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$(\varphi, \alpha, \delta, \lambda, \Omega)_p$-Neighborhood for some classes of multivalent functions

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Abstract
In the present paper, we obtain some interesting results for neighborhoods of multivalent functions. Furthermore, we give an application of Miller and Mocanu's lemma.

MSC: 30C45

Keywords: neighborhood; multivalent function; Miller and Mocanu's lemma

1 Introduction and definitions

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $A(p,n)$ the class of functions f of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (n, p \in \mathbb{N} = \{1, 2, \ldots\})$$

which are analytic and multivalent in the open unit disk U.

The concept of neighborhood for $f \in A$ was first given by Goodman [1]. The concept of δ-neighborhoods $N_\delta(f)$ of analytic functions $f \in A$ was first introduced by Ruscheweyh [2]. Walker [3] defined a neighborhood of analytic functions having positive real part. Owa *et al.* [4] generalized of the results given by Walker. In 1996, Altıntaş and Owa [5] gave (n,δ)-neighborhoods for functions $f \in A$ with negative coefficients. In 2007, new definitions for neighborhoods of analytic functions $f \in A$ were considered by Orhan *et al.* [6]. The authors gave the following definition of neighborhoods:

For $f, g \in A$, f is said to be (α, δ)-neighborhood for g if it satisfies

$$\left|f'(z) - e^{i\alpha} g'(z)\right| < \delta \quad (z \in U)$$

for some $-\pi \le \alpha \le \pi$ and $\delta > \sqrt{2(1-\cos\alpha)}$. They denote this neighborhood by $(\alpha,\delta) - N(g)$.

Also, they saw that $f \in (\alpha,\delta) - M(g)$ if it satisfies

$$\left|\frac{f(z)}{z} - e^{i\alpha}\frac{g(z)}{z}\right| < \delta \quad (z \in U)$$

for some $-\pi \le \alpha \le \pi$ and $\delta > \sqrt{2(1-\cos\alpha)}$.

In 2009, Altuntaş *et al.* [7] gave the following definition for neighborhood of analytic functions $f \in A(p,n)$.

For $f, g \in A(p,n)$, f is said to be $(\alpha,\delta)_p$-neighborhood for g if it satisfies

$$\left|\frac{f'(z)}{z^{p-1}} - e^{i\alpha}\frac{g'(z)}{z^{p-1}}\right| < \delta \quad (z \in U)$$

for some $-\pi \le \alpha \le \pi$ and $\delta > p\sqrt{2(1-\cos\alpha)}$. They denote this neighborhood by $(\alpha,\delta)_p - N(g)$.

Also, they saw that $f \in (\alpha,\delta)_p - M(g)$ if it satisfies

$$\left|\frac{f(z)}{z^p} - e^{i\alpha}\frac{g(z)}{z^p}\right| < \delta \quad (z \in U)$$

for some $-\pi \le \alpha \le \pi$ and $\delta > \sqrt{2(1-\cos\alpha)}$.

Recently, Frasin [8] introduced the following definition of (α,β,δ)-neighborhood for analytic function f in the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \ge 0). \tag{1.1}$$

Let f be defined by (1.1). Then f is said to be (α,β,δ)-neighborhood for $g = z - \sum_{n=2}^{\infty} b_n z^n$ $(b_n \ge 0)$ if it satisfies

$$\left|e^{i\alpha}\left(D^k f(z)\right)' - e^{i\beta}\left(D^k g(z)\right)'\right| < \delta$$

for some $-\pi \le \alpha, \beta \le \pi$ and $\delta > \sqrt{2(1-\cos(\alpha-\beta))}$.

The differential operator D^k was introduced by Salagean [9].

Now, we give the following equalities for the functions $f \in A(p,n)$

$$\begin{aligned}
&D^0 f(z) = f(z),\\
&D^1 f(z) = z\left(D^0 f(z)\right)' = pz^p + \sum_{k=n}^{\infty}(p+k)a_{k+p}z^{k+p},\\
&\vdots\\
&D^\Omega f(z) = D\left(D^{\Omega-1} f(z)\right) = p^\Omega z^p + \sum_{k=n}^{\infty}(p+k)^\Omega a_{k+p}z^{k+p}.
\end{aligned}$$

We define $\wp : A(p,n) \to A(p,n)$ such that

$$\wp\bigl(f(z)\bigr) = \left(\frac{1}{p^{\Omega}} - \lambda\right) D^{\Omega} f(z) + \frac{\lambda}{p} z\bigl(D^{\Omega} f(z)\bigr)' \quad \left(0 \le \lambda \le \frac{1}{p^{\Omega}}, \Omega \in \mathbb{N} \cup \{0\}\right). \tag{1.2}$$

We denote by $\wp_{(\Omega,\lambda)}$ the class of analytic functions of the form (1.2) in U.

