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Research Article

Near-Exact Distributions for Likelihood Ratio Statistics Used in the Simultaneous Test of Conditions on Mean Vectors and Patterns of Covariance Matrices

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The authors address likelihood ratio statistics used to test simultaneously conditions on mean vectors and patterns on covariance matrices. Tests for conditions on mean vectors, assuming or not a given structure for the covariance matrix, are quite common, since they may be easily implemented. But, on the other hand, the practical use of simultaneous tests for conditions on the mean vectors and a given pattern for the covariance matrix is usually hindered by the nonmanageability of the expressions for their exact distribution functions. The authors show the importance of being able to adequately factorize the c.f. of the logarithm of likelihood ratio statistics in order to obtain sharp and highly manageable near-exact distributions, or even the exact distribution in a highly manageable form. The tests considered are the simultaneous tests of equality or nullity of means and circularity, compound symmetry, or sphericity of the covariance matrix. Numerical studies show the high accuracy of the near-exact distributions and their adequacy for cases with very small samples and/or large number of variables. The exact and near-exact quantiles computed show how the common chi-square asymptotic approximation is highly inadequate for situations with small samples or large number of variables.

1. Introduction

Testing conditions on mean vectors is a common procedure in multivariate statistics. Often a given structure is assumed for the covariance matrix, without testing it, or otherwise this test to the covariance structure is carried out apart. This is often due to the fact that the exact distribution of the test statistics used to test simultaneously conditions on mean vectors and patterns on covariance matrices is too elaborate to be used in practice. The authors show how this problem may be overcome with the development of very sharp and manageable near-exact distributions for the test statistics. These distributions may be obtained from adequate factorizations of the characteristic function (c.f.) of the logarithm of the likelihood ratio (l.r.) statistics used for these tests.

The conditions tested on mean vectors are

(i) the equality of all the means in the mean vector,

(ii) the nullity of all the means in the mean vector

and the patterns tested on covariance matrices are

(i) circularity,

(ii) compound symmetry,

(iii) sphericity.

Let $\underline{X} = [X_1, \dots, X_p]'$ be a random vector with $\mathrm{Var}(\underline{X}) = \Sigma_c$. The covariance matrix Σ_c is said to be circular, or circulant, if $\Sigma_c = [\sigma_{ij}]$, $i, j = 1, \dots, p$, with

$$\sigma_{ii} = \mathrm{Var}(X_i) = \sigma_0^2,$$
$$\sigma_{i,i+k} = \sigma_{i+k,i} = \mathrm{Cov}(X_i, X_{i+k}) = \sigma_0^2 \rho_k, \quad (1)$$

where $\rho_k = \rho_{p-k} = \mathrm{Corr}(X_i, X_{i+k})$, for $i = 1, \dots, p$; $k = 1, \dots, p - i$.

For example, for $p = 6$ and $p = 7$, we have

$$\Sigma_c = \sigma_0^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix},$$

$$\Sigma_c = \sigma_0^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \rho_3 & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 & \rho_3 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 & \rho_3 \\ \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 & \rho_3 \\ \rho_3 & \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 & \rho_2 \\ \rho_2 & \rho_3 & \rho_3 & \rho_2 & \rho_1 & 1 & \rho_1 \\ \rho_1 & \rho_2 & \rho_3 & \rho_3 & \rho_2 & \rho_1 & 1 \end{bmatrix}. \quad (2)$$

Besides the almost obvious area of times series analysis, there is a wealth of other areas and research fields where circular or circulant matrices arise, such as statistical signal processing, information theory and cryptography, biological sciences, psychometry, quality control, and signal detection, as well as spatial statistics and engineering, when observations are made on the vertices of a regular polygon.

We say that a positive-definite $p \times p$ covariance matrix Σ_{cs} is compound-symmetric if we can write

$$\Sigma_{cs} = bE_{pp} + (a-b)\, I_p = aI_p + b\left(E_{pp} - I_p\right), \quad \text{with } -\frac{a}{(p-1)} < b < a. \quad (3)$$

For example, for $p = 4$, we have

$$\Sigma_{cs} = \begin{bmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{bmatrix}. \quad (4)$$

If, in (3), $b = 0$, we say that the matrix is spheric.

The l.r. tests for equality and nullity of means, assuming circularity, and the l.r. tests for the simultaneous test of equality or nullity of means and circularity of the covariance matrix were developed by [1], while the test for equality of means, assuming compound symmetry, and the test for equality of means and compound symmetry were formulated by [2] and the test for nullity of the means, assuming compound symmetry, and the simultaneous test for nullity of the means and compound symmetry of the covariance matrix were worked out by [3]. The exact distribution for the l.r. test statistic for the simultaneous test of equality of means and circularity of the covariance matrix was obtained in [4] and is briefly referred to in Section 2, for the sake of completeness, while near-exact distributions for the l.r. test statistic for the simultaneous test of nullity of the means and circularity of the covariance matrix are developed in Section 3. Near-exact distributions for the l.r. test statistics for the simultaneous test of equality and nullity of the means and compound symmetry of the covariance matrix are developed in Sections 4 and 5, using a different approach from the one used in Section 3. The l.r. statistics for the tests of equality and nullity of all means, assuming sphericity of the covariance matrix, may be analyzed in Appendix C and the l.r. statistics for the simultaneous tests of equality and nullity of all means and sphericity, together with the development of near-exact distributions for these statistics, may be examined in Sections 6 and 7.

Since, as referred above, the exact distributions for the statistics for the simultaneous tests of conditions on means vectors and patterns of covariance matrices are too elaborate to be used in practice, the authors propose in this paper the use of near-exact distributions for these statistics. These are asymptotic distributions which are built using a different concept in approximating distributions which combines an adequately developed decomposition of the c.f. of the statistic or of its logarithm, most often a factorization, with the action of keeping then most of this c.f. unchanged and replacing the remaining smaller part by an adequate asymptotic approximation [5, 6]. All this is done in order to obtain a manageable and very well-fitting approximation, which may be used to compute near-exact quantiles or p values. These distributions are much useful in situations where it is not possible to obtain the exact distribution in a manageable form and the common asymptotic distributions do not display the necessary precision. Near-exact distributions show very good performances for very small samples, and when correctly developed for statistics used in Multivariate Analysis, near-exact distributions display a sharp asymptotic behavior both for increasing sample sizes and for increasing number of variables.

In Sections 3–7, near-exact distributions are obtained using different techniques and results, according to the structure of the exact distribution of the statistic.

In order to study, in each case, the proximity between the near-exact distributions developed and the exact distribution, we will use the measure

$$\Delta = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \frac{\Phi_W(t) - \Phi_W^*(t)}{t} \right| dt, \quad (5)$$

with

$$\max_w \left| F_W(w) - F_W^*(w) \right| = \max_\ell \left| F_\Lambda(\ell) - F_\Lambda^*(\ell) \right| \le \Delta, \quad (6)$$

where Λ represents the l.r. statistic, $\Phi_W(t)$ is the exact c.f. of $W = -\log \Lambda$, $\Phi_W^*(t)$ is the near-exact c.f., and $F_W(\cdot)$, $F_\Lambda(\cdot)$, $F_W^*(\cdot)$, and $F_\Lambda^*(\cdot)$ are the exact and near-exact c.d.f.'s of W and Λ.

This measure is particularly useful, since in our cases we do not have the exact c.d.f. of Λ or W in a manageable form, but we have both the exact and near-exact c.f.'s for $W = -\log \Lambda$.

2. The Likelihood Ratio Test for the Simultaneous Test of Equality of Means and the Circularity of the Covariance Matrix

Let $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, where $\underline{\mu} = [\mu_1, \ldots, \mu_p]'$. Then, for a sample of size n, the $(2/n)$th power of the l.r. statistic to test the null hypothesis

$$H_0 : \mu_1 = \cdots = \mu_p; \quad \Sigma = \Sigma_c, \tag{7}$$

is

$$\Lambda_1 = 2^{2(p-m-1)} n^p \frac{|A|}{v_1} \prod_{j=2}^{p} \frac{1}{v_j + w_j} = 2^{2(p-m-1)} \frac{|V|}{v_1} \prod_{j=2}^{p} \frac{1}{v_j + w_j}, \tag{8}$$

where $m = \lfloor p/2 \rfloor$, A is the maximum likelihood estimator (m.l.e.) of Σ, $V = nU'AU$, where U is the matrix with running element

$$u_{ij} = \frac{1}{\sqrt{p}} \left\{ \cos\left(\frac{2\pi (j-1)(i-1)}{p} \right) + \sin\left(\frac{2\pi (j-1)(i-1)}{p} \right) \right\},$$
$$v_j = \begin{cases} v_{jj}, & j = 1, \text{ and also } j = m+1 \text{ if } p \text{ is even}, \\ v_{jj} + v_{p-j+2,p-j+2}, & j = 2, \ldots, m, \text{ and also } j = m+1 \text{ if } p \text{ is odd}, \end{cases} \tag{9}$$

with $v_j = v_{p-j+2}$ $(j = 2, \ldots, m)$, and where v_{jj} is the jth diagonal element of V, and

$$w_j = \begin{cases} y_j^2, & j = 1, \text{ and also } j = m+1 \text{ if } p \text{ is even}, \\ y_j^2 + y_{p-j+2}^2, & j = 2, \ldots, m, \text{ and also } j = m+1 \text{ if } p \text{ is odd}, \end{cases} \tag{10}$$

with $\underline{Y} = [y_j] = \sqrt{n}\,\overline{\underline{X}}U$, where $\overline{\underline{X}}$ is the vector of sample means.

This test statistic was derived by [1, sec. 5.2], where the expression for the l.r. test statistic has to be slightly corrected.

According to [1],

$$\Lambda_1 \stackrel{d}{\equiv} \prod_{j=2}^{p} Y_j, \tag{11}$$

where

$$Y_j \sim \begin{cases} \text{Beta}\left(\frac{n-j}{2}, \frac{j}{2} \right), & j = 2, \ldots, m+1, \\ \text{Beta}\left(\frac{n-j}{2}, \frac{j+1}{2} \right), & j = m+2, \ldots, p, \end{cases} \tag{12}$$

are a set of $p-1$ independent r.v.'s.

From this fact we may write the c.f. of $W_1 = -\log \Lambda_1$ as

$$\begin{aligned} \Phi_{W_1}(t) &= E\left(e^{itW_1}\right) = E\left(e^{-it\log\Lambda_1}\right) = E\left(\Lambda_1^{-it}\right) \\ &= \prod_{j=2}^{m+1} \frac{\Gamma(n/2)\,\Gamma((n-j)/2 - it)}{\Gamma((n-j)/2)\,\Gamma(n/2 - it)} \\ &\quad \cdot \prod_{j=m+2}^{p} \frac{\Gamma((n+1)/2)\,\Gamma((n-j)/2 - it)}{\Gamma((n-j)/2)\,\Gamma((n+1)/2 - it)}. \end{aligned} \tag{13}$$

By adequately handling this c.f., the exact distribution of W_1 is obtained in [4] as a Generalized Integer Gamma (GIG) distribution (see [7] for the GIG distribution), since we may write

$$\Phi_{W_1}(t) = \prod_{j=1}^{p} \left(\frac{n-j}{2} \right)^{r_j} \left(\frac{n-j}{2} - it \right)^{-r_j}, \tag{14}$$

for

$$r_j = \begin{cases} \dfrac{p-2+p \bmod 2}{2}, & j = 1, \\ \dfrac{p - p \bmod 2}{2} - \left\lfloor \dfrac{j-1-p \bmod 2}{2} \right\rfloor, & j = 2, \ldots, p. \end{cases} \tag{15}$$

A popular asymptotic approximation for the distribution of nW_1 is the chi-square asymptotic distribution with a number of degrees of freedom equal to the difference of the number of unknown parameters under the alternative hypothesis and the number of parameters under the null hypothesis, which gives for $nW_1 = -n\log\Lambda_1$, for Λ_1 in (8), a chi-square asymptotic distribution with $p(p+3)/2 - \lfloor (p+2)/2 \rfloor - 1$ degrees of freedom. Although this is a valid approximation for large sample sizes, in practical terms, this approximation is somewhat useless given the fact that it gives quantiles that are much lower than the exact ones, as it may be seen from the quantiles in Table 1, namely, for small samples or when the number of variables involved is somewhat large.

From the values in Table 1 we may see that even for quite large sample sizes and rather small number of variables as in the case of $p = 10$ and $n = 460$, the asymptotic chi-square quantile does not even match the units digit of the exact quantile, a difference that gets even larger as the number of variables increases. The chi-square asymptotic quantiles are always smaller than the exact ones, their use leading to an excessive number of rejections of the null hypotheses, a problem that becomes a grievous one when we use smaller samples or larger numbers of variables.

