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Exponential convexity of Petrović and related functional

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Abstract

We consider functionals due to the difference in Petrović and related inequalities and prove the log-convexity and exponential convexity of these functionals by using different families of functions. We construct positive semi-definite matrices generated by these functionals and give some related results. At the end, we give some examples.

Keywords: convex functions, divided difference, exponentially convex, functionals, log-convex functions, positive semi-definite

1 Introduction

First time exponentially convex functions are introduced by Bernstein [1]. Independently of Bernstein, but some what later Widder [2] introduced these functions, as a sub-class of convex functions in a given interval (a, b), and denoted this class by $W_{a,b}$. After the initial development, there is a big gap in time before applications and examples of interest were constructed. One of the reasons is that, aside from absolutely monotone functions and completely monotone functions, as special classes of exponentially convex functions, there is no operative criteria to recognize exponential convexity of functions.

Definition 1. [[3], p. 373] A function $f : (a, b) \to \mathbb{R}$ is exponentially convex if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j f(x_i + x_j) \geq 0 \tag{1}$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$ and $x_i + x_j \in (a, b)$, $1 \leq i, j \leq n$.

Proposition 1.1. *Let $f : (a, b) \to \mathbb{R}$. The following propositions are equivalent.*

(i) *f is exponentially convex.*

(ii) *f is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0$$

for every $\xi_i \in \mathbb{R}$ and every $x_i \in (a, b)$, $1 \leq i \leq n$.

Proposition 1.2. *If f is exponentially convex, then the matrix*

$$\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}$$

is positive semi-definite. In particular,

$$\det\left[f\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{n}\geq 0$$

for every $n \in \mathbb{N}$, $x_i \in (a, b)$, $i = 1, ..., n$.

Proposition 1.3. *If* $f : (a, b) \to (0, \infty)$ *is an exponentially convex function, then f is log-convex which means that for every* $x, y \in (a, b)$ *and all* $\lambda \in (0, 1)$

$$f(\lambda x+(1-\lambda)y)\leq f(x)^{\lambda}f(y)^{1-\lambda}.$$

We consider functionals due to the differences in the Petrović and related inequalities. These inequalities are given in the following theorems [[4], pp. 152-159].

Theorem 1.4. *Let* $I = (0, a] \subseteq \mathbb{R}$ *be an interval,* $(x_1, ..., x_n) \in I^n$, *and* $(p_1, ..., p_n)$ *be a non-negative n-tuple such that*

$$\sum_{i=1}^{n}p_ix_i\in I \quad \text{and} \quad \sum_{i=1}^{n}p_ix_i\geq x_j \text{ for } j=1,\ldots,n. \tag{2}$$

If $f : I \to \mathbb{R}$ *be a function such that* $f(x)/x$ *is an increasing for* $x \in I$, *then*

$$f\left(\sum_{i=1}^{n}p_ix_i\right)\geq\sum_{i=1}^{n}p_if(x_i). \tag{3}$$

Remark 1.5. *Let us note that if* $f(x)/x$ *is a strictly increasing function for* $x \in I$, *then equality in* (3) *is valid if we have equalities in* (2) *instead of the inequalities, that is,* $x_1 = ... = x_n$ *and* $\sum_{i=1}^{n} p_i = 1$.

Theorem 1.6. *Let* $I = (0, a] \subseteq \mathbb{R}$ *be an interval,* $(x_1, ..., x_n) \in I^n$, *such that* $0 < x_1 \leq ... \leq x_n$, $(p_1, ..., p_n)$ *be a non-negative n-tuple and* $f : I \to \mathbb{R}$ *be a function such that* $f(x)/x$ *is an increasing for* $x \in I$.

(i) *If there exists an* $m(\leq n)$ *such that*

$$0\leq\bar{P}_1\leq\bar{P}_2\leq\cdots\leq\bar{P}_m\leq 1,\quad \bar{P}_{m+1}=\cdots=\bar{P}_n=0, \tag{4}$$

where $P_k=\sum_{i=1}^{k}p_i$, $\bar{P}_k=P_n-P_{k-1}$ $(k=2,\ldots,n)$ *and* $\bar{P}_1 = P_n$, *then* (3) *holds.*

(ii) *If there exists an* $m(\leq n)$ *such that*

$$\bar{P}_1\geq\bar{P}_2\geq\cdots\geq\bar{P}_m\geq 1,\quad \bar{P}_{m+1}=\cdots=\bar{P}_n=0, \tag{5}$$

then the reverse of inequality in (3) *holds.*

Theorem 1.7. *Let* $I = (0, a] \subseteq \mathbb{R}$ *be an interval,* $(x_1, ..., x_n) \in I^n$, *and* $x_1 - x_2 - ... - x_n \in I$. *Also let* $f : I \to \mathbb{R}$ *be a function such that* $f(x)/x$ *is an increasing for* $x \in I$. *Then*

$$f\left(x_1 - \sum_{i=2}^{n} x_i\right) \leq f(x_1) - \sum_{i=2}^{n} f(x_i). \tag{6}$$

Remark 1.8. *If $f(x)/x$ is a strictly increasing function for $x \in I$, then strict inequality holds in* (6).

Theorem 1.9. *Let $I = (0, a] \subseteq \mathbb{R}$ be an interval, $(x_1, ..., x_n) \in I^n$, $(p_1, ..., p_n)$ and $(q_1, ..., q_n)$ be non-negative n-tuples such that* (2) *holds. If $f : I \to \mathbb{R}$ be an increasing function, then*

$$\sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} q_i f(x_i). \tag{7}$$

Remark 1.10. *If f is a strictly increasing function on I and all x_i's are not equal, then we obtain strict inequality in* (7).

Theorem 1.11. *Let $I = [0, a] \subseteq \mathbb{R}$ be an interval, $(x_1, ..., x_n) \in I^n$, and $(p_1, ..., p_n)$ be a non-negative n-tuple such that* (2) *holds.*

If f is a convex function on I, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \sum_{i=1}^{n} p_i f(x_i) + \left(1 - \sum_{i=1}^{n} p_i\right) f(0). \tag{8}$$

Remark 1.12. *In the above theorem, if f is a strictly convex, then inequality in* (8) *is strict, if all x_i's are not equal or $\sum_{i=1}^{n} p_i \neq 1$.*

Theorem 1.13. *Let $I \subseteq \mathbb{R}$ be an interval, $0 \in I$, f be a convex function on I, $h : [a.b] \to I$ be continuous and monotonic with $h(t_0) = 0$, $t_0 \in [a, b]$ be fixed, g be a function of bounded variation and*

$$G(t) := \int_a^t dg(x), \quad \overline{G}(t) := \int_t^b dg(x).$$

(*a*) *If* $\int_a^b h(t)dg(t) \in I$*and*

$$0 \leq G(t) \leq 1 \quad for \quad a \leq t \leq t_0, \quad 0 \leq \overline{G}(t) \leq 1 \quad for \quad t_0 < t \leq b, \tag{9}$$

then we have

$$\int_a^b f(h(t))dg(t) \geq f\left(\int_a^b h(t)dg(t)\right) + \left(\int_a^b dg(t) - 1\right) f(0). \tag{10}$$