For $f, g \in \wp_{(\Omega,\lambda)}$, f is said to be $(\varphi, \alpha, \delta, \lambda, \Omega)_p$-neighborhood for g if it satisfies

$$\left| e^{i\varphi} \frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha} \frac{\wp'(g(z))}{z^{p-1}} \right| < \delta \quad (z \in U)$$

for some $-\pi \le \varphi - \alpha \le \pi$ and $\delta > p\sqrt{2(1 - \cos(\varphi - \alpha))}$. We denote this neighborhood by $(\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$.

Also, we say that $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{M}(g)$ if it satisfies

$$\left| e^{i\varphi} \frac{\wp(f(z))}{z^{p}} - e^{i\alpha} \frac{\wp(g(z))}{z^{p}} \right| < \delta \quad (z \in U)$$

for some $-\pi \le \varphi - \alpha \le \pi$ and $\delta > \sqrt{2(1 - \cos(\varphi - \alpha))}$.

We discuss some properties of f belonging to $(\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$ and $(\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{M}(g)$.

2 Main results

Theorem 2.1 *If $f \in \wp_{(\Omega,\lambda)}$ satisfies*

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) \bigl| e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p} \bigr| \le \delta - p\sqrt{2\bigl\{1 - \cos(\varphi - \alpha)\bigr\}} \tag{2.1}$$

for some $-\pi \le \varphi - \alpha \le \pi$ and $\delta > p\sqrt{2(1 - \cos(\varphi - \alpha))}$, then $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$.

Proof By virtue of (1.2), we can write

$$\begin{aligned}
&\left| e^{i\varphi} \frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha} \frac{\wp'(g(z))}{z^{p-1}} \right| \\
&\quad = \Biggl| p e^{i\varphi} + e^{i\varphi} \sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) a_{k+p} z^{k} - p e^{i\alpha} \\
&\qquad - \sum_{k=n}^{\infty} e^{i\alpha} \left(\frac{k+p}{p}\right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) b_{k+p} z^{k} \Biggr| \\
&\quad < p\sqrt{2\bigl\{1 - \cos(\varphi - \alpha)\bigr\}} + \sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) \bigl| e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p} \bigr|.
\end{aligned}$$

If

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) \bigl| a_{k+p} - e^{i\alpha} b_{k+p} \bigr| \le \delta - p\sqrt{2\bigl\{1 - \cos(\varphi - \alpha)\bigr\}},$$

then we see that

$$\left| e^{i\varphi} \frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha} \frac{\wp'(g(z))}{z^{p-1}} \right| < \delta \quad (z \in U).$$

Thus, $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$. □

Example 2.2 For given

$$g(z) = z^p + \sum_{k=n}^{\infty} B_{k+p}(\varphi, \alpha, \delta, \lambda, \Omega) z^{k+p} \in \wp_{(\Omega,\lambda)} \quad \bigl(n, p \in \mathbb{N} = \{1, 2, \ldots\}\bigr)$$

we consider

$$f(z) = z^p + \sum_{k=n}^{\infty} A_{k+p}(\varphi, \alpha, \delta, \lambda, \Omega) z^{k+p} \in \wp_{(\Omega,\lambda)} \quad \bigl(n, p \in \mathbb{N} = \{1, 2, \ldots\}\bigr)$$

with

$$A_{k+p} = \frac{p^{\Omega}(\delta - p\sqrt{2(1 - \cos(\varphi - \alpha))})}{(k+p)^{\Omega+2}(1 + \lambda k p^{\Omega-1})(k+p-1)}(n+p-1)e^{-i\varphi} + e^{i(\alpha-\varphi)} B_{k+p}.$$

Then we have that

$$\begin{aligned} &\sum_{k=n}^{\infty} \left(\frac{k+p}{p} \right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) \bigl| e^{i\varphi} A_{k+p} - e^{i\alpha} B_{k+p} \bigr| \\ &\quad = (n+p-1)\Bigl(\delta - p\sqrt{2\bigl(1 - \cos(\varphi - \alpha)\bigr)}\Bigr) \sum_{k=n}^{\infty} \frac{1}{(k+p)(k+p-1)}. \end{aligned} \tag{2.2}$$

Finally, in view of the telescopic sum, we can write

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{1}{(k+p)(k+p-1)} &= \lim_{q \to \infty} \sum_{k=n}^{q} \left\{ \frac{1}{(k+p-1)} - \frac{1}{(k+p)} \right\} \\ &= \lim_{q \to \infty} \left\{ \frac{1}{(n+p-1)} - \frac{1}{(p+q)} \right\} \\ &= \frac{1}{n+p-1}. \end{aligned} \tag{2.3}$$

Using (2.3) in (2.2), we have

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p} \right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) \bigl| e^{i\varphi} A_{k+p} - e^{i\alpha} B_{k+p} \bigr| = \Bigl(\delta - p\sqrt{2\bigl(1 - \cos(\varphi - \alpha)\bigr)}\Bigr).$$

Therefore, $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$.

