=1}^{\infty}$ is nondecreasing.*

Lemma 3.2 *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing function. Let $\{\rho_{n,i}\}_{n=1}^{\infty}$, $i = 0,1,2,3,4,5$, be sequences in $[0,1)$. Let $\{g_n\}_{n=1}^{\infty}$, $\{g_{n,2}\}_{n=1}^{\infty}$ and $\{g_{n,4}\}_{n=1}^{\infty}$ (resp. $\{h_n\}_{n=1}^{\infty}$ and $\{h_{n,3}\}_{n=1}^{\infty}$) be defined by (3.1) (resp. (3.2)). Then the following hold:*

- (i) *if $f(g_1) < g_1$, then $g_n \geq g_{n,2} \geq g_{n,4} \geq g_{n+1}$;*
- (ii) *if $f(g_1) > g_1$, then $g_n \leq g_{n,2} \leq g_{n,4} \leq g_{n+1}$;*
- (iii) *if $f(h_1) < h_1$, then $h_n \geq h_{n,3} \geq h_{n+1}$;*
- (iv) *if $f(h_1) > h_1$, then $h_n \leq h_{n,3} \leq h_{n+1}$.*

Proof Since IshikawaIII (resp. NoorII) iteration is three-step Ishikawa (resp. two-step Noor) iteration, $\{g_n\}_{n=1}^{\infty}$, $\{g_{n,2}\}_{n=1}^{\infty}$, $\{g_{n,4}\}_{n=1}^{\infty}$ (resp. $\{h_n\}_{n=1}^{\infty}$, $\{h_{n,3}\}_{n=1}^{\infty}$) are subsequences of $\{s_n\}_{n=1}^{\infty}$ (resp. $\{w_n\}_{n=1}^{\infty}$). From Lemma 3.1, Lemma 3.2 follows. □

Lemma 3.3 *Let E be a closed interval on the real line and let $f: E \to E$ be a continuous and nondecreasing function. Let $\{\rho_{n,i}\}_{n=1}^{\infty}$, $i = 0,1,2,3,4,5$, be sequences in* $[0,1)$. *Let $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ be defined by* (3.1) *and* (3.2), *respectively. Then the following hold*:

(i) *if $f(g_1) < g_1$, then $f(g_n) < g_n$ for all $n \geq 1$ and $\{g_n\}_{n=1}^{\infty}$ is nonincreasing*;
(ii) *if $f(g_1) > g_1$, then $f(g_n) > g_n$ for all $n \geq 1$ and $\{g_n\}_{n=1}^{\infty}$ is nondecreasing*;
(iii) *if $f(h_1) < h_1$, then $f(h_n) < h_n$ for all $n \geq 1$ and $\{h_n\}_{n=1}^{\infty}$ is nonincreasing*;
(iv) *if $f(h_1) > h_1$, then $f(h_n) > h_n$ for all $n \geq 1$ and $\{h_n\}_{n=1}^{\infty}$ is nondecreasing.*

Proof The sequence $\{g_n\}_{n=1}^{\infty}$ (resp. $\{h_n\}_{n=1}^{\infty}$) can be considered as a subsequence of $\{s_n\}_{n=1}^{\infty}$ (resp. $\{w_n\}_{n=1}^{\infty}$), since IshikawaIII (resp. NoorII) iteration is three-step Ishikawa (resp. two-step Noor) iteration. So, Lemma 3.3 follows from Lemma 3.1.

For comparing the rate of convergence of Ishikawa and Noor iterations, here we make the following assumption:

(H) $\rho_{n,2} \leq \rho_{n,1} \frac{h_n - f(h_{n,1})}{h_n - f(h_{n,2})}$ for all $n \geq 1$. □

Lemma 3.4 *Let E be a closed interval on the real line and let $f: E \to E$ be a continuous and nondecreasing function. Let $\{\rho_{n,i}\}_{n=1}^{\infty}$, $i = 0,1,2,\ldots,5$ be sequences in* $[0,1)$ *satisfying* (H). *For $g_1 = h_1 \in E$, let $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ be the sequences defined by* (3.1) *and* (3.2), *respectively. Then the following are satisfied*:

(i) *if $f(g_1) < g_1$, then $g_n \leq h_n$ for all $n \geq 1$*;
(ii) *if $f(g_1) > g_1$, then $g_n \geq h_n$ for all $n \geq 1$.*