(*b*) *If* $\int_a^b h(t)dg(t) \in I$*and either* *there exists an* $s \le t_0$ *such that* $G(t) \le 0$ *for* $t < s$,

$$G(t) \geq 1 \quad for \quad s \leq t \leq t_0 \quad and \quad \overline{G}(t) \leq 0 \quad for \quad t > t_0 \tag{11}$$

or
there exists an $s \ge t_0$ *such that* $G(t) \le 0$ *for* $t < t_0$,

$$\overline{G}(t) \geq 1 \text{ for } t_0 < t < s, \text{ and } \overline{G}(t) \leq 0 \text{ for } t \geq s, \tag{12}$$

then the reverse of the inequality in (10) *holds.*

In this paper, we consider certain families of functions to prove log-convexity and exponential convexity of functionals due to the differences in inequalities given in Theorems 1.4-1.13. We construct positive semi-definite matrices generated by these functionals. Also by using log-convexity of these functionals, we prove monotonicity of the expressions introduced by these functionals. At the end, we give some examples.

2 Main results

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then for distinct points $u_i \in I$, $i = 0, 1, 2$, the divided differences in first and second order are defined as follows:

$$\begin{aligned} \left[u_i, u_{i+1}, f\right] &= \frac{f(u_{i+1}) - f(u_i)}{u_{i+1} - u_i} \quad (i = 0, 1), \\ \left[u_0, u_1, u_2, f\right] &= \frac{\left[u_1, u_2, f\right] - \left[u_0, u_1, f\right]}{u_2 - u_0}. \end{aligned} \tag{13}$$

The values of the divided differences are independent of the order of the points u_0, u_1, u_2 and may be extended to include the cases when some or all points are equal, that is

$$[u_0, u_0, f] = \lim_{u_1 \to u_0} [u_0, u_1, f] = f'(u_0),$$

provided that f' exists.

Now passing through the limit $u_1 \to u_0$ and replacing u_2 by u in (13), we have [[4], p. 16]

$$[u_0, u_0, u, f] = \lim_{u_1 \to u_0} [u_0, u_1, u, f] = \frac{f(u) - f(u_0) - (u - u_0) f'(u_0)}{(u - u_0)^2}, \quad u \neq u_0,$$

provided that f' exists. Also passing to the limit $u_i \to u$ ($i = 0, 1, 2$) in (13), we have

$$[u, u, u, f] = \lim_{u_i \to u} [u_0, u_1, u_2, f] = \frac{f''(u)}{2},$$

provided that f'' exists.

One can note that if for all $u_0, u_1 \in I$, $[u_0, u_1, f] \ge 0$, then f is increasing on I and if for all $u_0, u_1, u_2 \in I$, $[u_0, u_1, u_2, f] \ge 0$, then f is convex on I.

(M_1) Under the assumptions of Theorem 1.4, with all x_i's not equal, we define a linear functional as

$$\mathcal{P}_1(f) = f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i).$$

(M_2) Under the assumptions of Theorem 1.6, with all x_i's not equal and (4) is valid, we define a linear functional as

$$\mathcal{P}_2(f) = \mathcal{P}_1(f).$$

(M_3) Under the assumptions of Theorem 1.6, with all x_i's not equal and (5) is valid, we define a linear functional as

$$\mathcal{P}_3(f) = -\mathcal{P}_1(f).$$

(M_4) Under the assumptions of Theorem 1.7, with all x_i's not equal, we define a linear functional as

$$\mathcal{P}_4(f) = f(x_1) - \sum_{i=2}^{n} f(x_i) - f\left(x_1 - \sum_{i=2}^{n} x_i\right).$$

(M_5) Under the assumptions of Theorem 1.9, with all x_i's not equal, we define a linear functional as

$$\mathcal{P}_5(f) = \sum_{i=1}^{n} q_i f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} q_i f(x_i).$$

(M_6) Under the assumptions of Theorem 1.11, with all x_i's not equal, we define a linear functional as

$$\mathcal{P}_6(f) = f\left(\sum_{i=1}^{n} p_i x_i\right) - \sum_{i=1}^{n} p_i f(x_i) - \left(1 - \sum_{i=1}^{n} p_i\right) f(0).$$

(M_7) Under the assumptions of Theorem 1.13, such that (9) is valid, we define a linear functional as

$$\mathcal{P}_7(f) = \int_a^b f(h(t))\mathrm{d}g(t) - f\left(\int_a^b h(t)\mathrm{d}g(t)\right) - \left(\int_a^b \mathrm{d}g(t) - 1\right) f(0).$$

(M_8) Under the assumptions of Theorem 1.13, such that (11) or (12) is valid, we define a linear functional as

$$\mathcal{P}_8(f) = -\mathcal{P}_7(f).$$

Remark 2.1. *Under the assumptions of* (M_k) *for* $k = 1, 2, 3, 4$, *if* $f(u)/u$ *is an increasing function for* $u \in I$, *then*

$$\mathcal{P}_k(f) \geq 0, \text{ for } k = 1, 2, 3, 4.$$

If $f(u)/u$ *is strictly increasing for* $u \in I$ *and all* x_i*'s are not equal or* $\sum_1^n p_i \neq 1$*then strict inequality holds in the above expression.*

Remark 2.2. *Under the assumptions of* (M_5), *if* f *is an increasing function on I, then*

$$\mathcal{P}_5(f) \geq 0.$$

If f is strictly increasing function on I and all x_i's are not equal, then we obtain strict inequality in the above expression.

Remark 2.3. *Under the assumptions of* (M_k) *for* $k = 6, 7, 8$, *if f is a convex function on I, then*

$$\mathcal{P}_k(f) \geq 0 \text{ for } k = 6, 7, 8.$$

If f is strictly increasing function on I and all x_i's are not equal, then we obtain strict inequality in the above expression for $\mathcal{P}_6(f)$.

The following lemma is nothing more than the discriminant test for the non-negativity of second-order polynomials.

Lemma 2.4. *Let* $I \subseteq \mathbb{R}$ *be an interval. A function* $f : I \to (0, \infty)$ *is log-convex in J-sense on I, that is, for each* $r, t \in I$

$$f(r)f(t) \geq f^2\left(\frac{t+r}{2}\right)$$

if and only if, the relation

$$m^2 f(t) + 2mnf\left(\frac{t+r}{2}\right) + n^2 f(r) \geq 0 \tag{14}$$

holds for each $m, n \in \mathbb{R}$ *and* $r, t \in I$.