Corollary 2.3 *If $f \in \wp_{(\Omega,\lambda)}$ satisfies*

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p} \right)^{\Omega} (k+p)\bigl(1 + \lambda k p^{\Omega-1}\bigr) \bigl| |a_{k+p}| - |b_{k+p}| \bigr| \le \delta - p\sqrt{2\bigl(1 - \cos(\varphi - \alpha)\bigr)}$$

for some $-\pi \leq \varphi - \alpha \leq \pi$, $\delta > p\sqrt{2\{1-\cos(\varphi-\alpha)\}}$, *and* $\arg(a_{k+p}) - \arg(b_{k+p}) = \alpha - \varphi$ $(n, p \in \mathbb{N} = \{1, 2, \ldots\})$, *then* $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$.

Proof By Theorem 2.1, we see the inequality (2.1) which implies that $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$.

Since $\arg(a_{k+p}) - \arg(b_{k+p}) = \alpha - \varphi$, if $\arg(a_{k+p}) = \varphi_{k+p}$, we see $\arg(b_{k+p}) = \varphi_{k+p} - \alpha + \varphi$. Therefore,

$$e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p} = e^{i\varphi}|a_{k+p}|e^{i\varphi_{k+p}} - e^{i\alpha}|b_{k+p}|e^{i(\varphi_{k+p}-\alpha+\varphi)} = \big(|a_{k+p}| - |b_{k+p}|\big)e^{i(\varphi_{k+p}+\varphi)}$$

implies that

$$\left|e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p}\right| = \big||a_{k+p}| - |b_{k+p}|\big|. \tag{2.4}$$

Using (2.4) in (2.1), the proof of the corollary is complete. □

Theorem 2.4 *If* $f \in \wp_{(\Omega,\lambda)}$ *satisfies*

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} \big(1 + \lambda k p^{\Omega-1}\big)\left|e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p}\right| \leq \delta - \sqrt{2\big(1 - \cos(\alpha - \varphi)\big)}$$

for some $-\pi \leq \varphi - \alpha \leq \pi$ *and* $\delta > \sqrt{2\{1-\cos(\varphi-\alpha)\}}$, *then* $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{M}(g)$.

The proof of this theorem is similar with Theorem 2.1.

Corollary 2.5 *If* $f \in \wp_{(\Omega,\lambda)}$ *satisfies*

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} \big(1 + \lambda k p^{\Omega-1}\big)\big||a_{k+p}| - |b_{k+p}|\big| \leq \delta - \sqrt{2\big(1 - \cos(\varphi - \alpha)\big)}$$

for some $-\pi \leq \varphi - \alpha \leq \pi$, $\delta > \sqrt{2\{1-\cos(\varphi-\alpha)\}}$, *and* $\arg(a_{k+p}) - \arg(b_{k+p}) = \alpha - \varphi$, *then* $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{M}(g)$.

Next, we derive the following theorem.

Theorem 2.6 *If* $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$, $0 \leq \varphi < \alpha \leq \pi$ *and* $\arg(e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p}) = k\varphi$, *then*

$$\sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} (k+p)\big(1 + \lambda k p^{\Omega-1}\big)\left|e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p}\right| \leq \delta - p\{\cos\varphi - \cos\alpha\}.$$

Proof For $f \in (\varphi, \alpha, \delta, \lambda, \Omega)_p - \mathcal{N}(g)$, we have

$$\begin{aligned} &\left|e^{i\varphi}\frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha}\frac{\wp'(g(z))}{z^{p-1}}\right| \\ &\quad = \left|p\big(e^{i\varphi} - e^{i\alpha}\big) + \sum_{k=n}^{\infty} \left(\frac{k+p}{p}\right)^{\Omega} (k+p)\big(1 + \lambda k p^{\Omega-1}\big)\big(e^{i\varphi} a_{k+p} - e^{i\alpha} b_{k+p}\big)z^k\right| \end{aligned}$$

$$= \left| p\left(e^{i\varphi} - e^{i\alpha}\right) + \sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)\left(1+\lambda kp^{\Omega-1}\right)\left|e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}\right|e^{ik\varphi}z^{k}\right|$$
$$< \delta.$$