TABLE 1: Exact and asymptotic 0.95 and 0.99 quantiles for nW_1 where $W_1 = -\log\Lambda_1$ for the statistic Λ_1 in (8), for different values of p and samples of size $n = p+1, 50, 450$.

p	n	exact	Asymptotic-χ^2
		$\alpha = 0.95$	
10			76.77780315606147980433710659
	11	184.84579506364826855487849906	
	60	82.86779631112725385496956047	
	460	77.50088072977322094345813820	
15			153.19790274395621072198817490
	16	356.83946609433702153375390686	
	65	169.23132191434840041430238602	
	465	155.17420633635277455721974156	
25			379.74587752919253597245376194
	26	853.62442647392551959929457598	
	75	437.12290346767321994020024210	
	475	387.31925318201483716457949700	
50			1382.92839770564012472044120417
	51	2983.52950629554250120199516974	
	100	1719.07640203276757900109720368	
	500	1434.09183007253302711800352147	
		$\alpha = 0.99$	
10			85.95017624510346845181671517
	11	221.13637373719535956938670312	
	60	92.78317859393323169599466291	
	460	86.75984117402977424037787646	
15			165.84100085082047675645088502
	16	409.92566639020778120425384446	
	65	183.23718212829228159346647123	
	465	167.98095654076846741881112565	
25			399.22970790268112734530953113
	26	940.55141434060365229501805667	
	75	459.68274728064743270409254333	
	475	407.19370031104569525049581690	
50			1419.46244733465596475819616876
	51	3156.01716925813527651187643029	
	100	1765.17588807596249988258749774	
	500	1471.99072215013613268320536543	

3. The Likelihood Ratio Test for the Simultaneous Test of Nullity of Means and the Circularity of the Covariance Matrix

For a sample of size n, the $(2/n)$th power of the l.r. test statistic to test the null hypothesis

$$H_0 : \underline{\mu} = \underline{0}; \quad \Sigma = \Sigma_c \tag{16}$$

is

$$\Lambda_2 = 2^{2(p-m-1)} n^p \frac{|A|}{v_1} \prod_{j=1}^{p} \frac{1}{v_j + w_j} = 2^{2(p-m-1)} \frac{|V|}{v_1} \prod_{j=1}^{p} \frac{1}{v_j + w_j}, \tag{17}$$

where m, v_j, and w_j, as well as the matrices A and V, are defined as in the previous section.

According to [1],

$$\Lambda_2 \stackrel{d}{\equiv} \prod_{j=1}^{p} Y_j, \tag{18}$$

where

$$Y_j \sim \begin{cases} \text{Beta}\left(\dfrac{n-j}{2}, \dfrac{j}{2}\right), & j = 1, \ldots, m+1, \\ \text{Beta}\left(\dfrac{n-j}{2}, \dfrac{j+1}{2}\right), & j = m+2, \ldots, p, \end{cases} \tag{19}$$

are a set of p independent r.v.'s.

Taking $W_2 = -\log \Lambda_2$ and following similar steps to the ones used in [4] to handle the c.f. of W_1, we may write the c.f. of W_2 as

$$\Phi_{W_2}(t) = \frac{\Gamma(n/2)\Gamma((n-1)/2-it)}{\Gamma((n-1)/2)\Gamma(n/2-it)} \cdot \prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j}, \tag{20}$$

for r_j given by (15).

This shows that the exact distribution of W_2 is the same as that of the sum of GIG distributions of depth p with an independent Logbeta$((n-1)/2, 1/2)$ distributed r.v.

But then, using the result in expression (3) of [8], we know that we can replace asymptotically a Logbeta(a, b) distribution by an infinite mixture of $\Gamma(b-2j, a+(b-1)/2)$ distributions ($j = 0, 1, \ldots$), for large values of a. This means that we can replace asymptotically

$$\frac{\Gamma(n/2)\Gamma((n-1)/2-it)}{\Gamma((n-1)/2)\Gamma(n/2-it)}$$

$$\text{by } \sum_{j=0}^{\infty}\pi_j\left(\frac{n-1}{2}+\frac{1/2-1}{2}\right)^{1/2+2j} \cdot \left(\frac{n-1}{2}+\frac{1/2-1}{2}-it\right)^{-(1/2+2j)}. \tag{21}$$

As such, in order to obtain a very sharp and manageable near-exact distribution for W_2, we will use, as near-exact c.f. for W_2,

$$\begin{aligned}\Phi^*_{W_2}(t) &= \left\{\sum_{j=0}^{m}\pi_j\left(\frac{n-1}{2}+\frac{1/2-1}{2}\right)^{1/2+2j} \cdot \left(\frac{n-1}{2}+\frac{1/2-1}{2}-it\right)^{-(1/2+2j)}\right\}\left\{\prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j} \cdot \left(\frac{n-j}{2}-it\right)^{-r_j}\right\} \\ &= \sum_{j=0}^{m}\pi_j\left(\frac{n-1}{2}+\frac{1/2-1}{2}\right)^{1/2+2j}\left(\frac{n-1}{2}+\frac{1/2-1}{2}-it\right)^{-(1/2+2j)} \cdot \prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j},\end{aligned} \tag{22}$$

where the weights π_j, $j = 0, \ldots, m-1$, will be determined in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\Phi_{W_2}(t)\right|_{t=0} = \left.\frac{\partial^h}{\partial t^h}\Phi^*_{W_2}(t)\right|_{t=0}, \quad h = 1, \ldots, m, \tag{23}$$

with $\pi_m = 1 - \sum_{j=0}^{m-1}\pi_j$.

$\Phi^*_{W_2}(t)$ is the c.f. of a mixture of $m+1$ Generalized Near-Integer Gamma (GNIG) distributions of depth $p+1$ (see [5] for the GNIG distribution).

As such, using the notation for the p.d.f. and c.d.f. of the GNIG distribution used in Section 3 of [6], the near-exact p.d.f.'s and c.d.f.'s for $W_2 = -\log \Lambda_2$ and Λ_2 are

$$\begin{aligned} f^*_{W_2}(w) &= \sum_{j=0}^{m}\pi_j f^{\text{GNIG}}\left(w \mid r_1, \ldots, r_p, \frac{1}{2}+2j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \frac{n-1}{2}+\frac{1/2-1}{2}; p+1\right), \\ F^*_{W_2}(w) &= \sum_{j=0}^{m}\pi_j F^{\text{GNIG}}\left(w \mid r_1, \ldots, r_p, \frac{1}{2}+2j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \frac{n-1}{2}+\frac{1/2-1}{2}; p+1\right), \\ f^*_{\Lambda_2}(\ell) &= \sum_{j=0}^{m}\pi_j f^{\text{GNIG}}\left(-\log\ell \mid r_1, \ldots, r_p, \frac{1}{2}+2j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \frac{n-1}{2}+\frac{1/2-1}{2}; p+1\right)\frac{1}{\ell}, \\ F^*_{\Lambda_2}(\ell) &= \sum_{j=0}^{m}\pi_j\left(1 - F^{\text{GNIG}}\left(-\log\ell \mid r_1, \ldots, r_p, \frac{1}{2}+2j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \frac{n-1}{2}+\frac{1/2-1}{2}; p+1\right)\right), \end{aligned} \tag{24}$$

with $r_1, \ldots, r_p$ given by (15).

In Table 2 we may analyze values of the measure Δ in (5) for the near-exact distributions developed in this section, for different values of p and different sample sizes. We may see how these near-exact distributions display very low values of the measure Δ, indicating an extremely good proximity to the exact distribution, even for very small sample sizes, and how they display a sharp asymptotic behavior for increasing values of p and n.

In Table 3 we may analyze the asymptotic quantiles for nW_2 for the common chi-square asymptotic approximation for l.r. statistics, here with $p(p+3)/2 - \lfloor(p+2)/2\rfloor$ degrees of freedom and the quantiles for the near-exact distributions that equate 2, 6, and 10 exact moments. These quantiles are shown with 26 decimal places in order to make it possible to identify the number of correct decimal places for the quantiles of the near-exact distributions that match 2 and 6 exact moments. We should note that the quantiles of the near-exact distributions that match 10 exact moments always have much more than 26 decimal places that are correct. Also for the statistic in this section, we may see the lack of precision of the asymptotic chi-square quantiles.

4. The Likelihood Ratio Test for the Simultaneous Test of Equality of Means and Compound Symmetry of the Covariance Matrix

Let us assume that $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, with $\underline{\mu} = [\mu_1, \ldots, \mu_p]'$. We are interested in testing the hypothesis

$$\begin{aligned} H_0 &: \mu_1 = \cdots = \mu_p; \\ &\Sigma = \Sigma_{cs}, \end{aligned} \tag{25}$$

Table 2: Values of the measure Δ in (5), for the near-exact distributions of the l.r. test statistic Λ_2 in (17), which match m exact moments, for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	m				
		2	4	6	10	20
10	11	2.12×10^{-10}	5.95×10^{-14}	1.23×10^{-16}	1.35×10^{-20}	1.02×10^{-26}
	60	3.12×10^{-14}	2.32×10^{-20}	1.46×10^{-25}	2.80×10^{-34}	1.41×10^{-50}
	460	1.52×10^{-19}	3.22×10^{-29}	5.78×10^{-38}	9.53×10^{-54}	1.70×10^{-87}
15	16	4.12×10^{-12}	9.90×10^{-17}	2.09×10^{-20}	4.06×10^{-26}	1.32×10^{-35}
	65	5.95×10^{-15}	1.60×10^{-21}	3.86×10^{-27}	1.28×10^{-36}	1.59×10^{-54}
	465	4.76×10^{-20}	5.11×10^{-30}	4.92×10^{-39}	2.69×10^{-55}	5.39×10^{-90}
25	26	3.00×10^{-14}	2.91×10^{-20}	2.75×10^{-25}	1.56×10^{-33}	1.83×10^{-48}
	75	5.13×10^{-16}	2.85×10^{-23}	1.47×10^{-29}	2.52×10^{-40}	3.53×10^{-61}
	475	9.86×10^{-21}	3.91×10^{-31}	1.45×10^{-40}	1.30×10^{-57}	4.82×10^{-94}
50	51	4.11×10^{-17}	4.95×10^{-25}	6.07×10^{-32}	6.98×10^{-44}	2.02×10^{-67}
	100	9.04×10^{-18}	3.48×10^{-26}	1.27×10^{-33}	1.16×10^{-46}	5.22×10^{-73}
	500	9.25×10^{-22}	7.77×10^{-33}	6.23×10^{-43}	2.77×10^{-61}	7.62×10^{-101}

where Σ_{cs} represents a compound symmetric matrix, as defined in (3).

For a sample of size n, the $(2/n)$th power of the l.r. test statistic is (see [2])

$$\Lambda_3 = \frac{|A|}{\left(\hat{a} + (p-1)\hat{b}\right)\left(\hat{a} - \hat{b} + S^{*2}\right)^{p-1}}, \quad (26)$$

where

$$A = \left[a_{jk}\right] = X'\left(I_n - \frac{1}{n}E_{nn}\right)X, \quad (27)$$

with X being the $n \times p$ sample matrix and E_{nn} a matrix of 1's of dimension $n \times n$,

$$\hat{a} = \frac{1}{p}\sum_{j=1}^{p} a_{jj}, \qquad \hat{b} = \frac{2}{p(p-1)}\sum_{j=1}^{p-1}\sum_{k=j+1}^{p} a_{jk}, \quad (28)$$

$$S^{*2} = \frac{1}{p-1}\sum_{j=1}^{p}\left(\overline{X}_j - \overline{X}\right)^2, \quad (29)$$

with

$$\overline{X}_j = \frac{1}{n}\sum_{i=1}^{n} X_{ji}, \qquad \overline{X} = \frac{1}{p}\sum_{j=1}^{p}\overline{X}_j. \quad (30)$$

Wilks [2] has also shown that

$$\Lambda_3 \stackrel{d}{\equiv} \prod_{j=2}^{p} Y_j, \quad (31)$$

where

$$Y_j \sim \text{Beta}\left(\frac{n-j}{2}, \frac{j-2}{p-1} + \frac{j}{2}\right), \quad (32)$$

form a set of $p-1$ independent r.v.'s.

As such, the hth moment of Λ_3 may be written as

$$E\left(\Lambda_3^h\right) = \prod_{j=2}^{p} \frac{\Gamma(n/2 + (j-2)/(p-1))\,\Gamma((n-j)/2 + h)}{\Gamma((n-j)/2)\,\Gamma(n/2 + (j-2)/(p-1) + h)}, \quad (33)$$

$$\left(h > -\frac{n-p}{2}\right).$$

Since the expression in (33) remains valid for any complex h, we may write the c.f. of $W_3 = -\log \Lambda_3$ as

$$\Phi_{W_3}(t) = E\left(\Lambda_3^{-it}\right) = \prod_{j=2}^{p} \frac{\Gamma(n/2 + (j-2)/(p-1))\,\Gamma((n-j)/2 - it)}{\Gamma((n-j)/2)\,\Gamma(n/2 + (j-2)/(p-1) - it)}, \quad (34)$$

which may be rewritten as

$$\Phi_{W_3}(t) = \underbrace{\left\{\prod_{j=2}^{p} \frac{\Gamma(n/2 + (j-2)/(p-1))\,\Gamma((n-j)/2 + \lfloor (j-2)/(p-1) + j/2 \rfloor - it)}{\Gamma((n-j)/2 + \lfloor (j-2)/(p-1) + j/2 \rfloor)\,\Gamma(n/2 + (j-2)/(p-1) - it)}\right\}}_{\Phi_{W_{3,1}}(t)} \underbrace{\left\{\prod_{j=2}^{p} \frac{\Gamma((n-j)/2 + \lfloor (j-2)/(p-1) + j/2 \rfloor)\,\Gamma((n-j)/2 - it)}{\Gamma((n-j)/2)\,\Gamma((n-j)/2 + \lfloor (j-2)/(p-1) + j/2 \rfloor - it)}\right\}}_{\Phi_{W_{3,2}}(t)}. \quad (35)$$