To define different families of functions, let $I \subseteq \mathbb{R}$ and $(c, d) \subseteq \mathbb{R}$ be intervals. For distinct points $u_0, u_1, u_2 \in I$ we suppose

$\mathbf{D}_1 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_1, F_t]$ is log-convex in J-sense, where $F_t(u) = f_t(u)/u\}$.

$\mathbf{D}_2 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, F_t]$ is log-convex in J-sense, where $F_t(u) = f_t(u)/u$ and F_t' exists$\}$.

$\mathbf{D}_3 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_1, f_t]$ is log-convex in J-sense$\}$.

$\mathbf{D}_4 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, f_t]$ is log-convex in J-sense, where f_t' exists$\}$.

$\mathbf{D}_5 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_1, u_2, f_t]$ is log-convex in J-sense$\}$.

$\mathbf{D}_6 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, u_2, f_t]$ is log-convex in J-sense, where f_t' exists$\}$.

$\mathbf{D}_7 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, u_0, f_t]$ is log-convex in J-sense, where f_t'' exists$\}$.

In this theorem, we prove log-convexity in J-sense, log-convexity and related results of the functionals associated with their respective families of functions.

Theorem 2.5. *Let* $\mathcal{P}_k$ *be the linear functionals defined in* (M_k), *associate the functionals with* $\mathbf{D}_i$ *in such a way that, for* $k = 1, 2, 3, 4$, $f_t \in \mathbf{D}_i$, $i = 1, 2$, *for* $k = 5$, $f_t \in \mathbf{D}_i$, $i = 3, 4$ *and for* $k = 6, 7, 8$, $f_t \in \mathbf{D}_i$, $i = 5, 6, 7$. *Also for* $k = 7, 8$, *assume that the linear functionals are positive. Then, the following statements are valid:*

(*a*) *The functions* $t \mapsto \mathcal{P}_k(f_t)$ *are log-convex in J-sense on* (c, d).

(*b*) *If the functions* $t \mapsto \mathcal{P}_k(f_t)$ *are continuous on* (c, d), *then the functions* $t \mapsto \mathcal{P}_k(f_t)$ *are log-convex on* (c, d).

(*c*) *If the functions* $t \mapsto \mathcal{P}_k(f_t)$ *are derivable on* (c, d), *then for* $t, r, u, v \in (c, d)$ *such that* $t \leq u$, $r \leq v$, *we have*

$$\mathfrak{B}_{k,i}(t, r; f_t) \leq \mathfrak{B}_{k,i}(u, v; f_t),$$

where

$$\mathfrak{B}_{k,i}(t, r; f_t) = \begin{cases} \left(\frac{\mathcal{P}_k(f_t)}{\mathcal{P}_k(f_r)}\right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\frac{\frac{d}{d_t}(\mathcal{P}_k(f_t))}{\mathcal{P}_k(f_t)}\right), & t = r. \end{cases} \tag{15}$$

Proof. (*a*) First, we prove log-convexity in J-sense of $t \mapsto \mathcal{P}_k(f_t)$ for $k = 1, 2, 3, 4$. For this, we consider the families of functions defined in $\mathbf{D}_1$ and $\mathbf{D}_2$.

Choose any $m, n \in \mathbb{R}$, and $t, r \in (c, d)$, we define the function

$$h(u) = m^2 f_t(u) + 2mn f_{\frac{t+r}{2}}(u) + n^2 f_r(u).$$

This gives

$$[u_0, u_1, H] = m^2 [u_0, u_1, F_t] + 2mn \left[u_0, u_1, F_{\frac{t+r}{2}}\right] + n^2 [u_0, u_1, F_r],$$

where $H(u) = h(u)/u$ and $F_t(u) = f_t(u)/u$.

Since $t \mapsto [u_0, u_1, F_t]$ is log-convex in J-sense, by Lemma 2.4 the right-hand side of above expression is non-negative. This implies $h(u)/u$ is an increasing function for $u \in I$.

Thus by Remark 2.1

$$\mathcal{P}_k(h) \geq 0 \text{ for } k = 1, 2, 3, 4,$$

this implies

$$m^2 \mathcal{P}_k(f_t) + 2mn \mathcal{P}_k(f_{\frac{t+r}{2}}) + n^2 \mathcal{P}_k(f_r) \geq 0. \tag{16}$$

Now $[u_0, u_1, F_t] > 0$ as it is log-convex, this implies $f_t(u)/u$ is strictly increasing for all $u \in I$ and $t \in (c, d)$. Also all x_i's are not equal and therefore by Remark 2.1, $\mathcal{P}_k(f_t)$ are positive valued, and hence, by Lemma 2.4, the inequality (16) implies log-convexity in J-sense of the functions $t \mapsto \mathcal{P}_k(f_t)$ *for* $k = 1, 2, 3, 4$.

Now we prove log-convexity in J-sense of $t \mapsto \mathcal{P}_5(f_t)$. For this, we consider the families of functions defined in $\mathbf{D}_3$ and $\mathbf{D}_4$. Following the same steps as above and having $H(u) = h(u)$, we have the log-convexity in J-sense of $\mathcal{P}_5(f_t)$ by using Remark 2.2 and Lemma 2.4.

At last, we prove log-convexity in J-sense of $t \mapsto \mathcal{P}_k(f_t)$ for $k = 6, 7, 8$. For this, we consider the families of Functions defined in $\mathbf{D}_i$ *for* $i = 5, 6, 7$.

Choose any $m, n \in \mathbb{R}$, and $t, r \in (c, d)$, we define the function

$$h(u) = m^2 f_t(u) + 2mn f_{\frac{t+r}{2}}(u) + n^2 f_r(u).$$

This gives

$$\left[u_0, u_1, u_2, h\right] = m^2 \left[u_0, u_1, u_2, f_t\right] + 2mn \left[u_0, u_1, u_2, f_{\frac{t+r}{2}}\right] + n^2 \left[u_0, u_1, u_2, f_r\right].$$

Since $t \mapsto [u_0, u_1, u_2, f_t]$ is log-convex in J-sense, by Lemma 2.4 the right-hand side of above expression is non-negative. This implies h is a strictly convex function on I.

Thus by Remark 2.3

$$\mathcal{P}_k(h) \geq 0 \text{ for } k = 6, 7, 8,$$

this implies

$$m^2\mathcal{P}_k(f_t) + 2mn\mathcal{P}_k(f_{\frac{t+r}{2}}) + n^2\mathcal{P}_k(f_r) \geq 0. \tag{17}$$

Since $\mathcal{P}_k(f_t)$ are positive valued, we have by Lemma 2.4 and inequality (17) the log-convexity in J-sense of the functions $t \mapsto \mathcal{P}_k(f_t)$ for $k = 6, 7, 8$.