Let us consider z such that $\arg z = -\varphi$. Then $z^k = |z|^k e^{-ik\varphi}$. For such a point $z \in U$, we see that

$$\left|e^{i\varphi}\frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha}\frac{\wp'(g(z))}{z^{p-1}}\right|$$
$$= \left| p\left(e^{i\varphi} - e^{i\alpha}\right) + \sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)\left(1+\lambda kp^{\Omega-1}\right)\left|e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}\right||z|^{k}\right|$$
$$= \left\{\left[\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)\left(1+\lambda kp^{\Omega-1}\right)\left|e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}\right||z|^{k} + p(\cos\varphi - \cos\alpha)\right]^2 + p^2(\sin\varphi - \sin\alpha)^2\right\}^{\frac{1}{2}}$$
$$< \delta.$$

This implies that

$$\left\{\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)\left(1+\lambda kp^{\Omega-1}\right)\left|e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}\right||z|^{k} + p(\cos\varphi - \cos\alpha)\right\}^2 < \delta^2$$

or

$$p(\cos\varphi - \cos\alpha) + \sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)\left(1+\lambda kp^{\Omega-1}\right)\left|e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}\right||z|^{k} < \delta$$

for $z \in U$. Letting $|z| \to 1^-$, we have that

$$\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}(k+p)\left(1+\lambda kp^{\Omega-1}\right)\left|e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}\right| \le \delta - p(\cos\varphi - \cos\alpha).$$

□

Theorem 2.7 *$f \in (\varphi,\alpha,\delta,\lambda,\Omega)_p - \mathcal{M}(g)$, $0 \le \varphi < \alpha \le \pi$ and $\arg(e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}) = k\varphi$, then*

$$\sum_{k=n}^{\infty}\left(\frac{k+p}{p}\right)^{\Omega}\left(1+\lambda kp^{\Omega-1}\right)\left|e^{i\varphi}a_{k+p} - e^{i\alpha}b_{k+p}\right| \le \delta + \cos\alpha - \cos\varphi.$$

The proof of this theorem is similar with Theorem 2.6.

Remark 2.8 Taking $\varphi = 0$, $\Omega = 0$, $\lambda = 0$ and $p = 1$ in Theorem 2.6, we obtain the following theorem due to Orhan *et al.* [6].

Theorem 2.9 *If $f \in (\alpha,\delta) - \mathcal{N}(g)$ and $\arg(a_n - e^{i\alpha}b_n) = (n-1)\varphi$ $(n = 2,3,4,\ldots)$, then*

$$\sum_{n=2}^{\infty} n\left|a_n - e^{i\alpha}b_n\right| \le \delta + \cos\alpha - 1.$$

Remark 2.10 Taking $\varphi = 0$, $\Omega = 0$ and $\lambda = 0$ in Theorem 2.6, we obtain the following theorem due to Altuntaş *et al.* [7].

Theorem 2.11 *If* $f \in (\alpha, \delta)_p - \mathcal{N}(g)$ *and* $\arg(a_{k+p} - e^{i\alpha} b_{k+p}) = k\varphi$, *then*

$$\sum_{k=n}^{\infty}(k+p)\left|a_{k+p} - e^{i\alpha} b_{k+p}\right| \leq \delta - p(1-\cos\alpha).$$

We give an application of following lemma due to Miller and Mocanu [10].

Lemma 2.12 *Let the function*

$$w(z) = b_n z^n + b_{n+1} z^{n+1} + b_{n+2} z^{n+2} + \cdots \quad (n \in \mathbb{N})$$

be regular in the unit disk U *with* $w(z) \not\equiv 0$ $(z \in U)$. *If* $z_0 = r_0 e^{i\theta_0}$ $(r_0 < 1)$ *and* $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$, *then* $z_0 w'(z_0) = m w(z_0)$ *where* m *is real and* $m \geq n \geq 1$.