Proof (i) We use mathematical induction. Firstly, it holds $g_1 = h_1$. Assume that $g_k \leq h_k$. Thus $f(g_k) \leq f(h_k)$. We obtain $g_{k,1} - h_{k,1} = (1 - \rho_{k,0})(g_k - h_k) + \rho_{k,0}(f(g_k) - f(h_k)) \leq 0$, so $g_{k,1} \leq h_{k,1}$, which implies $f(g_{k,1}) \leq f(h_{k,1})$. Similarly, we get

$$g_{k,2} \leq h_{k,2}, \tag{3.3}$$

which implies $f(g_{k,2}) \leq f(h_{k,2})$. From Lemma 3.2(i), it follows $g_{k,2} \leq g_k$ and thus $g_{k,2} \leq h_k$. So, we have $g_{k,3} - h_{k,3} = (1 - \rho_{k,2})(g_{k,2} - h_k) + \rho_{k,2}(f(g_{k,2}) - f(h_{k,2})) \leq 0$, *i.e.*,

$$g_{k,3} \leq h_{k,3}, \tag{3.4}$$

which implies $f(g_{k,3}) \leq f(h_{k,3})$. Using the condition (H) and (3.2), we obtain $h_{k,3} - h_{k,2} = \rho_{k,1}(h_k - f(h_{k,1})) - \rho_{k,2}(h_k - f(h_{k,2})) \geq 0$ and thus $h_{k,3} \geq h_{k,2}$. Combining with (3.3), we have

$$h_{k,3} \geq g_{k,2}. \tag{3.5}$$

So, $g_{k,4} - h_{k,4} = (1 - \rho_{k,3})(g_{k,2} - h_{k,3}) + \rho_{k,3}(f(g_{k,3}) - f(h_{k,3})) \leq 0$, *i.e.*,

$$g_{k,4} \leq h_{k,4}, \tag{3.6}$$

which implies $f(g_{k,4}) \leq f(h_{k,4})$. From Lemma 3.2(i), it follows $g_{k,4} \leq g_{k,2}$. Combining with (3.5), we get $g_{k,4} \leq h_{k,3}$. We have $g_{k,5} - h_{k,5} = (1 - \rho_{k,4})(g_{k,4} - h_{k,3}) + \rho_{k,4}(f(g_{k,4}) - f(h_{k,4})) \leq 0$, *i.e.*, $g_{k,5} \le h_{k,5}$, which implies $f(g_{k,5}) \le f(h_{k,5})$. Similarly, we obtain $g_{k+1} - h_{k+1} = g_{k,6} - h_{k,6} = (1 - \rho_{k,5})(g_{k,4} - h_{k,3}) + \rho_{k,5}(f(g_{k,5}) - f(h_{k,5})) \le 0$, *i.e.*,

$$g_{k+1} \le h_{k+1}. \tag{3.7}$$

By induction, we get $g_n \le h_n$ for all $n \ge 1$.

(ii) Following the line of (i), we can show that $g_n \le h_n$ for all $n \ge 1$. □

Theorem 3.1 *Let E be a closed interval on the real line and let $f : E \to E$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. For $g_1 = h_1 \in E$, let $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ be the sequences defined by* (3.1) *and* (3.2), *respectively. Let $\{\rho_{n,i}\}_{n=1}^{\infty}$, $i = 0, 1, \ldots, 5$, be sequences in* $[0, 1)$ *satisfying* (H). *Then IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if NoorII iteration $\{h_n\}_{n=1}^{\infty}$ converges to p. Moreover, IshikawaIII iteration is better than NoorII iteration.*

Proof Firstly, if IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$, then set $\rho_{n,i} = 0$, $i = 0, 1, 2, 3$, and we get the convergence of Ishikawa iteration. On the other hand, assume that Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$. Let $s_1 = g_1$, $\beta_{3n-2} = \rho_{n,0}$, $\alpha_{3n-2} = \rho_{n,1}$, $\beta_{3n-1} = \rho_{n,2}$, $\alpha_{3n-1} = \rho_{n,3}$, $\beta_{3n} = \rho_{n,4}$ and $\alpha_{3n} = \rho_{n,5}$ for all $n \ge 1$, then $\{g_n\}_{n=1}^{\infty}$ is a subsequence of $\{s_n\}_{n=1}^{\infty}$ and thus converges to p. So, IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to p. Similarly, we get NoorIII iteration $\{h_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to p. From Theorem 3.7(ii) in [8], we have that Ishikawa iteration $\{s_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if Noor iteration $\{w_n\}_{n=1}^{\infty}$ converges to p. Therefore, IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ converges to $p \in F(f)$ if and only if NoorII iteration $\{h_n\}_{n=1}^{\infty}$ converges to p.

Next we prove that IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ is better than NoorII iteration $\{h_n\}_{n=1}^{\infty}$. Put $L = \inf\{p \in E : p = f(p)\}$ and $U = \sup\{p \in E : p = f(p)\}$. We divide our proof into the following three cases:

Case 1: $g_1 = h_1 > U$,

Case 2: $g_1 = h_1 < L$,

Case 3: $L \le g_1 = h_1 \le U$.

Case 1: $g_1 = h_1 > U$. By Proposition 3.5 in [8], we get $f(g_1) < g_1$ and $f(h_1) < h_1$. Using Lemma 3.4(i), we obtain $g_n \le h_n$ for all $n \ge 1$. Following the line of the proof of Theorem 3.7 in [8], we have $U \le g_n$ for all $n \ge 1$. Then we get $0 \le g_n - p \le h_n - p$, and thus

$$|g_n - p| \le |h_n - p| \quad \text{for all } n \ge 1. \tag{3.8}$$

We can see that IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ is better than NoorII iteration $\{h_n\}_{n=1}^{\infty}$.