Table 3: Quantiles of orders $\alpha = 0.95$ and $\alpha = 0.99$ for the chi-square approximation and for the near-exact distributions that match $m = 2, 6$, or 10 exact moments, of $nW_2 = -n \log \Lambda_2$ for the l.r. statistic Λ_2 in (17), for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	Near-exact distributions m 2	6	10	χ^2
			$\alpha = 0.95$		
10					77.930523805230422216265519134
	11	186.05876112572581565724084671	186.05432193047513242686020674	186.05432193047314525997232991	
	60	84.041251059992766671865246443	84.041084586092313489433382111	84.041084586092311095379705400	
	460	78.656341655065000240841998222	78.656338927217021040792286588	78.656338927217021040792041600	
15					154.30151616535021445238246295
	16	357.97669643528506074060759290	357.97507556215083974984357415	357.97507556215080939585842368	
	65	170.35237275637365647369739224	170.35226008885071534567845906	170.35226008885071540240672638	
	465	156.28032082820735784475589844	156.28031868060615422818671849	156.28031868060615422818672130	
25					380.80932798326309965351661308
	26	854.70444721767252577468152620	854.70394734650287271287479556	854.70394734650287269133917757	
	75	438.20003415718340026159076324	438.19996662815447062882655526	438.19996662815447063138401124	
	475	388.38496822730515375937449496	388.38496656458350909591459832	388.38496656458350909591459983	
50					1383.96068056800040468077204943
	51	2984.56855943704772921090729785	2984.56844789958359734870454884	2984.56844789958359734870371056	
	100	1720.11786940941582246751263410	1720.11783827238527706678307676	1720.11783827238527706678545071	
	500	1435.12611970183181982922209152	1435.12611845907109035050829355	1435.12611845907109035050829357	
			$\alpha = 0.99$		
10					87.165711399787569396055355555
	11	222.35460048933098793717199428	222.34999712883572718401028732	222.34999712881867010107412453	
	60	94.016160901770129818664368862	94.015919129082880104232426388	94.015919129082887386489759488	
	460	87.977747841956314228225493700	87.977743790406488758187722444	87.977743790406488758188277633	
15					166.98739013667388353875528529
	16	411.06647044542264851519342470	411.06480936217954062715053117	411.06480936217950312171917698	
	65	184.39761554112671734787489011	184.39747019675240094311740291	184.39747019675240137066064190	
	465	169.12943676298651871191909376	169.12943392987951859377231013	169.12943392987951859377239666	
25					400.31941006633016981735907054
	26	941.63369479396272075223880115	941.63318710455464318301861791	941.63318710455464316544796384	
	75	460.78287302204290413806922579	460.78279435481890211302276154	460.78279435481890211439671412	
	475	408.28522573430588726946677528	408.28522375722866010999960584	408.28522375722866010999960757	
50					1420.50810052822545308190548728
	51	3157.05746081267985238368475051	3157.05734831670317027719825824	3157.05734831670317027719755606	
	100	1766.22802929701017442904996266	1766.22799581952383557719092088	1766.22799581952383557718851667	
	500	1473.03793287996522918580780119	1473.03793152230567784068549373	1473.03793152230567784068549371	

Then, we may apply on $\Phi_{W_3,2}(t)$ the relation

$$\frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{\ell=0}^{n-1}(a+\ell), \quad \forall a \in \mathbb{C},\ n \in \mathbb{N}, \tag{36}$$

to obtain

$$\begin{aligned}\Phi_{W_3}(t) &= \left\{\prod_{j=3}^{p}\frac{\Gamma(n/2+(j-2)/(p-1))\,\Gamma((n-j)/2+\lfloor (j-2)/(p-1)+j/2\rfloor - it)}{\Gamma((n-j)/2+\lfloor (j-2)/(p-1)+j/2\rfloor)\,\Gamma(n/2+(j-2)/(p-1)-it)}\right\} \\ &\cdot \left\{\prod_{j=2}^{p}\prod_{\ell=0}^{\lfloor (j-2)/(p-1)+j/2\rfloor-1}\left(\frac{n-j}{2}+\ell\right)\left(\frac{n-j}{2}+\ell-it\right)^{-1}\right\} \\ &= \underbrace{\left\{\prod_{j=3}^{p}\frac{\Gamma(n/2+(j-2)/(p-1))\,\Gamma((n-j)/2+\lfloor (j-2)/(p-1)+j/2\rfloor - it)}{\Gamma((n-j)/2+\lfloor (j-2)/(p-1)+j/2\rfloor)\,\Gamma(n/2+(j-2)/(p-1)-it)}\right\}}_{\Phi_{W_3,1}(t)} \underbrace{\left\{\prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j}\right\}}_{\Phi_{W_3,2}(t)},\end{aligned} \tag{37}$$

with

$$r_j = \begin{cases}\dfrac{p}{4}-1, & j=1,\ \text{if } (p \bmod 4)=0, \\ \left\lfloor\dfrac{(p+1)}{4}\right\rfloor & j=1,\ \text{if } (p \bmod 4)\neq 0, \\ \left\lfloor\dfrac{(p-j+2)}{2}\right\rfloor & j=2,\dots,p.\end{cases} \tag{38}$$

Expression (37) shows that the exact distribution of W_3 is the same as that of the sum of GIG distributed r.v.'s of depth p with an independent sum of $p-2$ independent Logbeta$((n-j)/2+\lfloor (j-2)/(p-1)+j/2\rfloor, (j-2)/(p-1)+j/2-\lfloor (j-2)/(p-1)+j/2\rfloor)$ r.v.'s.

Our aim in building the near-exact distribution will be to keep $\Phi_{W_3,2}(t)$ unchanged and approximate asymptotically $\Phi_{W_3,1}(t)$.

In order to obtain this asymptotic approximation, we will need to use a different approach from the one used in the previous section. We will use the result in sec. 5 of [9], which implies that a Logbeta(a,b) distribution may be asymptotically replaced by an infinite mixture of $\Gamma(b+j,a)$ $(j=0,1,\dots)$ distributions.

Using a somewhat heuristic approach, we will thus approximate $\Phi_{W_3,1}(t)$ by a mixture of $\Gamma(r+j,\lambda^*)$ distributions where

$$\begin{aligned} r &= \sum_{j=2}^{p}\frac{j-2}{p-1}+\frac{j}{2}-\left\lfloor\frac{j-2}{p-1}+\frac{j}{2}\right\rfloor \\ &= \frac{p-3}{2}+\frac{((p+1) \bmod 2)+((p+1) \bmod 4)}{2^{(p+1) \bmod 4}}\end{aligned} \tag{39}$$

is the sum of the second parameters of the Logbeta r.v.'s in $\Phi_{W_3,1}(t)$ and λ^* is the common rate parameter in the mixture of two Gamma distributions that matches the first 4 moments of $\Phi_{W_3,1}(t)$, that is, λ^* in

$$\begin{aligned}&\frac{d^h}{dt^h}\left(p(\lambda^*)^{r_1}(\lambda^*-it)^{-r_1}\right. \\ &\left.\left.+(1-p)(\lambda^*)^{r_2}(\lambda^*-it)^{-r_2}\right)\right|_{t=0} = \frac{d^h}{dt^h} \\ &\cdot \Phi_{W_3,1}(t)\Big|_{t=0}, \quad h=1,\dots,4.\end{aligned} \tag{40}$$

As such, in order to build the near-exact distributions for W_3, we will use, as near exact c.f. for W_3,

$$\begin{aligned}\Phi^*_{W_3}(t) &= \left\{\sum_{j=0}^{m}\pi_j(\lambda^*)^{r+j}(\lambda^*-it)^{-(r+j)}\right\} \\ &\cdot \left\{\prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j}\right\} \\ &= \sum_{j=0}^{m}\pi_j(\lambda^*)^{r+j}(\lambda^*-it)^{-(r+j)} \\ &\cdot \prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j},\end{aligned} \tag{41}$$

where the weights π_j, $j=0,\dots,m-1$, will be determined in such a way that

$$\left.\frac{\partial^h}{\partial t^h}\Phi_{W_3}(t)\right|_{t=0} = \left.\frac{\partial^h}{\partial t^h}\Phi^*_{W_3}(t)\right|_{t=0}, \quad h=1,\dots,m, \tag{42}$$

with $\pi_m = 1-\sum_{j=0}^{m-1}\pi_j$.

The c.f. in (41) is, for integer r, the c.f. of a mixture of $m+1$ GIG distributions of depth $p+1$ or, for noninteger r, the

Table 4: Values of the measure Δ in (5), for the near-exact distributions of the l.r. test statistic Λ_3 in (26), which match m exact moments, for different values of p and samples of size $n = p+1, 50, 450$.

p	n	m = 2	4	6	10	20
10	11	9.68×10^{-9}	3.84×10^{-13}	1.07×10^{-15}	1.36×10^{-20}	5.71×10^{-29}
	60	3.01×10^{-10}	1.24×10^{-15}	3.49×10^{-19}	6.90×10^{-26}	8.11×10^{-40}
	460	6.80×10^{-13}	3.93×10^{-20}	1.95×10^{-25}	9.75×10^{-36}	7.94×10^{-58}
15	16	9.99×10^{-10}	1.62×10^{-14}	4.49×10^{-19}	2.91×10^{-27}	3.19×10^{-41}
	65	9.85×10^{-11}	3.36×10^{-16}	2.24×10^{-21}	9.91×10^{-31}	1.76×10^{-49}
	465	2.81×10^{-13}	2.09×10^{-20}	3.35×10^{-27}	9.96×10^{-40}	3.22×10^{-62}
25	26	6.12×10^{-11}	1.73×10^{-16}	6.85×10^{-22}	1.67×10^{-32}	1.52×10^{-50}
	75	1.87×10^{-11}	2.86×10^{-17}	5.76×10^{-23}	1.12×10^{-33}	2.66×10^{-55}
	475	7.06×10^{-14}	3.38×10^{-21}	2.21×10^{-28}	6.35×10^{-42}	2.53×10^{-65}
50	51	1.78×10^{-12}	5.44×10^{-19}	2.24×10^{-25}	9.80×10^{-37}	8.74×10^{-61}
	100	2.35×10^{-12}	9.83×10^{-19}	4.94×10^{-25}	2.44×10^{-36}	1.64×10^{-58}
	500	2.25×10^{-14}	5.94×10^{-22}	1.76×10^{-29}	2.05×10^{-43}	8.50×10^{-72}

c.f. of a mixture of $m+1$ GNIG distributions of depth $p+1$, with shape parameters $r_1, \ldots, r_p, r+j$ ($j = 0, \ldots, m$) and rate parameters $(n-1)/2, \ldots, (n-p)/2, \lambda^*$.

This will yield, for noninteger r, near-exact distributions whose p.d.f.'s and c.d.f.'s for $W_3 = -\log \Lambda_3$ and Λ_3 are

$$\begin{aligned}
f^*_{W_3}(w) &= \sum_{j=0}^{m} \pi_j f^{\mathrm{GNIG}}\left(w \mid r_1, \ldots, r_p, r+j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \lambda^*; p+1\right), \\
F^*_{W_3}(w) &= \sum_{j=0}^{m} \pi_j F^{\mathrm{GNIG}}\left(w \mid r_1, \ldots, r_p, r+j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \lambda^*; p+1\right), \\
f^*_{\Lambda_3}(\ell) &= \sum_{j=0}^{m} \pi_j f^{\mathrm{GNIG}}\left(-\log \ell \mid r_1, \ldots, r_p, r+j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \lambda^*; p+1\right) \frac{1}{\ell}, \\
F^*_{\Lambda_3}(\ell) &= \sum_{j=0}^{m} \pi_j \left(1 - F^{\mathrm{GNIG}}\left(-\log \ell \mid r_1, \ldots, r_p, r+j; \frac{n-1}{2}, \ldots, \frac{n-p}{2}, \lambda^*; p+1\right)\right),
\end{aligned} \tag{43}$$

with $r_1, \ldots, r_p$ given by (38). For integer r, we will only have to replace in the above expressions the GNIG p.d.f. and c.d.f. by the GIG p.d.f. and c.d.f., respectively.

In Table 4, we may analyze values of the measure Δ in (5) for the near-exact distributions developed in this section, for different values of p and different sample sizes. We may see how these near-exact distributions display, once again, very low values of the measure Δ even for very small sample sizes, indicating an extremely good proximity to the exact distribution and how, once again, they display a sharp asymptotic behavior for increasing values of p and n, although for large values of p, namely, for $p = 50$ in Table 4, one may have to consider larger values of n in order to be able to observe the asymptotic behavior in terms of sample size.

The asymptotic quantiles for nW_3 in Table 5, for the common chi-square asymptotic approximation for l.r. statistics, now with $p(p+3)/2-3$ degrees of freedom, display again, as in the previous sections, an almost shocking lack of precision, mainly for small sample sizes and/or larger numbers of variables. On the other hand, the near-exact quantiles show a steady evolution towards the exact quantiles for increasing number of exact moments matched, with the quantiles for the near-exact distributions that match 6 exact moments displaying more than 20 correct decimal places, for the larger sample sizes.