(b) If $t \mapsto \mathcal{P}_k(f_t)$ are additionally continuous for $k = 1, ..., 8$ and $\mathbf{D}_i$'s associated with them, then these are log-convex, since J-convex continuous functions are convex functions.

(c) Since the functions $\log \mathcal{P}_k(f_t)$ are convex for $k = 1, ..., 8$, and $\mathbf{D}_i$'s associated with them, therefore for $t \leq u$, $r \leq v$, $t \neq r$, $u \neq v$, we have [[4], p.2],

$$\frac{\log \mathcal{P}_k(f_t) - \log \mathcal{P}_k(f_r)}{t - r} \leq \frac{\log \mathcal{P}_k(f_u) - \log \mathcal{P}_k(f_v)}{u - v},$$

concluding

$$\mathfrak{B}_{k,i}(t, r; f_t) \leq \mathfrak{B}_{k,i}(u, v; f_t).$$

Now if $t = r \leq u$, we apply $\lim_{r \to t}$, concluding,

$$\mathfrak{B}_{k,i}(t, t; f_t) \leq \mathfrak{B}_{k,i}(u, v; f_t).$$

Other possible cases are treated similarly.

In order to define different families of functions related to exponential convexity, let $I \subseteq \mathbb{R}$ and $(c, d) \subseteq \mathbb{R}$ be any intervals. For distinct points $u_0, u_1, u_2 \in I$ we suppose

$\mathbf{E}_1 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_1, F_t]$ is exponentially convex, where $F_t(u) = f_t(u)/u\}$.

$\mathbf{E}_2 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, F_t]$ is exponentially convex, where $F_t(u) = f_t(u)/u$ and F_t' exists$\}$.

$\mathbf{E}_3 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_1, f_t]$ is exponentially convex$\}$.

$\mathbf{E}_4 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, f_t]$ is exponentially convex, where f_t' exists$\}$.

$\mathbf{E}_5 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_1, u_2, f_t]$ is exponentially convex$\}$.

$\mathbf{E}_6 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, u_2, f_t]$ is exponentially convex, where f_t' exists$\}$.

$\mathbf{E}_7 = \{f_t : I \to \mathbb{R} \mid t \in (c, d)$ and $t \mapsto [u_0, u_0, u_0, f_t]$ is exponentially convex, where f_t'' exists$\}$.

In this theorem, we prove the exponential convexity of the functionals associated with their respective families of functions. Also we define positive semi-definite matrices for these functionals and give some related results.

Theorem 2.6. *Let $\mathcal{P}_k$ be the linear functionals defined in (M_k), associate the functionals with $\mathbf{E}_i$ in such a way that, for $k = 1, 2, 3, 4$, $f_t \in \mathbf{E}_i$, $i = 1, 2$, for $k = 5$, $f_t \in \mathbf{E}_i$, $i = 3, 4$ and for $k = 6, 7, 8$, $f_t \in \mathbf{E}_i$, $i = 5, 6, 7$. Then, the following statements are valid:*

(a) If $t \mapsto \mathcal{P}_k(f_t)$ *are continuous on (c, d), then the functions* $t \mapsto \mathcal{P}_k(f_t)$, *are exponentially convex on (c, d).*

(b) For every $q \in \mathbb{N}$ *and* $t_1, ..., t_q \in (c, d)$, *the matrices*

$$\left[\mathcal{P}_k(f_{\frac{t_l+t_m}{2}})\right]_{l,m=1}^{q}$$

are positive semi-definite. In particular

$$\det\left[\mathcal{P}_k(f_{\frac{t_l+t_m}{2}})\right]_{l,m=1}^{s} \geq 0 \text{ for } s = 1, 2, \ldots, q.$$

(c) If $t \mapsto \mathcal{P}_k(f_t)$ *are positive derivable on (c, d), then for* $t, r, u, v \in (c, d)$ *such that* $t \leq u$, $r \leq v$, we have

$$\mathfrak{C}_{k,i}(t, r; f_t) \leq \mathfrak{C}_{k,i}(u, v; f_t)$$

where $\mathfrak{C}_{k,i}(t, r; f_t)$ *is defined similarly as in* (15).

Proof. (*a*) First, we prove exponential convexity of $t \mapsto \mathcal{P}_k(f_t)$ for $k = 1, 2, 3, 4$. For this, we consider the families of functions defined in $\mathbf{E}_1$ and $\mathbf{E}_2$.

For any $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $t_i \in (c, d)$, $i = 1, ..., n$, we define

$$h(u) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i + t_j}{2}}(u).$$

This gives

$$[u_0, u_1, H] = \sum_{i,j=1}^{n} \xi_i \xi_j \left[u_0, u_1, F_{\frac{t_i+t_j}{2}}\right],$$

where $H(u) = h(u)/u$ and $F_t(u) = f_t(u)/u$.

Since $t \mapsto [u_0, u_1, F_t]$ is exponentially convex, right-hand side of the above expression is non-negative, which implies $h(u)/u$ is an increasing function on I.

Thus by Remark 2.1, we have

$$\mathcal{P}_k(h) \geq 0, \text{ for } k = 1, 2, 3, 4,$$

thus

$$\sum_{i,j=1}^{n} \xi_i \xi_j \mathcal{P}_k\left(f_{\frac{t_i+t_j}{2}}\right) \geq 0.$$

Hence $t \mapsto \mathcal{P}_k(f_t)$ is exponentially convex for $k = 1, 2, 3, 4$.

Now we prove exponential convexity of $t \mapsto \mathcal{P}_5(f_t)$. For this, we consider the families of functions defined in $\mathbf{E}_3$ and $\mathbf{E}_4$. Following the same steps as above and having $H(u) = h(u)$, we have the exponential convexity of the $\mathcal{P}_5(f_t)$ by using Remark 2.2.

At last, we prove exponential convexity of $t \mapsto \mathcal{P}_k(f_t)$ for $k = 6, 7, 8$. For this, we consider the families of functions defined in $\mathbf{E}_i$ *for* $i = 5, 6, 7$.

For any $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $t_i \in (c, d)$, $i = 1, ..., n$, we define

$$h(u) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{t_i+t_j}{2}}(u).$$

This gives

$$[u_0, u_1, u_2, h] = \sum_{i,j=1}^{n} \xi_i \xi_j \left[u_0, u_1, u_2, f_{\frac{t_i+t_j}{2}}\right].$$

Since $t \mapsto [u_0, u_1, u_2, f_t]$ is exponentially convex therefore right-hand side of the above expression is non-negative, which implies $h(u)$ is a strictly convex function on I.

Thus by Remark 2.3, we have

$$\mathcal{P}_k(h) \geq 0 \quad \text{for} \quad k = 6, 7, 8,$$

thus

$$\sum_{i,j=1}^{n} \xi_i \xi_j \mathcal{P}_k\left(f_{\frac{t_i+t_j}{2}}\right) \geq 0.$$

Hence $t \mapsto \mathcal{P}_k(f_t)$ are exponentially convex for $k = 6, 7, 8$.