Case 2: $g_1 = h_1 < L$. By Proposition 3.6 in [8], we get $f(g_1) > g_1$ and $f(h_1) > h_1$. Using Lemma 3.4(ii), we obtain $g_n \ge h_n$ for all $n \ge 1$. Following the line of the proof of Theorem 3.7 in [8], we have $g_n \le L$ for all $n \ge 1$. Then we get $h_n - p \le g_n - p \le 0$, and thus

$$|g_n - p| \le |h_n - p| \quad \text{for all } n \ge 1. \tag{3.9}$$

We can see that IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ is better than NoorII iteration $\{h_n\}_{n=1}^{\infty}$.

Case 3: $L \le g_1 = h_1 \le U$. Suppose that $f(g_1) \ne g_1$. If $f(g_1) < g_1$, we have by Lemma 3.3(i) that $\{g_n\}_{n=1}^{\infty}$ is nonincreasing with limit p. By Lemma 3.4(i), we get $p \le g_n \le h_n$ for all $n \ge 1$.

Table 1 Comparison of rate of convergence of IshikawaIII and NoorII iterations

n	IshikawaIII			NoorII		
	g_n	$\lvert f(g_n) - g_n\rvert$	$\frac{\lvert g_n - p\rvert}{\lvert g_{n-1} - p\rvert}$	h_n	$\lvert f(h_n) - h_n\rvert$	$\frac{\lvert h_n - p\rvert}{\lvert h_{n-1} - p\rvert}$
10	1.01890784	1.4150E-02	6.9325E-01	1.04663866	3.4788E-02	7.0099E-01
⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	1.00336759	2.5247E-03	7.1604E-01	1.00856596	6.4180E-03	7.1901E-01
16	1.00242177	1.8158E-03	7.1914E-01	1.00618210	4.6332E-03	7.2170E-01
17	1.00174843	1.3111E-03	7.2196E-01	1.00447705	3.3560E-03	7.2420E-01
18	1.00126682	9.4997E-04	7.2454E-01	1.00325264	2.4386E-03	7.2651E-01
19	1.00092089	6.9059E-04	7.2693E-01	1.00237012	1.7771E-03	7.2868E-01

It follows that $|g_n - p| \leq |h_n - p|$ for all $n \geq 1$. Hence, we obtain that IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ is better than NoorII iteration $\{h_n\}_{n=1}^{\infty}$. If $f(g_1) > g_1$, we have by Lemma 3.3(ii) that $\{g_n\}_{n=1}^{\infty}$ is nondecreasing with limit p. By Lemma 3.4(ii), we have $p \geq g_n \geq h_n$ for all $n \geq 1$. It follows that $|g_n - p| \leq |h_n - p|$ for all $n \geq 1$. Hence, we have that IshikawaIII iteration $\{g_n\}_{n=1}^{\infty}$ is better than NoorII iteration $\{h_n\}_{n=1}^{\infty}$. □

Remark 3.3 From Theorem 3.1 and Proposition 3.1(iii), we come to a conclusion that, under the same computational cost, Mann iteration is better than Ishikawa and Noor iterations, Ishikawa iteration is better than Noor iteration if the condition (H) is satisfied.

Next, we present a numerical example. Set $\rho_{n,i} = \frac{1}{n^2+1}$, $i = 0, 1, 3, 4$, $\rho_{n,2} = \rho_{n,1}\frac{h_n - f(h_{n,1})}{h_n - f(h_{n,2})}$, $\rho_{n,5} = \frac{1}{n^{0.2}+1}$, for which the condition (H) is obviously satisfied.

Example 3.4 Let $f : [0, 8] \to [0, 8]$ be defined by $f(x) = \frac{x^2+\sqrt{x}+8}{10}$. Then f is a continuous and nondecreasing function. Take initial points $g_1 = h_1 = 4$. Table 1 illustrates the comparison of the convergence rate of IshikawaIII and NoorII iterations to the exact fixed point $p = 1$, and we observe that IshikawaII iteration is better than NoorII iteration, which verifies theoretical results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
QLD made theoretical derivation and completed the paper. SH provided useful suggestions for the Theorem 3.1. XL participated in the program to calculate the numerical example. All authors read and approved the final manuscript.

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Acknowledgements
The research was supported by National Natural Science Foundation of China (No. 11201476), supported in part by the Foundation of Tianjin Key Lab for Advanced Signal Processing and it was also supported by Fundamental Research Funds for the Central Universities (No. ZXH2012K001).
The authors would like to express their thanks to the referees, whose careful reading and constructive suggestions led to improvements in the presentation of the results.

Received: 15 October 2012 Accepted: 9 May 2013 Published: 30 May 2013

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doi:10.1186/1029-242X-2013-269
Cite this article as: Dong et al.: **Rate of convergence of Mann, Ishikawa and Noor iterations for continuous functions on an arbitrary interval.** *Journal of Inequalities and Applications* 2013 **2013**:269.