5. The Likelihood Ratio Test for the Simultaneous Test of Nullity of Means and Compound Symmetry of the Covariance Matrix

Let us assume now that $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$. We are interested in testing the hypothesis

$$\begin{aligned} H_0 : \underline{\mu} &= \underline{0}; \\ \Sigma &= \Sigma_{cs}. \end{aligned} \tag{44}$$

We may write

$$H_0 \equiv H_{02|01} \circ H_{01}, \tag{45}$$

where

$$\begin{aligned} H_{02|01} &: \mu = \underline{0}, \ \text{assuming } \Sigma = \Sigma_{cs}, \\ H_{01} &: \Sigma = \Sigma_{cs}. \end{aligned} \tag{46}$$

While, for a sample of size n, the $(2/n)$th power of the l.r. statistic to test $H_{02|01}$ may be shown to be (see Appendix A for details)

$$\Lambda_{2|1} = \frac{\left(\hat{a} - \hat{b}\right)^{p-1}\left(\hat{a} + (p-1)\hat{b}\right)}{\left(\hat{a}_0 - \hat{b}_0\right)^{p-1}\left(\hat{a}_0 + (p-1)\hat{b}_0\right)}, \tag{47}$$

Table 5: Quantiles of orders $\alpha = 0.95$ and $\alpha = 0.99$ for the chi-square approximation and for the near-exact distributions that match $m = 2, 6$, or 10 exact moments, of $-n \log \Lambda_3$ for the l.r. statistic Λ_3 in (26), for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	Near-exact distributions m 2	6	10	χ^2
			$\alpha = 0.95$		
10					81.381015188899104508431120785
	11	189.114696480423388015515754393	189.114697238826289320495944292	189.114697238826277646711500921	
	60	87.443617040369847972033082381	87.443617050273318041561932389	87.443617050273318056851954690	
	460	82.100788323800277056582786653	82.100788323820173111875365635	82.100788323820173111875372688	
15					159.813546850997802977506766879
	16	363.087896604766629226939377100	363.087896734501151262354131399	363.087896734501151251811277829	
	65	175.805353263604236148015901820	175.805353269595285156952328009	175.805353269595285157155780468	
	465	161.784540428765594956506807022	161.784540428780828916102538276	161.784540428780828916102538558	
25					391.438719112192616721837671209
	26	864.891282269279116189851387185	864.891282284150965012514477481	864.891282284150965012477034037	
	75	448.748369857001308907641564890	448.748369859173975960465295344	448.748369859173975960473758210	
	475	399.002746812117152321633172649	399.002746812124288653675424722	399.002746812124288653675424753	
50					1407.69978493252055471492271999
	51	3007.81592652688874585708833686	3007.81592652788317452613238449	3007.81592652788317452613235100	
	100	1743.73166206895650376506106613	1743.73166206957708782548197403	1743.73166206957708782548209645	
	500	1458.84419233676150570472235786	1458.84419233676635867029771753	1458.84419233676635867029771754	
			$\alpha = 0.99$		
10					90.801532030838687874709640406
	11	225.421470930453454966690527975	225.421472009727318561373265594	225.421472009727311663857900086	
	60	97.582829539160859251491143413	97.582829585977762108505255066	97.582829585977762137425094394	
	460	91.604850363213970123907600250	91.604850363315475545131468587	91.604850363315475545131487705	
15					172.710824396692046362684067121
	16	416.192057135707772720106372243	416.192057320949648013699720809	416.192057320949648008342780372	
	65	190.035075104220296563348087039	190.035075126350713655861714506	190.035075126350713655903270996	
	465	174.841471634002327233344093337	174.841471634061877548052071914	174.841471634061877548052072072	
25					411.209208332240728524234613091
	26	951.840639147683569759583191462	951.840639169340274602907166879	951.840639169340274602887597489	
	75	471.550355225483649333455494949	471.550355232304350604344289953	471.550355232304350604337719505	
	475	419.157414390428896468907403837	419.157414390452257215547282031	419.157414390452257215547282016	
50					1444.55331478957048828683472593
	51	3180.33179707887525737674116194	3180.33179708038051828516553103	3180.33179708038051828516550812	
	100	1790.07969304544686923930101362	1790.07969304718352521730352610	1790.07969304718352521730331125	
	500	1497.05022951059504341326946354	1497.05022951060908489974722503	1497.05022951060908489974722503	

where $\widehat{a}$ and $\widehat{b}$ are given by (28) and

$$\widehat{a}_0 = \frac{1}{p}\sum_{j=1}^{p} a_{0(jj)}, \qquad \widehat{b}_0 = \frac{2}{p(p-1)}\sum_{j=1}^{p-1}\sum_{k=j+1}^{p} a_{0(jk)}, \tag{48}$$

with

$$A_0 = \left[a_{0(jk)}\right] = X'X, \tag{49}$$

where X is the $n \times p$ sample matrix, the l.r. test statistic to test H_{01} is shown by [2] to be

$$\Lambda_1 = \frac{|A|}{\left(\widehat{a}-\widehat{b}\right)^{p-1}\left(\widehat{a}+(p-1)\widehat{b}\right)}, \tag{50}$$

again with $\widehat{a}$ and $\widehat{b}$ given by (28) and A given by (27).

The l.r. test to test H_0 in (44) is thus

$$\Lambda_4 = \Lambda_1\Lambda_{2|1} = \frac{|A|}{\left(\widehat{a}_0-\widehat{b}_0\right)^{p-1}\left(\widehat{a}_0+(p-1)\widehat{b}_0\right)}. \tag{51}$$

For a sample of size n, $\Lambda_{2|1}$, the $(2/n)$th power of the l.r. test statistic to test $H_{02|01}$, may be shown to be distributed as (see [3] and Appendix A for details) $Y_1^{p-1}Y_2$, where Y_1 and Y_2 are independent, with

$$Y_1 \sim \text{Beta}\left(\frac{(n-1)(p-1)}{2}, \frac{p-1}{2}\right), \qquad Y_2 \sim \text{Beta}\left(\frac{n-1}{2}, \frac{1}{2}\right), \tag{52}$$

while [2] shows that the $(2/n)$th power of the l.r. statistic to test H_{01} is distributed as $\prod_{j=2}^{p} Y_j^*$, where Y_j^* are independent, with

$$Y_j^* \sim \text{Beta}\left(\frac{n-j}{2}, \frac{j-2}{p-1}+\frac{j-1}{2}\right). \tag{53}$$

Based on Theorem 5 in [10], it is then possible to show that the l.r. statistics to test H_{01} and $H_{02|01}$ are independent, since Λ_1 is independent of $(\widehat{a}-\widehat{b})^{p-1}(\widehat{a}+(p-1)\widehat{b})$ and $\Lambda_{2|1}$ is built only on this statistic, since $(\widehat{a}_0-\widehat{b}_0)^{p-1}(\widehat{a}_0+(p-1)\widehat{b}_0)$ is the same statistic in a constrained subspace.

From this fact, we may show that the $(2/n)$th power of the l.r. statistic to test H_0 in (44), Λ_4, is distributed as (see Appendix B for details)

$$\left\{\prod_{j=2}^{p} Y_j^{**}\right\} Y, \tag{54}$$

where all r.v.'s are independent, with

$$Y_j^{**} \sim \text{Beta}\left(\frac{n-j}{2}, \frac{j-2}{p-1}+\frac{j}{2}\right), \qquad Y \sim \text{Beta}\left(\frac{n-1}{2}, \frac{1}{2}\right). \tag{55}$$

We note that the r.v.'s Y_j^{**} are the same as the r.v.'s Y_j in (31) and (32).

As such, the c.f. of $W_4 = -\log\Lambda_4$ may be written as

$$\Phi_{W_4}(t) = \underbrace{\left\{\prod_{j=3}^{p}\frac{\Gamma(n/2+(j-2)/(p-1))\,\Gamma((n-j)/2+\lfloor (j-2)/(p-1)+j/2\rfloor - it)}{\Gamma((n-j)/2+\lfloor (j-2)/(p-1)+j/2\rfloor)\,\Gamma(n/2+(j-2)/(p-1)-it)}\right\}\frac{\Gamma(n/2)\,\Gamma((n-1)/2-it)}{\Gamma((n-1)/2)\,\Gamma(n/2-it)}}_{\Phi_{W_4,1}(t)}\underbrace{\left\{\prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j}\right\}}_{\Phi_{W_4,2}(t)}, \tag{56}$$

with r_j given by (38).

Then, following a similar approach to the one used for W_3 and Λ_3, in the previous section, we obtain near-exact distributions with a similar structure to those in that section, now with

$$r = \frac{1}{2}+\sum_{j=2}^{p}\frac{j-2}{p-1}+\frac{j}{2}-\left\lfloor\frac{j-2}{p-1}+\frac{j}{2}\right\rfloor = \frac{p-2}{2}+\frac{((p+1)\bmod 2)+((p+1)\bmod 4)}{2^{(p+1)\bmod 4}} \tag{57}$$

and with λ^* determined as the solution of a system of equations similar to the one in (40), with $\Phi_{W_3,1}$ replaced by $\Phi_{W_4,1}$.

This will yield for Λ_4 and W_4 near-exact distributions with p.d.f.'s and c.d.f.'s given by (43), now with r given by (57).

We should note that as it happens with Λ_3 and W_3, also for W_4 and Λ_4, r may be either an integer or a half-integer, so that, in those cases where r is an integer, the near-exact distributions are mixtures of GIG distributions, while when r is noninteger, they are mixtures of GNIG distributions.

In Table 6 we may analyze values of the measure Δ in (5) for the near-exact distributions developed in this section, for different values of p and different sample sizes. We may see how these near-exact distributions display similar properties to those of the near-exact distributions developed for Λ_3 in the previous section.

Table 6: Values of the measure Δ in (5), for the near-exact distributions of the l.r. test statistic Λ_4 in (51), which match m exact moments, for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	m				
		2	4	6	10	20
10	11	2.54×10^{-9}	5.91×10^{-13}	9.61×10^{-16}	2.01×10^{-20}	8.06×10^{-29}
	60	2.80×10^{-10}	3.92×10^{-16}	4.36×10^{-19}	7.29×10^{-26}	1.73×10^{-40}
	460	7.70×10^{-13}	1.42×10^{-20}	2.88×10^{-25}	1.38×10^{-35}	9.04×10^{-58}
15	16	1.36×10^{-10}	1.88×10^{-14}	5.28×10^{-18}	3.92×10^{-25}	9.44×10^{-36}
	65	4.44×10^{-11}	1.52×10^{-16}	2.70×10^{-20}	9.22×10^{-28}	2.19×10^{-43}
	465	2.05×10^{-13}	8.43×10^{-23}	3.28×10^{-26}	4.38×10^{-37}	1.62×10^{-60}
25	26	3.13×10^{-11}	1.07×10^{-16}	2.08×10^{-21}	2.25×10^{-29}	1.37×10^{-45}
	75	7.48×10^{-13}	1.25×10^{-18}	2.38×10^{-22}	6.06×10^{-31}	5.83×10^{-49}
	475	3.30×10^{-14}	1.93×10^{-21}	1.03×10^{-27}	1.73×10^{-39}	9.08×10^{-65}
50	51	1.49×10^{-12}	4.43×10^{-19}	1.32×10^{-25}	9.24×10^{-37}	2.37×10^{-60}
	100	1.71×10^{-12}	6.92×10^{-19}	2.41×10^{-25}	2.36×10^{-36}	1.62×10^{-58}
	500	1.24×10^{-14}	3.26×10^{-22}	6.45×10^{-30}	2.06×10^{-43}	1.20×10^{-70}

In Table 7, the asymptotic chi-square quantiles are made available for the common chi-square asymptotic approximation for l.r. statistics, now with $p(p+3)/2-2$ degrees of freedom, as well as the near-exact quantiles for nW_4. Similar conclusions to those drawn in the previous sections apply here.

6. The Likelihood Ratio Test for the Simultaneous Test of Equality of Means and Sphericity of the Covariance Matrix

If $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, where $\underline{\mu} = [\mu_1, \ldots, \mu_p]'$, and we are interested in testing the null hypothesis

$$H_0 : \mu_1 = \cdots = \mu_p; \qquad \Sigma = \sigma^2 I_p \quad \left(\text{with } \sigma^2 \text{ unspecified}\right), \tag{58}$$

we may write

$$H_0 \equiv H_{02|01} \circ H_{01}, \tag{59}$$

where

$$H_{02|01} : \mu_1 = \cdots = \mu_p, \text{ assuming } \Sigma = \sigma^2 I_p, \qquad H_{01} : \Sigma = \sigma^2 I_p, \tag{60}$$

where, for a sample of size n, the $(2/n)$th power of the l.r. statistic to test $H_{02|01}$, versus an alternative hypothesis that assumes sphericity for the covariance matrix and no structure for the mean vector, may be shown to be (see Appendix C for details)

$$\Lambda_{2|1} = \left(\frac{\operatorname{tr}(A)}{\operatorname{tr}(A_0)}\right)^p, \tag{61}$$

where A is the matrix in (27) and

$$A_0 = \left(X - E_{n1}\underline{\hat{\mu}}'\right)'\left(X - E_{n1}\underline{\hat{\mu}}'\right), \tag{62}$$

with

$$\underline{\hat{\mu}} = \frac{1}{p}E_{p1}E_{1p}\underline{\overline{X}} = \frac{1}{p}E_{pp}\underline{\overline{X}}. \tag{63}$$

We have

$$\operatorname{tr}(A) = \sum_{j=1}^{p}\sum_{i=1}^{n}\left(X_{ij} - \overline{X}_j\right)^2, \tag{64}$$

$$\begin{aligned}\operatorname{tr}(A_0) &= \sum_{j=1}^{p}\sum_{i=1}^{n}\left(X_{ij} - \overline{X}\right)^2 \\ &= \underbrace{\sum_{j=1}^{p}\sum_{i=1}^{n}\left(X_{ij} - \overline{X}_j\right)^2}_{A^*} + \underbrace{n\sum_{j=1}^{p}\left(\overline{X}_j - \overline{X}\right)^2}_{B^*},\end{aligned} \tag{65}$$

where $X = [X_{ij}]$ $(i = 1, \ldots, n;\ j = 1, \ldots, p)$ is the sample matrix and

$$\overline{X} = \frac{1}{p}\sum_{j=1}^{p}\overline{X}_j, \quad \text{with } \overline{X}_j = \frac{1}{n}\sum_{i=1}^{n}X_{ij}. \tag{66}$$