(b) It follows by Proposition 1.2.

(c) Since $t \mapsto \mathcal{P}_k(f_t)$ are positive derivable for $k = 1, ..., 8$ with $\mathbf{E}_i$'s associated with them, we have our conclusion using part (c) of the Theorem 2.5.

3 Examples

In this section, we will vary on choices of families of functions in order to construct different examples of log and exponentially convex functions and related results.

Example 1. Let $t \in \mathbb{R}$ and $\phi_t : (0, \infty) \to \mathbb{R}$ be the function defined as

$$\varphi_t(u) = \begin{cases} \frac{u^t}{t-1}, & t \neq 1, \\ u \log u, & t = 1. \end{cases} \tag{18}$$

Then $\phi_t(u)/u$ is strictly increasing on $(0, \infty)$ for each $t \in \mathbb{R}$. One can note that $t \mapsto [u_0, u_0, \phi_t(u)/u]$ is log-convex for all $t \in \mathbb{R}$. If we choose $f_t = \phi_t$ in Theorem 2.5, we get log-convexity of the functionals $\mathcal{P}_k(\varphi_t)$ for $k = 1, 2, 3, 4$, which have been proved in [5,6].

Since $\phi_t(u)/u)' = u^{t-2} = e^{(t-2)\log u}$, the mapping $t \mapsto (\phi_t(u)/u)'$ is exponentially convex [7]. If we choose $f_t = \phi_t$ in Theorem 2.6, we get results that have been proved in [6,8]. Also we get $\mathfrak{C}_{1,2}(t, r; \varphi_t) = A^1_{t,r}(\mathbf{x}; \mathbf{p})$ for $t, r \neq 1$. By making substitution $x_i \mapsto x_i^s$, $t \mapsto t/s$, $r \mapsto r/s$ and $s \neq 0$, $t, r \neq s$, we get $\mathfrak{C}_{1,2}(t, r; \varphi_t) = A^s_{t,r}(\mathbf{x}; \mathbf{p})$ for $t, r \neq s$, where $A^s_{t,r}(\mathbf{x}; \mathbf{p})$ is defined in [5].

Similarly, $\mathfrak{C}_{4,2}(t, r; \varphi_t) = C^1_{t,r}(\mathbf{x})$ *for* $t, r \neq 1$, and by substitution used above $\mathfrak{C}_{4,2}(t, r; \varphi_t) = C^s_{t,r}(\mathbf{x})$ for $t, r \neq s$, where $C^s_{t,r}(\mathbf{x})$ is defined in [6].

Example 2. Let $t \in \mathbb{R}$ and $\beta_t : (0, \infty) \to \mathbb{R}$ be the function defined as

$$\beta_t(u) = \begin{cases} \frac{u^t}{t}, & t \neq 0, \\ \log u, & t = 0. \end{cases} \tag{19}$$

Then, β_t is strictly increasing on $(0, \infty)$ for each $t \in \mathbb{R}$. One can note that $t \mapsto [u_0, u_0, \beta_t]$ is log-convex for all $t \in \mathbb{R}$. If we choose $f_t = \beta_t$ in Theorem 2.5, we get log-convexity of the functional $\mathcal{P}_5(\beta_t)$, which have been proved in [9].

Since $\beta_t'(u) = u^{t-1} = e^{(t-1)\log u}$, the mapping $t \mapsto \beta_t'$ is exponentially convex [7]. If we choose $f_t = \beta_t$ in Theorem 2.6, we get results that have been proved in [9]. Also we get $\mathfrak{C}_{5,4}(t, r; \beta_t) = H_{t,r}(\mathbf{x}; \mathbf{p}; \mathbf{q})$ for $t, r \neq 0$, where $H_{t,r}(\mathbf{x}; \mathbf{p}; \mathbf{q})$ is defined in [9].

Example 3. Let $t \in (0, \infty)$ and $\delta_t : [0, \infty) \to \mathbb{R}$ be the function defined as

$$\delta_t(u) = \begin{cases} \frac{u^t}{t(t-1)}, & t \neq 1, \\ u \log u, & t = 1, \end{cases} \tag{20}$$

with a convention that 0 log 0 = 0. Then δ_t is convex on $[0, \infty)$ for each $t \in (0, \infty)$. One can note that $t \mapsto [u_0, u_0, u_0, \delta_t]$ is log-convex for all $t \in (0, \infty)$. If we choose $f_t = \delta_t$ in Theorem 2.5, we get log-convexity of the functionals $\mathcal{P}_k(\delta_t)$ for $k = 6, 7, 8$, which have been proved in [10].

Since $\delta_t''(u) = u^{t-2} = e^{(t-2)\log u}$, the mapping $t \mapsto \delta_t''$ is exponentially convex [7]. If we choose $f_t = \delta_t$ in Theorem 2.6, we get results that have been proved in [8,10]. Also we get $\mathfrak{C}_{6,7}(t, r; \delta_t) = B^1_{t,r}(\mathbf{x}; \mathbf{p})$ for $t, r \neq 1$. By making substitution $x_i \mapsto x_i^s$, $t \mapsto t/s$, $r \mapsto r/s$ and $s \neq 0$, $t, r \neq s$, we get $\mathfrak{C}_{6,7}(t, r; \delta_t) = B^s_{t,r}(\mathbf{x}; \mathbf{p})$ for $t, r \neq s$, where $B^s_{t,r}(\mathbf{x}; \mathbf{p})$ is defined in [10].

Similarly, $\mathfrak{C}_{7,7}(t, r; \delta_t) = F^1_{t,r}(a, b, h, g)$ for $t, r \neq 1$ and by substitution used above $\mathfrak{C}_{7,7}(t, r; \delta_t) = F^s_{t,r}(a, b, h, g)$ for $t, r \neq s$, where $F^s_{t,r}(a, b, h, g)$ is defined in [6].

Example 4. Let $t \in (0, \infty)$ and $\zeta_t : (0, \infty) \to \mathbb{R}$ be the function defined as

$$\zeta_t(u) = \begin{cases} \frac{ut^{-u}}{-\log t}, & t \neq 1, \\ u^2, & t = 1. \end{cases} \tag{21}$$

One can note that $t \mapsto [u_0, u_0, \zeta_t(u)/u]$ is log-convex for all $t \in (0, \infty)$. If we choose $f_t = \zeta_t$ in Theorem 2.5, we get log-convexity of the functionals $\mathcal{P}_k(\zeta_t)$ for $k = 1, 2, 3, 4$.