Theorem 2.13 *If* $f \in \wp_{(\Omega,\lambda)}$ *satisfies*

$$\left|e^{i\varphi}\frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha}\frac{\wp'(g(z))}{z^{p-1}}\right| < \delta(p+n) - p\sqrt{2\bigl(1-\cos(\varphi-\alpha)\bigr)}$$

for some $-\pi \leq \varphi - \alpha \leq \pi$ *and* $\delta > (\frac{p}{p+n})\sqrt{2(1-\cos(\varphi-\alpha))}$, *then*

$$\left|e^{i\varphi}\frac{\wp(f(z))}{z^{p}} - e^{i\alpha}\frac{\wp(g(z))}{z^{p}}\right| < \delta + \sqrt{2\bigl(1-\cos(\varphi-\alpha)\bigr)} \quad (z \in U).$$

Proof Let us define $w(z)$ by

$$e^{i\varphi}\frac{\wp(f(z))}{z^{p}} - e^{i\alpha}\frac{\wp(g(z))}{z^{p}} = e^{i\varphi} - e^{i\alpha} + \delta w(z). \tag{2.5}$$

Then $w(z)$ is analytic in U and $w(0) = 0$. By logarithmic differentiation, we obtain from (2.5) that

$$\frac{e^{i\varphi}\wp'(f(z)) - e^{i\alpha}\wp'(g(z))}{e^{i\varphi}\wp(f(z)) - e^{i\alpha}\wp(g(z))} - \frac{p}{z} = \frac{\delta w'(z)}{e^{i\varphi} - e^{i\alpha} + \delta w(z)}.$$

Since

$$\frac{e^{i\varphi}\wp'(f(z)) - e^{i\alpha}\wp'(g(z))}{z^p(e^{i\varphi} - e^{i\alpha} + \delta w(z))} = \frac{p}{z} + \frac{\delta w'(z)}{e^{i\varphi} - e^{i\alpha} + \delta w(z)},$$

we see that

$$e^{i\varphi}\frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha}\frac{\wp'(g(z))}{z^{p-1}} = p\bigl(e^{i\varphi} - e^{i\alpha}\bigr) + \delta w(z)\left(p + \frac{zw'(z)}{w(z)}\right).$$

This implies that

$$\left|e^{i\varphi}\frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha}\frac{\wp'(g(z))}{z^{p-1}}\right| = \left|p\bigl(e^{i\varphi} - e^{i\alpha}\bigr) + \delta w(z)\left(p + \frac{zw'(z)}{w(z)}\right)\right|.$$

We claim that

$$\left| e^{i\varphi} \frac{\wp'(f(z))}{z^{p-1}} - e^{i\alpha} \frac{\wp'(g(z))}{z^{p-1}} \right| < \delta(p+n) - p\sqrt{2\bigl(1-\cos(\varphi-\alpha)\bigr)}$$

in U.

Otherwise, there exists a point $z_0 \in U$ such that $z_0 w'(z_0) = mw(z_0)$ (by Miller and Mocanu's lemma) where $w(z_0) = e^{i\theta}$ and $m \geq n \geq 1$.

Therefore, we obtain that

$$\begin{aligned} \left| e^{i\varphi} \frac{\wp'(f(z_0))}{z_0^{p-1}} - e^{i\alpha} \frac{\wp'(g(z_0))}{z_0^{p-1}} \right| &= \left| p\bigl(e^{i\varphi} - e^{i\alpha}\bigr) + \delta e^{i\theta}(p+m) \right| \\ &\geq \delta(p+m) - \left| p\bigl(e^{i\varphi} - e^{i\alpha}\bigr) \right| \\ &\geq \delta(p+n) - p\sqrt{2\bigl(1-\cos(\varphi-\alpha)\bigr)}. \end{aligned}$$

This contradicts our condition in Theorem 2.13. □

Hence, there is no $z_0 \in U$ such that $|w(z_0)| = 1$. This implies that $|w(z)| < 1$ for all $z \in U$. Thus, we have that

$$\begin{aligned} \left| e^{i\varphi} \frac{\wp(f(z))}{z^{p}} - e^{i\alpha} \frac{\wp(g(z))}{z^{p}} \right| &= \left| \bigl(e^{i\varphi} - e^{i\alpha}\bigr) + \delta w(z) \right| \\ &\leq \left| e^{i\varphi} - e^{i\alpha} \right| + \delta \left| w(z) \right| \\ &< \delta + \sqrt{2\bigl(1-\cos(\varphi-\alpha)\bigr)}. \end{aligned}$$

Letting $\varphi = 0$, $\Omega = 0$, $\lambda = 0$ and $\alpha = \frac{\pi}{2}$ in Theorem 2.13, we can obtain the following corollary.

Corollary 2.14 *If $f \in A(p,n)$ satisfies*

$$\left| \frac{f'(z)}{z^{p-1}} - i\frac{g'(z)}{z^{p-1}} \right| < \delta(p+n) - p\sqrt{2} \quad (z \in U)$$

for some $\delta > \sqrt{2}(\frac{p}{p+n})$, then

$$\left| \frac{f(z)}{z^{p}} - i\frac{g(z)}{z^{p}} \right| < \delta + \sqrt{2} \quad (z \in U).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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