In (65), from standard theory on normal r.v.'s, since $X_{ij} \sim N(\mu_j, \sigma^2)$, independent for $i = 1, \ldots, n$,

$$\frac{\sum_{i=1}^{n}\left(X_{ij} - \overline{X}_j\right)^2}{\sigma^2} \sim \chi^2_{n-1}, \quad (j = 1, \ldots, p), \tag{67}$$

Table 7: Quantiles of orders $\alpha = 0.95$ and $\alpha = 0.99$ for the chi-square approximation and for the near-exact distributions that match $m = 2, 6$, or 10 exact moments, of $-n \log \Lambda_4$ for the l.r. statistic Λ_4 in (51), for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	Near-exact distributions m 2	6	10	χ^2
			$\alpha = 0.95$		
10					82.52872654147179311367711677
	11	190.32756331545176062637260063	190.32756312341501096656402680	190.32756312341502158305183289	
	60	88.61292917028511318678026436	88.61292916079636593558509765	88.61292916079636591605025196	
	460	83.25135114394460784736871943	83.25135114392165525365869519	83.25135114392165525365868449	
15					160.91477802694323358920183897
	16	364.22508588466772907687245064	364.22508590299516153552837462	364.22508590299516166009461668	
	65	176.92458777551078581129813667	176.92458777274377296466188482	176.92458777274377296219866202	
	465	162.88835115579315719900816060	162.88835115578197146107641778	162.88835115578197146107641501	
25					392.50115579019226559339157016
	26	865.97128758699096427158883014	865.97128759463038534181481168	865.97128759463038534192831533	
	75	449.82482829542519016999435848	449.82482829549700283679630302	449.82482829549700283676131146	
	475	400.06750221064414330250460763	400.06750221064079210854726486	400.06750221064079210854726472	
50					1408.73177341938097671969938629
	51	3008.85497566956079322838948966	3008.85497567039266902344201858	3008.85497567039266902344199893	
	100	1744.77297987136924686391338212	1744.77297987182100521614403517	1744.77297987182100521614409508	
	500	1459.87821714249309834106087509	1459.87821714249577662643142818	1459.87821714249577662643142818	
			$\alpha = 0.99$		
10					92.01002361413199132182815244
	11	226.63967070294372053364533245	226.63967043727843805017814101	226.63967043727844439241178794	
	60	98.81015393413627584334298342	98.81015389070350242742117425	98.81015389070350239170134559	
	460	92.81589286301973694456386019	92.81589286290397669965973982	92.81589286290397669965971186	
15					173.85385431432844593057930059
	16	417.33284870330949699905167521	417.33284873028538888270377699	417.33284873028538894546001788	
	65	191.19301774559926635976479112	191.19301773577866477925423064	191.19301773577866477873083849	
	465	175.98671397271816998582919704	175.98671397267474771413147408	175.98671397267474771413147254	
25					412.29747839373905678663071280
	26	952.92291432362881946300916855	952.92291433479892788245985114	952.92291433479892788251876064	
	75	472.64955572272794762005618804	472.64955572304282388887164444	472.64955572304282388889904345	
	475	420.24758865175907740683966303	420.24758865174822800295256496	420.24758865174822800295256503	
50					1445.59855177795512096240506173
	51	3181.37208704154942688955747637	3181.37208704280924834914384041	3181.37208704280924834914382694	
	100	1791.13162751307828413931876980	1791.13162751434597504689343387	1791.13162751434597504689332917	
	500	1498.09706702524552623910040811	1498.09706702525328984531816943	1498.09706702525328984531816942	

and, since, under $H_{02|01}$, we have $\overline{X}_j \sim N(\mu, \sigma^2/n)$, i.i.d. for $j = 1, \ldots, p$, under this null hypothesis,

$$\frac{\sum_{j=1}^{p} \left(\overline{X}_j - \overline{X}\right)^2}{\sigma^2/n} \sim \chi^2_{p-1}. \tag{68}$$

Thus, since the r.v.'s in (67) are independent for $j = 1, \ldots, p$,

$$\frac{A^*}{\sigma^2} \sim \chi^2_{(n-1)p}, \tag{69}$$

while, from (68),

$$\frac{B^*}{\sigma^2} \sim \chi^2_{p-1}. \tag{70}$$

Since A^* and B^* are independent, given that A^* is independent of all $\overline{X}_j$ $(j = 1, \ldots, p)$ and B^* is defined only from the $\overline{X}_j$, then

$$\frac{\operatorname{tr}(A)}{\operatorname{tr}(A_0)} = \frac{A^*}{A^* + B^*} \sim \operatorname{Beta}\left(\frac{(n-1)p}{2}, \frac{p-1}{2}\right), \tag{71}$$

$$\Lambda_{2|1} \sim \left(\operatorname{Beta}\left(\frac{(n-1)p}{2}, \frac{p-1}{2}\right)\right)^p. \tag{72}$$

From [6, 11] and [12, sec. 10.7], the $(2/n)$th power of the l.r. statistic to test H_{01} in (60) is given by

$$\Lambda_1 = \frac{|A|}{(\operatorname{tr}((1/p) A))^p}, \tag{73}$$

with (see [6])

$$\Lambda_1 \stackrel{d}{\equiv} \prod_{j=2}^{p} Y_j, \tag{74}$$

where, for $j = 2, \ldots, p$,

$$Y_j \sim \operatorname{Beta}\left(\frac{n-j}{2}, \frac{j-1}{p} + \frac{j-1}{2}\right) \tag{75}$$

are a set of $p-1$ independent r.v.'s.

The $(2/n)$th power of the l.r. statistic to test H_0 in (60) is thus

$$\Lambda_5 = \Lambda_{2|1}\Lambda_1 = \frac{|A|}{(\operatorname{tr}((1/p) A_0))^p}, \tag{76}$$

where, from Theorem 5 in [10], we may assure the independence between $\Lambda_{2|1}$ and Λ_1 and as such say that

$$\Lambda_5 \stackrel{d}{\equiv} \left\{\prod_{j=2}^{p} Y_j\right\} Y^*, \tag{77}$$

where all r.v.'s are independent, Y_j are the r.v.'s in (75), and Y^* is a r.v. with the same distribution as $\Lambda_{2|1}$ in (72).

Let us take $W_5 = -\log \Lambda_5$, $W_2 = -\log \Lambda_{2|1}$, and $W_1 = -\log \Lambda_1$. Then we will have

$$\Phi_{W_5}(t) = \Phi_{W_1}(t)\,\Phi_{W_2}(t), \tag{78}$$

where, using (36), we may write, for odd p,

$$\begin{aligned}
\Phi_{W_2}(t) &= E\left(e^{tW_2}\right) = E\left(e^{-t\log\Lambda_{2|1}}\right) = E\left(\Lambda_{2|1}^{-it}\right) \\
&= \frac{\Gamma((np-1)/2)\,\Gamma((np-p)/2 - pit)}{\Gamma((np-1)/2 - pit)\,\Gamma((np-p)/2)} \\
&= \prod_{\ell=0}^{(p-1)/2-1} \left(\frac{np-p}{2} + \ell\right)\left(\frac{np-p}{2} - pit + \ell\right)^{-1} \\
&= \prod_{j=0}^{(p-1)/2-1} \left(\frac{np-p+2j}{2}\right)\left(\frac{np-p+2j}{2} - pit\right)^{-1} \\
&= \prod_{j=0}^{(p-1)/2-1} \left(\frac{n-1+(2j)/p}{2}\right) \\
&\quad \cdot \left(\frac{n-1+(2j)/p}{2} - it\right)^{-1},
\end{aligned} \tag{79}$$

and, for even p, following similar steps,

$$\begin{aligned}
\Phi_{W_2}(t) &= \frac{\Gamma((np-1)/2)\,\Gamma((np-p)/2 - pit)}{\Gamma((np-1)/2 - pit)\,\Gamma((np-p)/2)} \\
&= \frac{\Gamma((np-1)/2)\,\Gamma((np-2)/2 - pit)}{\Gamma((np-1)/2 - pit)\,\Gamma((np-2)/2)} \frac{\Gamma((np-2)/2)\,\Gamma((np-p)/2 - pit)}{\Gamma((np-2)/2 - pit)\,\Gamma((np-p)/2)} \\
&= \frac{\Gamma((np-1)/2)\,\Gamma((np-2)/2 - pit)}{\Gamma((np-1)/2 - pit)\,\Gamma((np-2)/2)} \left\{\prod_{\ell=0}^{p/2-2} \left(\frac{np-p}{2} + \ell\right)\left(\frac{np-p}{2} - pit + \ell\right)^{-1}\right\} \\
&= \frac{\Gamma((np-1)/2)\,\Gamma((np-2)/2 - pit)}{\Gamma((np-1)/2 - pit)\,\Gamma((np-2)/2)} \left\{\prod_{j=0}^{p/2-2} \left(\frac{n-1+(2j)/p}{2}\right)\left(\frac{n-1+(2j)/p}{2} - it\right)^{-1}\right\}.
\end{aligned} \tag{80}$$

Taking for $\Phi_{W_1}(t)$ the expression for $\Phi_{W_3}(t)$ in (A.6) in [6], we may write

$$\Phi_{W_5}(t) = \underbrace{\left\{\prod_{j=1}^{p-k^*}\frac{\Gamma((n-1)/2+(j-1)/p)\Gamma((n-1)/2-it)}{\Gamma((n-1)/2+(j-1)/p-it)\Gamma((n-1)/2)}\right\}\left\{\prod_{j=p-k^*+1}^{p}\frac{\Gamma((n-1)/2+(j-1)/p)\Gamma(n/2-it)}{\Gamma((n-1)/2+(j-1)/p-it)\Gamma(n/2)}\right\}\left(\frac{\Gamma((np-1)/2)\Gamma((np-2)/2-pit)}{\Gamma((np-1)/2-pit)\Gamma((np-2)/2)}\right)^{(p+1)\bmod 2}}_{\Phi_{W_5,1}(t)}$$
$$\cdot\underbrace{\left\{\prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j}\right\}\left\{\prod_{j=1}^{\lfloor(p-1)/2\rfloor-1}\left(\frac{n-1+(2j)/p}{2}\right)\left(\frac{n-1+(2j)/p}{2}-it\right)^{-1}\right\}}_{\Phi_{W_5,2}(t)}, \quad (81)$$

where $k^* = \lfloor p/2 \rfloor$,

$$r_j = \begin{cases} 1, & j=1, \\ \left\lfloor \dfrac{p-j+2}{2} \right\rfloor, & j=2,\dots,p. \end{cases} \quad (82)$$

Then, following a similar approach to the one used for W_3 and Λ_3, in Section 4, we obtain near-exact distributions with a somewhat similar structure to those in that section, now with

$$r = \frac{(p+1)\bmod 2}{2} + \sum_{j=1}^{p-k^*}\frac{j-1}{p} + \sum_{j=p-k^*+1}^{p}\left(\frac{j-1}{p}-\frac{1}{2}\right) = \frac{p-p\bmod 2}{4} \quad (83)$$

and with λ^* determined as the solution of a system of equations similar to the one in (40), with $\Phi_{W_3,1}$ replaced by $\Phi_{W_5,1}$.

This will yield for W_5 near-exact distributions which are mixtures of $m+1$ GIG or GNIG distributions, according to the fact that r is an integer or a noninteger, of depth $p+\lfloor(p-1)/2\rfloor$, with shape parameters

$$\underbrace{1,\dots,1}_{\lfloor(p-1)/2\rfloor-1}, r_1,\dots,r_p, r+j \quad (j=0,\dots,m), \quad (84)$$

with r_j by (82) and r given by (83), and corresponding rate parameters

$$\underbrace{\frac{n-1+2/p}{2},\dots,\frac{n-1+2\lfloor(p-1)/2\rfloor-2}{2}}_{\lfloor(p-1)/2\rfloor-1,\text{ with step }2/p}, \underbrace{\frac{n-1}{2},\dots,\frac{n-p}{2}}_{p}, \lambda^*, \quad (85)$$

with p.d.f.'s and c.d.f.'s, respectively, given by

$$f^*_{W_5}(w) = \sum_{j=0}^{m}\pi_j f^{\text{GNIG}}\Bigg(w \mid \underbrace{1,\dots,1}_{\lfloor(p-1)/2\rfloor-1}, r_1,\dots,r_p, r+j; \underbrace{\frac{n-1+2/p}{2},\dots,\frac{n-1+2\lfloor(p-1)/2\rfloor-2}{2}}_{\lfloor(p-1)/2\rfloor-1,\text{ with step }2/p}, \underbrace{\frac{n-1}{2},\dots,\frac{n-p}{2}}_{p}, \lambda^*; p+\left\lfloor\frac{p-1}{2}\right\rfloor\Bigg),$$

$$F^*_{W_5}(w) = \sum_{j=0}^{m}\pi_j F^{\text{GNIG}}\Bigg(w \mid \underbrace{1,\dots,1}_{\lfloor(p-1)/2\rfloor-1}, r_1,\dots,r_p, r+j; \underbrace{\frac{n-1+2/p}{2},\dots,\frac{n-1+2\lfloor(p-1)/2\rfloor-2}{2}}_{\lfloor(p-1)/2\rfloor-1,\text{ with step }2/p}, \underbrace{\frac{n-1}{2},\dots,\frac{n-p}{2}}_{p}, \lambda^*; p+\left\lfloor\frac{p-1}{2}\right\rfloor\Bigg), \quad (86)$$

and for Λ_5 with p.d.f.'s and c.d.f.'s, respectively, given by

$$f^*_{\Lambda_5}(w) = \sum_{j=0}^{m}\pi_j f^{\text{GNIG}}\Bigg(-\log\ell \mid \underbrace{1,\dots,1}_{\lfloor(p-1)/2\rfloor-1}, r_1,\dots,r_p, r+j; \underbrace{\frac{n-1+2/p}{2},\dots,\frac{n-1+2\lfloor(p-1)/2\rfloor-2}{2}}_{\lfloor(p-1)/2\rfloor-1,\text{ with step }2/p}, \underbrace{\frac{n-1}{2},\dots,\frac{n-p}{2}}_{p}, \lambda^*; p+\left\lfloor\frac{p-1}{2}\right\rfloor\Bigg)\frac{1}{\ell},$$

$$F^*_{\Lambda_5}(w) = \sum_{j=0}^{m}\pi_j\Bigg(1-F^{\text{GNIG}}\Bigg(-\log\ell \mid \underbrace{1,\dots,1}_{\lfloor(p-1)/2\rfloor-1}, r_1,$$