Since $\zeta_t(u)/u)' = t^{-u}$, the mapping $t \mapsto (\zeta_t(u)/u)'$ is exponentially convex [7]. If we choose $f_t = \zeta_t$ in Theorem 2.6, we get exponential convexity of the functionals $\mathcal{P}_k(\zeta_t)$ for $k = 1, 2, 3, 4$.

For $\mathcal{P}_1(f_t)$ using the function ζ_t in Theorem 2.6, $\mathfrak{C}_{1,2}(t, r; \zeta_t)$ in this particular case looks like

$$\mathfrak{C}_{1,2}(t, r; \zeta_t) = \begin{cases} \left(\dfrac{\log r \left(\tilde{x}_n t^{-\tilde{x}_n} - \sum_{i=1}^n p_i x_i t^{-x_i} \right)}{\log t \left(\tilde{x}_n r^{-\tilde{x}_n} - \sum_{i=1}^n p_i x_i r^{-x_i} \right)} \right)^{\frac{1}{t-r}}, & t \neq r, \quad t, r \neq 1, \\ \left(\dfrac{\tilde{x}_n t^{-\tilde{x}_n} - \sum_{i=1}^n p_i x_i t^{-x_i}}{-\log t \left(\tilde{x}_n^2 - \sum_{i=1}^n p_i x_i^2 \right)} \right)^{\frac{1}{t-1}}, & t \neq r = 1, \\ \exp\left(\dfrac{-1}{t \log t} - \dfrac{\tilde{x}_n^2 t^{-\tilde{x}_n} - \sum_{i=1}^n p_i x_i^2 t^{-x_i}}{t \left(\tilde{x}_n t^{-\tilde{x}_n} - \sum_{i=1}^n p_i x_i t^{-x_i} \right)} \right), & t = r, \quad t, r \neq 1, \\ \exp\left(\dfrac{\tilde{x}_n^3 - \sum_{i=1}^n p_i x_i^3}{-2 \left(\tilde{x}_n^2 - \sum_{i=1}^n p_i x_i^2 \right)} \right), & t = r = 1, \end{cases}$$

where $\tilde{x}_n = \sum_{i=1}^n p_i x_i$.

For $\mathcal{P}_4(f_t)$ using the function ζ_t in Theorem 2.6, $\mathfrak{C}_{4,2}$ $(t, r; \zeta_t)$ in this particular case looks like

$$\mathfrak{C}_{4,2}(t,r;\zeta_t) = \begin{cases} \left(\dfrac{\log r \left(x_1 t^{-x_1} - \sum_{i=2}^{n} x_i t^{-x_i} - \hat{x}_n t^{-\hat{x}_n} \right)}{\log t \left(x_1 r^{-x_1} - \sum_{i=2}^{n} x_i r^{-x_i} - \hat{x}_n r^{-\hat{x}_n} \right)} \right)^{\frac{1}{t-r}}, & t \neq r, \quad t, r \neq 1, \\ \left(\dfrac{x_1 t^{-x_1} - \sum_{i=2}^{n} x_i t^{-x_i} - \hat{x}_n t^{-\hat{x}_n}}{-\log t \left(x_1^2 - \sum_{i=2}^{n} x_i^2 - \hat{x}_n^2 \right)} \right)^{\frac{1}{t-1}}, & t \neq r = 1, \\ \exp\left(\dfrac{-1}{t \log t} - \dfrac{x_1^2 t^{-x_1} - \sum_{i=2}^{n} x_i^2 t^{-x_i} - \hat{x}_n^2 t^{-\hat{x}_n}}{t\left(x_1 t^{-x_1} - \sum_{i=2}^{n} x_i t^{-x_i} - \hat{x}_n t^{-\hat{x}_n} \right)} \right), & t = r, \quad t, r \neq 1, \\ \exp\left(\dfrac{x_1^3 - \sum_{i=2}^{n} x_i^3 - \hat{x}_n^3}{-2\left(x_1^2 - \sum_{i=2}^{n} x_i^2 - \hat{x}_n^2 \right)} \right), & t = r = 1, \end{cases}$$

where $\hat{x}_n = (x_1 - \sum_{i=2}^{n} x_i)$.

Example 5. Let $t \in (0, \infty)$ and $\theta_t : (0, \infty) \to \mathbb{R}$ be the function defined as

$$\theta_t(u) = \begin{cases} \frac{t^{-u}}{-\log t}, & t \neq 1, \\ u, & t = 1. \end{cases} \tag{22}$$

One can note that $t \mapsto [u_0, u_0, \theta_t]$ is log-convex for all $t \in (0, \infty)$, and if we choose $f_t = \theta_t$ in Theorem 2.5, we get log-convexity of the functional $\mathcal{P}_5(\theta_t)$.

Since $\theta_t'(u) = t^{-u}$, the mapping $t \mapsto \theta_t'(u)$ is exponentially convex function [7]. If we choose $f_t = \theta_t$ in Theorem 2.6, we get exponential convexity of the functional $\mathcal{P}_5(\theta_t)$.

For $\mathcal{P}_5(f_t)$ using the function θ_t in Theorem 2.6, $\mathfrak{C}_{5,4}$ $(t, r; \theta_t)$ in this particular case looks like

$$\mathfrak{C}_{5,4}(t,r;\theta_t) = \begin{cases} \left(\dfrac{\log r \left(\sum_{i=1}^{n} q_i t^{-\tilde{x}_n} - \sum_{i=1}^{n} q_i t^{-x_i} \right)}{\log t \left(\sum_{i=1}^{n} q_i r^{-\tilde{x}_n} - \sum_{i=1}^{n} q_i r^{-x_i} \right)} \right)^{\frac{1}{t-r}}, & t \neq r, \quad t, r \neq 1, \\ \left(\dfrac{\sum_{i=1}^{n} q_i t^{-\tilde{x}_n} - \sum_{i=1}^{n} q_i t^{-x_i}}{-\log t \left(\sum_{i=1}^{n} q_i \tilde{x}_n - \sum_{i=1}^{n} q_i x_i \right)} \right)^{\frac{1}{t-1}}, & t \neq r = 1, \\ \exp\left(\dfrac{-1}{t \log t} - \dfrac{\sum_{i=1}^{n} q_i \tilde{x}_n t^{-\tilde{x}_n} - \sum_{i=1}^{n} q_i x_i t^{-x_i}}{-t\left(\sum_{i=1}^{n} q_i t^{-\tilde{x}_n} - \sum_{i=1}^{n} q_i t^{-x_i} \right)} \right), & t = r, \quad t, r \neq 1, \\ \exp\left(\dfrac{\sum_{i=1}^{n} q_i \tilde{x}_n^2 - \sum_{i=1}^{n} q_i x_i^2}{-2\left(\sum_{i=1}^{n} q_i \tilde{x}_n - \sum_{i=1}^{n} q_i x_i \right)} \right), & t = r = 1, \end{cases}$$

Where $\tilde{x}_n = \sum_{i=1}^{n} p_i x_i$.

Example 6. Let $t \in (0, \infty)$ and $\lambda_t : (0, \infty) \to \mathbb{R}$ be the function defined as

$$\lambda_t(u) = \frac{u e^{-u\sqrt{t}}}{-\sqrt{t}}. \tag{23}$$

One can note that $t \mapsto [u_0, u_0, \lambda_t(u)/u]$ is log-convex for all $t \in (0, \infty)$. If we choose $f_t = \lambda_t$ in Theorem 2.5, we get log-convexity of the functionals $\mathcal{P}_k(\lambda_t)$ for $k = 1, 2, 3, 4$.

Since $\left(\lambda_t(u)/u\right)' = e^{-u\sqrt{t}}$, the mapping $t \mapsto (\lambda_t(u)/u)'$ is exponentially convex function [7]. If we choose $f_t = \lambda_t$ in Theorem 2.6, we get exponential convexity of the functionals $\mathcal{P}_k(\lambda_t)$ for $k = 1, 2, 3, 4$.

For $\mathcal{P}_1(f_t)$ using the function λ_t in Theorem 2.6, $\mathfrak{C}_{1,2}(t, r; \lambda_t)$ in this particular case looks like

$$\mathfrak{C}_{1,2}(t,r;\lambda_t) = \begin{cases} \left(\dfrac{\sqrt{r}\left(\tilde{x}_n e^{-\tilde{x}_n\sqrt{t}} - \sum_{i=1}^{n} p_i x_i e^{-x_i\sqrt{t}}\right)}{\sqrt{t}\left(\tilde{x}_n e^{-\tilde{x}_n\sqrt{r}} - \sum_{i=1}^{n} p_i x_i e^{-x_i\sqrt{r}}\right)} \right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\dfrac{-1}{2t} - \dfrac{\tilde{x}_n^2 e^{-\tilde{x}_n\sqrt{t}} - \sum_{i=1}^{n} p_i x_i^2 e^{-x_i\sqrt{t}}}{2\sqrt{t}\left(\tilde{x}_n e^{-\tilde{x}_n\sqrt{t}} - \sum_{i=1}^{n} p_i x_i e^{-x_i\sqrt{t}}\right)} \right), & t = r, \end{cases}$$

Where $\tilde{x}_n = \sum_{i=1}^{n} p_i x_i$.

Now for $\mathcal{P}_4(f_t)$ using the function λ_t in Theorem 2.6, $\mathfrak{C}_{4,2}(t, r; \lambda_t)$ in this particular case looks like

$$\mathfrak{C}_{4,2}(t,r;\lambda_t) = \begin{cases} \left(\dfrac{\sqrt{r}\left(x_1 e^{-x_1\sqrt{t}} - \sum_{i=2}^{n} x_i e^{-x_i\sqrt{t}} - \hat{x}_n e^{-\hat{x}_n\sqrt{t}}\right)}{\sqrt{t}\left(x_1 e^{-x_1\sqrt{t}} - \sum_{i=2}^{n} x_i e^{-x_i\sqrt{r}} - \hat{x}_n e^{-\hat{x}_n\sqrt{r}}\right)} \right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\dfrac{-1}{2t} - \dfrac{x_1^2 e^{-x_1\sqrt{t}} - \sum_{i=2}^{n} x_i^2 e^{-x_i\sqrt{t}} - \hat{x}_n^2 e^{-\hat{x}_n\sqrt{t}}}{2\sqrt{t}\left(x_1 e^{-x_1\sqrt{t}} - \sum_{i=2}^{n} x_i e^{-x_i\sqrt{t}} - \hat{x}_n e^{-\hat{x}_n\sqrt{t}}\right)} \right), & t = r, \end{cases}$$

Where $\hat{x}_n = (x_1 - \sum_{i=2}^{n} x_i)$.

Example 7. Let $t \in (0, \infty)$ *and* $\xi_t : (0, \infty) \to \mathbb{R}$, be the function defined as

$$\xi_t(u) = \frac{e^{-u\sqrt{t}}}{-\sqrt{t}}. \tag{24}$$

One can note that $t \mapsto [u_0, u_0, \xi_t]$ is log-convex for all $t \in (0, \infty)$. If we choose $f_t = \xi_t$ in Theorem 2.5, we get log-convexity of the functional $\mathcal{P}_5(\xi_t)$.

Since $\xi_t'(u) = e^{-u\sqrt{t}}$, the mapping $t \mapsto \xi_t'(u)$ is exponentially convex function [7]. If we choose $f_t = \xi_t$ in Theorem 2.6 we get exponential convexity of the functional $\mathcal{P}_5(\xi_t)$.

For $\mathcal{P}_5(f_t)$ using the function ξ_t in Theorem 2.6, $\mathfrak{C}_{5,4}(t, r; \xi_t)$ in this particular case looks like

$$\mathfrak{C}_{5,4}(t,r;\xi_t) = \begin{cases} \left(\dfrac{\sqrt{r}\left(\sum_{i=1}^{n} q_i e^{-\tilde{x}_n\sqrt{t}} - \sum_{i=1}^{n} q_i e^{-x_i\sqrt{t}}\right)}{\sqrt{t}\left(\sum_{i=1}^{n} q_i e^{-\tilde{x}_n\sqrt{r}} - \sum_{i=1}^{n} q_i e^{-x_i\sqrt{r}}\right)} \right)^{\frac{1}{t-r}}, & t \neq r, \\ \exp\left(\dfrac{-1}{2t} - \dfrac{\sum_{i=1}^{n} q_i \tilde{x}_n e^{-\tilde{x}_n\sqrt{t}} - \sum_{i=1}^{n} q_i x_i e^{-x_i\sqrt{t}}}{2\sqrt{t}\left(\sum_{i=1}^{n} q_i e^{-\tilde{x}_n\sqrt{t}} - \sum_{i=1}^{n} q_i e^{-x_i\sqrt{t}}\right)} \right), & t = r, \end{cases}$$

Where $\tilde{x}_n = \sum_{i=1}^{n} p_i x_i$.

Example 8. Let $t \in \mathbb{R}$ and $\psi_t : (0, \infty) \to \mathbb{R}$ be the function defined as

$$\psi_t(u) = \begin{cases} \frac{ue^{ut}}{t}, & t \neq 0, \\ u^2, & t = 0. \end{cases} \tag{25}$$

One can note that $t \mapsto [u_0, u_0, \psi_t(u)/u]$ is log-convex for all $t \in \mathbb{R}$. If we choose $f_t = \psi_t$ in Theorem2.5, we get log-convexity of the functionals $\mathcal{P}_k(\psi_t)$ for $k = 1, 2, 3, 4$.