Table 8: Values of the measure Δ in (5), for the near-exact distributions of the l.r. test statistic Λ_5 in (76), which match m exact moments, for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	m				
		2	4	6	10	20
10	11	1.64×10^{-8}	1.39×10^{-12}	4.04×10^{-16}	3.00×10^{-23}	9.74×10^{-33}
	60	7.64×10^{-10}	6.34×10^{-15}	1.40×10^{-19}	2.69×10^{-28}	1.99×10^{-45}
	460	1.92×10^{-12}	2.93×10^{-19}	1.15×10^{-25}	6.96×10^{-38}	2.56×10^{-63}
15	16	2.28×10^{-10}	5.87×10^{-15}	1.29×10^{-18}	4.62×10^{-24}	1.31×10^{-35}
	65	4.07×10^{-11}	4.01×10^{-16}	1.45×10^{-20}	1.66×10^{-28}	3.79×10^{-43}
	465	1.38×10^{-13}	3.10×10^{-20}	2.73×10^{-26}	3.09×10^{-37}	6.60×10^{-61}
25	26	1.35×10^{-11}	5.36×10^{-17}	5.76×10^{-22}	1.44×10^{-29}	2.91×10^{-45}
	75	7.41×10^{-12}	1.74×10^{-17}	1.62×10^{-22}	1.83×10^{-31}	5.01×10^{-49}
	475	4.09×10^{-14}	2.98×10^{-21}	9.09×10^{-28}	1.49×10^{-39}	3.64×10^{-65}
50	51	1.73×10^{-12}	2.51×10^{-19}	1.28×10^{-25}	7.57×10^{-38}	8.22×10^{-64}
	100	3.12×10^{-12}	5.88×10^{-19}	3.52×10^{-25}	2.54×10^{-37}	8.43×10^{-65}
	500	4.28×10^{-14}	4.66×10^{-22}	1.59×10^{-29}	3.78×10^{-44}	1.17×10^{-76}

$$\ldots, r_p, r+j;\ \underbrace{\frac{n-1+2/p}{2}, \ldots, \frac{n-1+2\lfloor (p-1)/2 \rfloor - 2}{2}}_{\lfloor (p-1)/2 \rfloor - 1,\text{ with step } 2/p},\ \underbrace{\frac{n-1}{2}, \ldots, \frac{n-p}{2}}_{p}, \lambda^*; p + \left\lfloor \frac{p-1}{2} \right\rfloor \Bigg)\Bigg). \tag{87}$$

We should note that as it happens with Λ_3 and W_3, also for W_5 and Λ_5, r may be either an integer or a half-integer, so that, in those cases where r is an integer, the near-exact distributions are mixtures of GIG distributions, while when r is noninteger, they are mixtures of GNIG distributions.

In Table 8 are displayed the values of the measure Δ in (5) for the near-exact distributions developed for W_5 and Λ_5 and in Table 9 we may find the chi-square asymptotic quantiles for nW_5, based on a chi-square distribution with $p(p+3)/2 - 2$ degrees of freedom and the quantiles for the near-exact distributions with $m = 2$, 6, and 10. Similar conclusions to those drawn for the asymptotic and near-exact distributions for the l.r. statistics in the previous sections also apply here.

7. The Likelihood Ratio Test for the Simultaneous Test of Nullity of Means and Sphericity of the Covariance Matrix

We now assume $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, and we now want to test the null hypothesis

$$H_0 : \underline{\mu} = \underline{0}; \quad \Sigma = \sigma^2 I_p \quad \left(\text{with } \sigma^2 \text{ unspecified}\right), \tag{88}$$

which may be written as

$$H_0 \equiv H_{02|01} \circ H_{01}, \tag{89}$$

where

$$H_{02|01} : \underline{\mu} = \underline{0},\ \text{assuming } \Sigma = \sigma^2 I_p, \qquad H_{01} : \Sigma = \sigma^2 I_p. \tag{90}$$

For a sample of size n, the $(2/n)$th power of the l.r. statistic to test $H_{02|01}$, versus an alternative hypothesis that assumes sphericity for the covariance matrix and no structure for the mean vector, may be shown to be (see Appendix D for details)

$$\Lambda_{2|1} = \left(\frac{\operatorname{tr}(A)}{\operatorname{tr}(A_0)} \right)^p, \tag{91}$$

where A is the matrix in (27) and now

$$A_0 = X' X, \tag{92}$$

which was already used in Section 5.

We now have

$$\operatorname{tr}(A_0) = \sum_{j=1}^{p} \sum_{i=1}^{n} \left(X_{ij}\right)^2 = \underbrace{\sum_{j=1}^{p} \sum_{i=1}^{n} \left(X_{ij} - \overline{X}_j\right)^2}_{A^*} + \underbrace{n \sum_{j=1}^{p} \left(\overline{X}_j\right)^2}_{B^{**}}, \tag{93}$$

where A^* is the r.v. defined in (65), $X = [X_{ij}]$ $(i = 1, \ldots, n;\ j = 1, \ldots, p)$ is the sample matrix, and

$$\overline{X}_j = \frac{1}{n} \sum_{i=1}^{n} X_{ij}. \tag{94}$$

Table 9: Quantiles of orders $\alpha = 0.95$ and $\alpha = 0.99$ for the chi-square approximation and for the near-exact distributions that match $m = 2, 6$, or 10 exact moments, of $-n \log \Lambda_5$ for the l.r. statistic Λ_5 in (76), for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	Near-exact distributions m = 2	Near-exact distributions m = 6	Near-exact distributions m = 10	χ^2
			$\alpha = 0.95$		
10					82.52872654147179311367711677
	11	190.34577352567912574418958472	190.34577224119045860164933939	190.34577224119046305269448848	
	60	88.61692747841779863573642579	88.61692745278611013951753747	88.61692745278611013324695622	
	460	83.25188239392602534312672397	83.25188239386900029725019512	83.25188239386900029725019084	
15					160.91477802694323358920183897
	16	364.24055367674081921965459933	364.24055364729180144370579916	364.24055364729180141324533927	
	65	176.92888004902098965390468116	176.92888004652298399068321382	176.92888004652298398935665716	
	465	162.88896091620593405370503267	162.88896091619839192532218731	162.88896091619839192532218500	
25					392.50115579019226559339157016
	26	865.98217181891738253073510565	865.98217181564421769408452503	865.98217181564421769411599634	
	75	449.82888066877263241039729095	449.82888066790844106600627651	449.82888066790844106598248210	
	475	400.06815103634197188114099440	400.06815103633783183795559146	400.06815103633783183795559133	
50					1408.73177341938097671969938629
	51	3008.86102439239945267715486761	3008.86102439143486337077345854	3008.86102439143486337077347761	
	100	1744.77616968248446695304159926	1744.77616968166143981696831893	1744.77616968166143981696823169	
	500	1459.87886196492823624335068318	1459.87886196491897609569196780	1459.87886196491897609569196779	
			$\alpha = 0.99$		
10					92.01002361413199132182815244
	11	226.65802233268160965656246440	226.65802050594440639840352032	226.65802050594440942100275312	
	60	98.81456601487869161798863928	98.81456589548535463063927369	98.81456589548535461913454163	
	460	92.81648438468645753157179639	92.81648438439817704231467008	92.81648438439817704231465890	
15					173.85385431432844593057930059
	16	417.34840709478673000029559780	417.34840705292241842439350332	417.34840705292241840622917724	
	65	191.19761414641994048857514633	191.19761413728546916412651829	191.19761413728546916385955487	
	465	175.98737199442095758355328501	175.98737199439157553407869592	175.98737199439157553407869464	
25					412.29747839373905678663071280
	26	952.93384227143576783559100319	952.93384226668216247367925905	952.93384226668216247369526304	
	75	472.65378089035668898963824328	472.65378088767304149154578277	472.65378088767304149156442088	
	475	420.24826946954476567253237555	420.24826946953125669939828606	420.24826946953125669939828612	
50					1445.59855177795512096240506173
	51	3181.37814988860070147625187269	3181.37814988714219606939294787	3181.37814988714219606939296092	
	100	1791.13488232226638659185402987	1791.13488231997376953941120636	1791.13488231997376953941135956	
	500	1498.09772807257343724270969706	1498.09772807254669066052003540	1498.09772807254669066052003541	

In (93), from standard theory on normal r.v.'s, since under $H_{02|01}$ we have $\overline{X}_j \sim N(0,\sigma^2/n)$, independent for $j=1,\ldots,p$, under this null hypothesis,

$$\frac{\sum_{j=1}^{p}\left(\overline{X}_j\right)^2}{\sigma^2/n} = \frac{B^{**}}{\sigma^2} \sim \chi^2_p. \tag{95}$$

Since A^* and B^{**} are independent, given that A^* is independent of all $\overline{X}_j$ $(j=1,\ldots,p)$ and B^{**} is defined only from $\overline{X}_j$, then, given the distribution of A^* in (69),

$$\frac{\operatorname{tr}(A)}{\operatorname{tr}(A_0)} = \frac{A^*}{A^*+B^{**}} \sim \operatorname{Beta}\left(\frac{(n-1)p}{2},\frac{p}{2}\right), \tag{96}$$

$$\Lambda_{2|1} \sim \left(\operatorname{Beta}\left(\frac{(n-1)p}{2},\frac{p}{2}\right)\right)^p. \tag{97}$$

Since H_{01} is the same hypothesis as H_{01} in Section 6, the $(2/n)$th power of the l.r. statistic to test H_0 in (90) is thus

$$\Lambda_6 = \Lambda_{2|1}\Lambda_1 = \frac{|A|}{\left(\operatorname{tr}\left((1/p)A_0\right)\right)^p}, \tag{98}$$

where A is given by (27) and A_0 by (92) and where, from Theorem 5 in [10], we may assure the independence between $\Lambda_{2|1}$ and Λ_1, so that

$$\Lambda_6 \stackrel{d}{\equiv} \left\{\prod_{j=2}^{p} Y_j\right\} Y^*, \tag{99}$$

where all r.v.'s are independent, Y_j are the r.v.'s in (75), and Y^* is a r.v. with the same distribution as $\Lambda_{2|1}$ in (97).

Then, if we take $W_6 = -\log\Lambda_6$, $W_2 = -\log\Lambda_{2|1}$, and $W_1 = -\log\Lambda_1$, we may write

$$\Phi_{W_6}(t) = \Phi_{W_1}(t)\,\Phi_{W_2}(t), \tag{100}$$

where, using (36), we may write, for even p,

$$\begin{aligned}
\Phi_{W_2}(t) &= E\left(e^{tW_2}\right) = E\left(e^{-t\log\Lambda_{2|1}}\right) = E\left(\Lambda_{2|1}^{-it}\right) \\
&= \frac{\Gamma(np/2)\,\Gamma((np-p)/2-pit)}{\Gamma(np/2-pit)\,\Gamma((np-p)/2)} \\
&= \prod_{\ell=0}^{p/2-1}\left(\frac{np-p}{2}+\ell\right)\left(\frac{np-p}{2}-pit+\ell\right)^{-1} \\
&= \prod_{j=0}^{p/2-1}\left(\frac{np-p+2j}{2}\right)\left(\frac{np-p+2j}{2}-pit\right)^{-1} \\
&= \prod_{j=0}^{p/2-1}\left(\frac{n-1+(2j)/p}{2}\right) \\
&\quad\cdot\left(\frac{n-1+(2j)/p}{2}-it\right)^{-1},
\end{aligned} \tag{101}$$

while, for odd p, we may write

$$\begin{aligned}
\Phi_{W_2}(t) &= \frac{\Gamma(np/2)\,\Gamma((np-p)/2-pit)}{\Gamma(np/2-pit)\,\Gamma((np-p)/2)} = \frac{\Gamma(np/2)\,\Gamma((np-1)/2-pit)}{\Gamma(np/2-pit)\,\Gamma((np-1)/2)}\,\frac{\Gamma((np-1)/2)\,\Gamma((np-p)/2-pit)}{\Gamma((np-1)/2-pit)\,\Gamma((np-p)/2)} \\
&= \frac{\Gamma(np/2)\,\Gamma((np-1)/2-pit)}{\Gamma(np/2-pit)\,\Gamma((np-1)/2)}\left\{\prod_{\ell=0}^{(p-1)/2-1}\left(\frac{np-p}{2}+\ell\right)\left(\frac{np-p}{2}-pit+\ell\right)^{-1}\right\} \\
&= \frac{\Gamma(np/2)\,\Gamma((np-1)/2-pit)}{\Gamma(np/2-pit)\,\Gamma((np-1)/2)}\left\{\prod_{j=0}^{(p-1)/2-1}\left(\frac{n-1+(2j)/p}{2}\right)\left(\frac{n-1+(2j)/p}{2}-it\right)^{-1}\right\}.
\end{aligned} \tag{102}$$

Following then a similar procedure to the one used in the previous section, in order to build near-exact distributions for W_6 and Λ_6, we take for $\Phi_{W_1}(t)$ the expression for $\Phi_{W_3}(t)$ in (A.6) in [6] and write

$$\begin{aligned}
&\Phi_{W_6}(t) \\
&= \underbrace{\left\{\prod_{j=1}^{p-k^*}\frac{\Gamma((n-1)/2+(j-1)/p)\,\Gamma((n-1)/2-it)}{\Gamma((n-1)/2+(j-1)/p-it)\,\Gamma((n-1)/2)}\right\}\left\{\prod_{j=p-k^*+1}^{p}\frac{\Gamma((n-1)/2+(j-1)/p)\,\Gamma(n/2-it)}{\Gamma((n-1)/2+(j-1)/p-it)\,\Gamma(n/2)}\right\}\left(\frac{\Gamma(np/2)\,\Gamma((np-1)/2-pit)}{\Gamma(np/2-pit)\,\Gamma((np-1)/2)}\right)^{p \bmod 2}}_{\Phi_{W_6,1}(t)} \\
&\quad\cdot\underbrace{\left\{\prod_{j=1}^{p}\left(\frac{n-j}{2}\right)^{r_j}\left(\frac{n-j}{2}-it\right)^{-r_j}\right\}\left\{\prod_{j=1}^{\lfloor p/2\rfloor-1}\left(\frac{n-1+(2j)/p}{2}\right)\left(\frac{n-1+(2j)/p}{2}-it\right)^{-1}\right\}}_{\Phi_{W_6,2}(t)},
\end{aligned} \tag{103}$$

where $k^* = \lfloor p/2\rfloor$, and the r_j are defined in (82).