Since $(\psi_t(u)/u)' = e^{ut}$, the mapping $t \mapsto (\psi_t(u)/u)'$ is exponentially convex function [7]. If we choose $f_t = \psi_t$ in Theorem 2.6 we get exponential convexity of the functionals $\mathcal{P}_k(\psi_t)$ for $k = 1, 2, 3, 4$.

For $\mathcal{P}_1(f_t)$ using the function ψ_t in Theorem 2.6, $\mathfrak{C}_{1,2}(t, r; \psi_t)$ in this particular case looks like

$$\mathfrak{C}_{1,2}(t, r; \psi_t) = \begin{cases} \left(\dfrac{r\left(\tilde{x}_n e^{\tilde{x}_n t} - \sum_{i=1}^{n} p_i x_i e^{x_i t}\right)}{t\left(\tilde{x}_n e^{\tilde{x}_n r} - \sum_{i=1}^{n} p_i x_i e^{x_i r}\right)} \right)^{\frac{1}{t-r}}, & t \neq r, \quad t, r \neq 0, \\ \left(\dfrac{\tilde{x}_n e^{\tilde{x}_n t} - \sum_{i=1}^{n} p_i x_i e^{x_i t}}{t\left(\tilde{x}_n^2 - \sum_{i=1}^{n} p_i x_i^2\right)} \right)^{\frac{1}{t-1}}, & t \neq r = 0, \\ \exp\left(\dfrac{-1}{t} + \dfrac{\tilde{x}_n^2 e^{\tilde{x}_n t} - \sum_{i=1}^{n} p_i x_i^2 e^{x_i t}}{\tilde{x}_n e^{\tilde{x}_n t} - \sum_{i=1}^{n} p_i x_i e^{x_i t}} \right), & t = r, \quad t, r \neq 0, \\ \exp\left(\dfrac{\tilde{x}_n^3 - \sum_{i=1}^{n} p_i x_i^3}{2\left(\tilde{x}_n^2 - \sum_{i=1}^{n} p_i x_i^2\right)} \right), & t = r = 0, \end{cases}$$

Where $\tilde{x}_n = \sum_{i=1}^{n} p_i x_i$.

Now for $\mathcal{P}_4(f_t)$ using the function ψ_t in Theorem 2.6, $\mathfrak{C}_{4,2}(t, r; \psi_t)$ in this particular case looks like

$$\mathfrak{C}_{4,2}(t, r; \psi_t) = \begin{cases} \left(\dfrac{r\left(x_1 e^{x_1 t} - \sum_{i=2}^{n} x_i e^{x_i t} - \hat{x}_n e^{\hat{x}_n}\right)}{t\left(x_1 e^{x_1 r} - \sum_{i=2}^{n} x_i e^{x_i r} - \hat{x}_n e^{\hat{x}_n r}\right)} \right)^{\frac{1}{t-r}}, & t \neq r, \quad t, r \neq 0, \\ \left(\dfrac{x_1 e^{x_1 t} - \sum_{i=2}^{n} x_i e^{x_i t} - \hat{x}_n e^{\hat{x}_n}}{t\left(x_1^2 - \sum_{i=2}^{n} x_i^2 - \hat{x}_n^2\right)} \right)^{\frac{1}{t-1}}, & t \neq r = 0, \\ \exp\left(\dfrac{-1}{t} + \dfrac{x_1^2 e^{x_1 t} - \sum_{i=2}^{n} x_i^2 e^{x_i t} - \hat{x}_n^2 e^{\hat{x}_n t}}{x_1 e^{x_1 t} - \sum_{i=2}^{n} x_i e^{x_i t} - \hat{x}_n e^{\hat{x}_n t}} \right), & t = r, \quad t, r \neq 0, \\ \exp\left(\dfrac{x_1^3 - \sum_{i=2}^{n} x_i^3 - \hat{x}_n^3}{2\left(x_1^2 - \sum_{i=2}^{n} x_i^2 - \hat{x}_n^2\right)} \right), & t = r = 0, \end{cases}$$

where $\hat{x}_n = (x_1 - \sum_{i=2}^{n} x_i)$.

Example 9. Let $t \in \mathbb{R}$ *and* $\omega_t : (0, \infty) \to \mathbb{R}$ be the function defined as

$$\omega_t(u) = \begin{cases} \frac{e^{ut}}{t}, & t \neq 0, \\ u, & t = 0. \end{cases} \tag{26}$$

One can note that $t \mapsto [u_0, u_0, \omega_t]$ is log-convex for all $t \in \mathbb{R}$. If we choose $f_t = \omega_t$ in Theorem 2.5, we get log-convexity of the functional $\mathcal{P}_5(\omega_t)$.

Since $\omega_t'(u) = e^{ut}$, the mapping $t \mapsto \omega_t'(u)$ is exponentially convex function [7]. If we choose $f_t = \omega_t$ in Theorem 2.6 we get exponential convexity of the functional $\mathcal{P}_5(\omega_t)$.

For $\mathcal{P}_5(f_t)$ using the function ω_t in Theorem 2.6, $\mathfrak{C}_{5,4}$ $(t, r; \omega_t)$ in this particular case looks like

$$\mathfrak{C}_{5,4}(t, r; \omega_t) = \begin{cases} \left(\dfrac{r\left(\sum_{i=1}^{n} q_i e^{\tilde{x}_n t} - \sum_{i=1}^{n} q_i e^{x_i t}\right)}{t\left(\sum_{i=1}^{n} q_i e^{\tilde{x}_n r} - \sum_{i=1}^{n} q_i e^{x_i r}\right)} \right)^{\frac{1}{t-r}}, & t \neq r, \quad t, r \neq 0, \\ \left(\dfrac{\sum_{i=1}^{n} q_i e^{\tilde{x}_n t} - \sum_{i=1}^{n} q_i e^{x_i t}}{t\left(\sum_{i=1}^{n} q_i \tilde{x}_n - \sum_{i=1}^{n} q_i x_i\right)} \right)^{\frac{1}{t-1}}, & t \neq r = 0, \\ \exp\left(\dfrac{-1}{t} + \dfrac{\sum_{i=1}^{n} q_i \tilde{x}_n e^{\tilde{x}_n t} - \sum_{i=1}^{n} q_i x_i e^{x_i t}}{\sum_{i=1}^{n} q_i e^{\tilde{x}_n t} - \sum_{i=1}^{n} q_i e^{x_i t}} \right), & t = r, \quad t, r \neq 0, \\ \exp \dfrac{\sum_{i=1}^{n} q_i \tilde{x}_n^2 - \sum_{i=1}^{n} q_i x_i^2}{2\left(\sum_{i=1}^{n} q_i \tilde{x}_n - \sum_{i=1}^{n} q_i x_i\right)}, & t = r = 0, \end{cases}$$

Where $\tilde{x}_n = \sum_{i=1}^{n} p_i x_i$.

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Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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The authors declare that they have no competing interests.

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