Then, we may obtain near-exact distributions for W_6 and Λ_6, with a similar structure to those in the previous section, now with

$$r = \frac{p \bmod 2}{2} + \sum_{j=1}^{p-k^*} \frac{j-1}{p} + \sum_{j=p-k^*+1}^{p} \left(\frac{j-1}{p} - \frac{1}{2} \right) = \frac{p - 2 + 3(p \bmod 2)}{4}, \tag{104}$$

with λ^* determined as the solution of a system of equations similar to the one in (40), with $\Phi_{W_3,1}$ replaced by $\Phi_{W_6,1}$.

This yields for W_6 near-exact distributions which are mixtures of $m+1$ GIG or GNIG distributions, according to the fact that r is an integer or a noninteger, of depth $p + \lfloor p/2 \rfloor$, with shape parameters

$$\underbrace{1, \ldots, 1}_{\lfloor p/2 \rfloor - 1}, r_1, \ldots, r_p, r + j \quad (j = 0, \ldots, m), \tag{105}$$

with r_j by (82) and r given by (104), and corresponding rate parameters

$$\underbrace{\frac{n-1+2/p}{2}, \ldots, \frac{n-1+2\lfloor p/2 \rfloor - 2}{2}}_{\lfloor p/2 \rfloor - 1, \text{ with step } 2/p}, \underbrace{\frac{n-1}{2}, \ldots, \frac{n-p}{2}}_{p}, \lambda^*, \tag{106}$$

with p.d.f.'s and c.d.f.'s, respectively, given by

$$f^*_{W_6}(w) = \sum_{j=0}^{m} \pi_j f^{\mathrm{GNIG}} \left(w \mid \underbrace{1, \ldots, 1}_{\lfloor p/2 \rfloor - 1}, r_1, \ldots, r_p, r + j; \underbrace{\frac{n-1+2/p}{2}, \ldots, \frac{n-1+2\lfloor p/2 \rfloor - 2}{2}}_{\lfloor p/2 \rfloor - 1, \text{ with step } 2/p}, \underbrace{\frac{n-1}{2}, \ldots, \frac{n-p}{2}}_{p}, \lambda^*; p + \left\lfloor \frac{p}{2} \right\rfloor \right),$$
$$F^*_{W_6}(w) = \sum_{j=0}^{m} \pi_j F^{\mathrm{GNIG}} \left(w \mid \underbrace{1, \ldots, 1}_{\lfloor p/2 \rfloor - 1}, r_1, \ldots, r_p, r + j; \underbrace{\frac{n-1+2/p}{2}, \ldots, \frac{n-1+2\lfloor p/2 \rfloor - 2}{2}}_{\lfloor p/2 \rfloor - 1, \text{ with step } 2/p}, \underbrace{\frac{n-1}{2}, \ldots, \frac{n-p}{2}}_{p}, \lambda^*; p + \left\lfloor \frac{p}{2} \right\rfloor \right), \tag{107}$$

and for Λ_6 with p.d.f.'s and c.d.f.'s, respectively, given by

$$f^*_{\Lambda_6}(w) = \sum_{j=0}^{m} \pi_j f^{\mathrm{GNIG}} \left(-\log \ell \mid \underbrace{1, \ldots, 1}_{\lfloor p/2 \rfloor - 1}, r_1, \ldots, r_p, r + j; \underbrace{\frac{n-1+2/p}{2}, \ldots, \frac{n-1+2\lfloor p/2 \rfloor - 2}{2}}_{\lfloor p/2 \rfloor - 1, \text{ with step } 2/p}, \underbrace{\frac{n-1}{2}, \ldots, \frac{n-p}{2}}_{p}, \lambda^*; p + \left\lfloor \frac{p}{2} \right\rfloor \right) \frac{1}{\ell},$$
$$F^*_{\Lambda_6}(w) = \sum_{j=0}^{m} \pi_j \left(1 - F^{\mathrm{GNIG}} \left(-\log \ell \mid \underbrace{1, \ldots, 1}_{\lfloor p/2 \rfloor - 1}, r_1, \ldots, r_p, r + j; \underbrace{\frac{n-1+2/p}{2}, \ldots, \frac{n-1+2\lfloor p/2 \rfloor - 2}{2}}_{\lfloor p/2 \rfloor - 1, \text{ with step } 2/p}, \underbrace{\frac{n-1}{2}, \ldots, \frac{n-p}{2}}_{p}, \lambda^*; p + \left\lfloor \frac{p}{2} \right\rfloor \right) \right). \tag{108}$$

We should note that as it happens with the statistics in Sections 3–6, also for W_6 and Λ_6, r may be either an integer or a half-integer, so that, in those cases where r is an integer, the near-exact distributions are mixtures of GIG distributions, while when r is noninteger, they are mixtures of GNIG distributions.

In Tables 10 and 11 are displayed the values of the measure Δ in (5) for the near-exact distributions developed for W_6 and Λ_6 and the chi-square asymptotic quantiles for nW_6, based on a chi-square distribution with $p(p+3)/2 - 1$ degrees of freedom, together with the quantiles for the near-exact distributions with $m = 2$, 6, and 10. In these tables we may observe the same developments discovered in previous sections. Although in Table 10 we may observe a slight increase in the values of the measure Δ when we go from $p = 10$ to $p = 15$ as well as when we compare the near-exact distributions that match only 2 exact moments for $p = 10$ and $p = 25$, the near-exact distributions developed end up having a sharp asymptotic behavior for increasing p, which is clearly visible when we compare the values of Δ for $p = 50$ with those for any other p.

8. Conclusions

The near-exact approximations developed in this paper allow the practical and precise implementation of simultaneous tests on conditions on mean vectors and of patterns on covariance matrices. These approximations are based on mixtures of Generalized Near-Integer Gamma or Generalized Integer Gamma distributions which are highly manageable and for which there are computational modules available on the Internet: https://sites.google.com/site/nearexactdistributions/home. Numerical studies show the quality and accuracy of

Table 10: Values of the measure Δ in (5), for the near-exact distributions of the l.r. test statistic Λ_6 in (98), which match m exact moments, for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	m				
		2	4	6	10	20
10	11	1.14×10^{-10}	5.65×10^{-14}	6.84×10^{-17}	3.17×10^{-22}	3.95×10^{-33}
	60	2.53×10^{-11}	2.02×10^{-16}	4.05×10^{-21}	7.58×10^{-30}	5.45×10^{-45}
	460	6.89×10^{-14}	1.10×10^{-20}	5.17×10^{-27}	1.29×10^{-38}	3.73×10^{-64}
15	16	2.69×10^{-9}	5.09×10^{-14}	5.26×10^{-18}	3.34×10^{-24}	3.53×10^{-36}
	65	4.00×10^{-10}	1.68×10^{-15}	4.58×10^{-20}	3.83×10^{-28}	2.62×10^{-43}
	465	1.33×10^{-12}	1.25×10^{-19}	7.40×10^{-26}	3.82×10^{-37}	9.27×10^{-61}
25	26	1.30×10^{-10}	3.24×10^{-16}	4.27×10^{-21}	7.05×10^{-30}	2.14×10^{-45}
	75	6.66×10^{-11}	9.02×10^{-17}	6.11×10^{-22}	2.92×10^{-31}	3.64×10^{-49}
	475	3.61×10^{-13}	1.52×10^{-20}	3.12×10^{-27}	1.76×10^{-39}	4.59×10^{-65}
50	51	1.78×10^{-13}	2.02×10^{-20}	5.12×10^{-27}	1.83×10^{-39}	1.14×10^{-63}
	100	3.32×10^{-13}	4.87×10^{-20}	1.59×10^{-26}	1.82×10^{-38}	7.39×10^{-64}
	500	4.68×10^{-15}	3.93×10^{-23}	7.64×10^{-31}	3.73×10^{-45}	6.76×10^{-77}

near-exact distributions developed, contrary to what happens, for example, with usual chi-square approximation. A natural extension of this work will be to develop approximations that will allow the implementation of simultaneous tests on the equality or nullity of several mean vectors and on the equality of the corresponding covariance matrices to a given matrix which may have a specific structure.

Appendices

A. On the Likelihood Ratio Test Statistic to Test the Nullity of Means Assuming Compound Symmetry of the Covariance Matrix

The l.r. statistic to test the null hypotheses

$$H_0 : \underline{\mu} = \underline{0}, \quad \Sigma = \Sigma_{cs} \quad \text{versus } H_1 : \Sigma = \Sigma_{cs} \ \left(\text{and any } \underline{\mu}\right) \tag{A.1}$$

is the l.r. statistic to test the null hypothesis $H_{02|01}$ in (46), which is

$$\Lambda^*_{2|1} = \frac{\max L_0}{\max L_1}, \tag{A.2}$$

where L_0, the likelihood function under H_0 in (A.1) above, is, for Σ_{cs}, defined as in (3), for which we have $|\Sigma_{cs}| = (a - b)^{p-1}(a + (p-1)b)$,

$$L_0 = (2\pi)^{-np/2} \left((a-b)^{p-1}(a+(p-1)b)\right)^{-n/2} \cdot e^{-(1/2)\operatorname{tr}[X'X\Sigma_{cs}^{-1}]}, \tag{A.3}$$

where X is the $n \times p$ sample matrix, and L_1, the likelihood function under H_1 in (A.1), is

$$L_1 = (2\pi)^{-np/2} \left((a-b)^{p-1}(a+(p-1)b)\right)^{-n/2} \cdot e^{-(1/2)\operatorname{tr}[(X-E_{n1}\underline{\mu}')'(X-E_{n1}\underline{\mu}')\Sigma_{cs}^{-1}]}, \tag{A.4}$$

where E_{n1} is an $n \times 1$ vector of 1's.

Since under H_1 we have $\widehat{a}$ and $\widehat{b}$, the m.l.e.'s of a and b given by (28), and the m.l.e. of Σ_{cs} is $\widehat{\Sigma}_{cs} = (1/n)(\widehat{a}I_p + \widehat{b}(E_{pp} - I_p))$, then

$$\widehat{\Sigma}_{cs}^{-1} = \frac{n}{\widehat{a}^2 + (p-2)\widehat{a}\widehat{b} - (p-1)\widehat{b}^2} \left((\widehat{a} + (p-2)\widehat{b}) \cdot I_p - \widehat{b}(E_{pp} - I_p)\right), \tag{A.5}$$

$$\max L_1 = (2\pi)^{-np/2} \left((\widehat{a}-\widehat{b})^{p-1}(\widehat{a}+(p-1)\widehat{b})\right)^{-n/2} \cdot e^{-(1/2)\operatorname{tr}[A\widehat{\Sigma}_{cs}^{-1}]}, \tag{A.6}$$

where $A = (X - E_{n1}\overline{\underline{X}}')(X - E_{n1}\overline{\underline{X}}')$ is the matrix defined in (27), with $\overline{\underline{X}}$, the vector sample means, which is the m.l.e. of $\underline{\mu}$.

In (A.6),

$$\operatorname{tr}\left(A\widehat{\Sigma}_{cs}^{-1}\right) = n \cdot \frac{1}{\widehat{a}^2 + (p-2)\widehat{a}\widehat{b} - (p-1)\widehat{b}^2} \left\{ \sum_{j=1}^{p} a_{jj} \left(\widehat{a} + (p-2)\widehat{b}\right) - \widehat{b} \sum_{\substack{k=1 \\ k \neq j}}^{p} a_{jk} \right\} = n$$

Table 11: Quantiles of orders $\alpha = 0.95$ and $\alpha = 0.99$ for the chi-square approximation and for the near-exact distributions that match $m = 2, 6$, or 10 exact moments, of $-n \log \Lambda_6$ for the l.r. statistic Λ_6 in (98), for different values of p and samples of size $n = p + 1, 50, 450$.

p	n	Near-exact distributions m 2	6	10	χ^2
		$\alpha = 0.95$			
10					83.675260742720985209859769504
	11	191.40417508216086413376362096	191.40417507456156657285101953	191.40417507456156582171789369	
	60	89.75551474727646979869969732	89.75551474640967644843717358	89.75551474640967644824901262	
	460	84.39744286268007803380889520	84.39744286267799553786765774	84.39744286267799553786765754	
15					162.015627915781020974421513240
	16	365.27598959457250439641270781	365.27598924568078011317902861	365.27598924568078023771489768	
	65	178.02159933390715505406007972	178.02159930927581887701221583	178.02159930927581887281868717	
	465	163.98875588508246868863047277	163.98875588500965129426910391	163.98875588500965129426909762	
25					393.563502697733190559496512362
	26	867.00163199459511920686859768	867.00163196313224586608216176	867.00163196313224586631572990	
	75	450.88354614389745101761538811	450.88354613612423832735039952	450.88354613612423832726057126	
	475	401.12939936195988958155797530	401.12939936192330144108737902	401.12939936192330144108737858	
50					1409.76374981471409967132946025
	51	3009.86996103494312689347231192	3009.86996103484414649104682359	3009.86996103484414649104682436	
	100	1745.80181083720649016504206416	1745.80181083711886310915937759	1745.80181083711886310915937366	
	500	1460.90974356899046369444463197	1460.90974356898945187656106283	1460.90974356898945187656106283	
		$\alpha = 0.99$			
10					93.216859660238415548016837195
	11	227.72045780686361924157771904	227.72045779716228291792684432	227.72045779716228232600823452	
	60	100.00766587721934070773055100	100.00766587324444064978254945	100.00766587324444064947003150	
	460	94.02162317660297597321472166	94.02162317659253570112582966	94.02162317659253570112582916	
15					174.996346513999368603209796319
	16	418.38682050544091438546913353	418.38682000788295279610953591	418.38682000788295286039688584	
	65	192.32704662492917996667148895	192.32704653485001159062688254	192.32704653485001158976312093	
	465	177.12815947623512459860873742	177.12815947595193142347546688	177.12815947595193142347546347	
25					413.385621639270785606729896736
	26	953.95532266934168704083850077	953.95532262361714954158839471	953.95532262361714954171095313	
	75	473.73022881819913551147338725	473.73022879405047104192607988	473.73022879405047104199617631	
	475	421.33473289045564827205481429	421.33473289033629484717703330	421.33473289033629484717703352	
50					1446.64377167058432609278629314
	51	3182.38825614178288689315436637	3182.38825614163324534715229586	3182.38825614163324534715229638	
	100	1792.17081609574757121169227449	1792.17081609550358487097066245	1792.17081609550358487097066936	
	500	1499.14133899336021309354248845	1499.14133899335729098476088160	1499.14133899335729098476088160	

$$\cdot \frac{1}{\widehat{a}^2+(p-2)\,\widehat{a}\widehat{b}-(p-1)\,\widehat{b}^2}\left\{p\widehat{a}\left(\widehat{a}+(p-2)\right.\right.$$
$$\left.\left.\cdot\,\widehat{b}\right)-\widehat{b}p\,(p-1)\,\widehat{b}\right\}=np, \tag{A.7}$$

so that

$$\max L_1 = (2\pi)^{-np/2}\left(\left(\widehat{a}-\widehat{b}\right)^{p-1}\left(\widehat{a}+(p-1)\,\widehat{b}\right)\right)^{-n/2}e^{-np/2}. \tag{A.8}$$

Under H_0 the m.l.e.'s of a and b are $\widehat{a}_0$ and $\widehat{b}_0$, given by (48), and the m.l.e. of Σ_{cs} is $\widehat{\Sigma}_{cs(0)}=(1/n)(\widehat{a}_0 I_p+\widehat{b}_0(E_{pp}-I_p))$, so that

$$\widehat{\Sigma}^{-1}_{cs(0)}=n\frac{1}{\widehat{a}_0^2+(p-2)\,\widehat{a}_0\widehat{b}_0-(p-1)\,\widehat{b}_0^2}\left(\left(\widehat{a}_0+(p-2)\,\widehat{b}_0\right)I_p-\widehat{b}_0\left(E_{pp}-I_p\right)\right),$$
$$\max L_0=(2\pi)^{-np/2}\left(\left(\widehat{a}_0-\widehat{b}_0\right)^{p-1}\left(\widehat{a}_0+(p-1)\,\widehat{b}_0\right)\right)^{-n/2}e^{-(1/2)\,\mathrm{tr}[A_0\widehat{\Sigma}^{-1}_{cs(0)}]}, \tag{A.9}$$

where A_0 is the matrix defined in (49).

Given the definition of $\widehat{a}_0$ and $\widehat{b}_0$ in (48), following similar steps to the ones used under H_1, $\mathrm{tr}[A_0\widehat{\Sigma}_{cs(0)}]=np$, so that finally

$$\max L_0=(2\pi)^{-np/2}\cdot\left(\left(\widehat{a}_0-\widehat{b}_0\right)^{p-1}\left(\widehat{a}_0+(p-1)\,\widehat{b}_0\right)\right)^{-n/2}\cdot e^{-np/2}. \tag{A.10}$$

As such, we have

$$\Lambda^*_{2|1}=\left(\frac{\left(\widehat{a}-\widehat{b}\right)^{p-1}\left(\widehat{a}+(p-1)\,\widehat{b}\right)}{\left(\widehat{a}_0-\widehat{b}_0\right)^{p-1}\left(\widehat{a}_0+(p-1)\,\widehat{b}_0\right)}\right)^{n/2} \tag{A.11}$$

and its $(2/n)$th power given by $\Lambda_{2|1}$ in (47).

Let H_p be a Helmert matrix of order p. Then

$$H_p\Sigma_{cs}H_p'=\Delta = \mathrm{diag}\left(a+(p-1)\,b,\underbrace{a-b,\ldots,a-b}_{p-1}\right), \tag{A.12}$$

and if we take $A^*=H_pAH_p'$, we will have

$$\widehat{a-b}=\widehat{a}-\widehat{b}=\frac{1}{p-1}\sum_{j=2}^{p}a^*_{jj},$$
$$a+\widehat{(p-1)}\,b=\widehat{a}+(p-1)\,\widehat{b}=a^*_{11}, \tag{A.13}$$

where a^*_{jj} $(j=1,\ldots,p)$ are the diagonal elements of A^*.

Since, for A in (27), we have

$$A\sim W_p\left(n-1,\Sigma_{cs}\right), \tag{A.14}$$

then

$$A^*=H_pAH_p'\sim W_p\left(n-1,\Delta\right), \tag{A.15}$$

so that the diagonal elements of A^* are independent, with

$$\frac{a^*_{jj}}{\Delta_j}\sim\chi^2_{n-1},\quad (j=1,\ldots,p), \tag{A.16}$$

where Δ_j is the jth diagonal element of Δ.

As such, $\widehat{a-b}=\widehat{a}-\widehat{b}$ is distributed as $(a-b)/(p-1)$ times a chi-square r.v. with $(n-1)(p-1)$ degrees of freedom, independently distributed from $a+\widehat{(p-1)}b=\widehat{a}+(p-1)\widehat{b}$ which is distributed as $a+(p-1)b$ times a chi-square with $n-1$ degrees of freedom.

Concerning the matrix A_0 in (49), we know that

$$A_0\sim W_p\left(n,\Sigma_{cs}\right), \tag{A.17}$$

so that

$$A_0^*=H_pA_0H_p'\sim W_p\left(n,\Delta\right), \tag{A.18}$$

which shows that the diagonal elements of A_0^*, $a^*_{0(jj)}$ $(j=1,\ldots,p)$, are independently distributed, with

$$\frac{a^*_{0(jj)}}{\Delta_j}\sim\chi^2_n. \tag{A.19}$$

But then, the m.l.e.'s of a and b, under H_0 in (A.1), are given by (48) or, equivalently,

$$\widehat{a_0-b_0}=\widehat{a}_0-\widehat{b}_0=\frac{1}{p-1}\sum_{j=2}^{p}a^*_{0(jj)},$$
$$a_0+\widehat{(p-1)}\,b_0=\widehat{a}_0+(p-1)\,\widehat{b}_0=a^*_{0(11)} \tag{A.20}$$

as such, with $\widehat{a_0-b_0}=\widehat{a}_0-\widehat{b}_0$ distributed as $(a-b)/(p-1)$ times a chi-square r.v. with $n(p-1)$ degrees of freedom, independently distributed from $a_0+\widehat{(p-1)}b_0=\widehat{a}_0+(p-1)\widehat{b}_0$ which is distributed as $a+(p-1)b$ times a chi-square with n degrees of freedom.

We can write

$$A = X'\left(I_n - \frac{1}{n}E_{nn}\right)X = X'X - \frac{1}{n}X'E_{nn}X$$
$$= A_0 - \frac{1}{n}X'E_{nn}X, \quad (A.21)$$

or

$$A_0 = A + \frac{1}{n}X'E_{nn}X, \quad (A.22)$$

where, by application of Cochran's Theorem (see, e.g., [12, Thm. 7.4.1]), it is easy to show that, under H_0 in (A.1), A and $(1/n)X'E_{nn}X$ are independent, since we can write

$$A = \sum_{i=1}^{n}\underline{X}_i\left(I_n - \frac{1}{n}E_{nn}\right)\underline{X}_i',$$
$$\frac{1}{n}X'E_{nn}X = \sum_{i=1}^{n}\frac{1}{n}\underline{X}_i E_{nn}\underline{X}_i', \quad (A.23)$$

where, under the null hypothesis in (A.1), the ith column of X' is

$$\underline{X}_i \sim N_p\left(\underline{0}, \Sigma_{cs}\right), \quad (A.24)$$

and we have

$$\left(I_n - \frac{1}{n}E_{nn}\right)\left(\frac{1}{n}E_{nn}\right) = \frac{1}{n}E_{nn} - \frac{1}{n^2}E_{nn}E_{nn}$$
$$= \frac{1}{n}E_{nn} - \frac{1}{n}E_{nn} = 0, \quad (A.25)$$

with

$$\operatorname{rank}\left(I_n - \frac{1}{n}E_{nn}\right) = \operatorname{tr}\left(I_n - \frac{1}{n}E_{nn}\right) = n - 1,$$
$$\operatorname{rank}\left(\frac{1}{n}E_{nn}\right) = 1, \quad (A.26)$$

which yields for A the distribution in (A.14) and

$$A_1 = \frac{1}{n}X'E_{nn}X \sim W_p\left(1, \Sigma_{cs}\right), \quad (A.27)$$

so that

$$A_1^* = H_p A_1 H_p' \sim W_p\left(1, \Delta\right), \quad (A.28)$$

so that, for $j = 1, \ldots, p$, each jth diagonal element of A_1^*, $a_{1(jj)}^*$ has a distribution such that

$$\frac{a_{1(jj)}^*}{\Delta_j} \sim \chi_1^2, \quad (A.29)$$

with

$$\widehat{a_0 - b_0} = \widehat{a - b} + \widehat{a_1 - b_1}, \quad (A.30)$$

where

$$\widehat{a_1 - b_1} = \frac{1}{p-1}\sum_{j=2}^{p}a_{1(jj)}^*, \quad (A.31)$$

so that $\widehat{a_1 - b_1}$ is independent of $\widehat{a - b}$ and it has a distribution which is that of a chi-square with $p - 1$ degrees of freedom, multiplied by $(a - b)/(p - 1)$, while we also have

$$a_0 + \widehat{(p-1)}\,b_0 = a + \widehat{(p-1)}\,b + a_1 + \widehat{(p-1)}\,b_1, \quad (A.32)$$

where $a_1 + \widehat{(p-1)}b_1$ is distributed as a chi-square with 1 degree of freedom, multiplied by $a + (p - 1)b$, and distributed independently of $a + \widehat{(p-1)}b$.

As such

$$\frac{\hat{a} - \hat{b}}{\hat{a}_0 - \hat{b}_0} = \frac{\widehat{a - b}}{\widehat{a_0 - b_0}}$$
$$\sim \operatorname{Beta}\left(\frac{(n-1)(p-1)}{2}, \frac{p-1}{2}\right), \quad (A.33)$$

which is independently distributed from

$$\frac{\hat{a} + (p-1)\hat{b}}{\hat{a}_0 + (p-1)\hat{b}_0} = \frac{a + \widehat{(p-1)}\,b}{a_0 + \widehat{(p-1)}\,b_0}$$
$$\sim \operatorname{Beta}\left(\frac{n-1}{2}, \frac{1}{2}\right), \quad (A.34)$$

yielding for $\Lambda_{2|1}$ in (47) the distribution stated in Section 5, as also stated by Geisser in [3], but where the expression for the l.r. statistic should be corrected to be stated as in (A.11).

B. On the Distribution of Λ_4 in (51)

In order to show the distribution of Λ_4 in (51) as mentioned in (54) and (55), all we need to do is to show that

$$\left\{\prod_{j=2}^{p}Y_j^*\right\}\left(Y_1\right)^{p-1} \stackrel{d}{\equiv} \left\{\prod_{j=2}^{p}Y_j^{**}\right\}, \quad (B.1)$$

for Y_j^* in (53), Y_1 in (52), and Y_j^{**} in (55), where Y_1 is independent of all Y_j^*.

Let us take Λ_1^* as a r.v. whose distribution is the same as that of the product of r.v.'s on the left hand side of (B.1) and Λ_2^* as a r.v. whose distribution is the same as that of the product of r.v.'s on the right hand side of (B.1).

Then, using the multiplication formula for the Gamma function

$$\Gamma(nz) = (2\pi)^{(1-n)/2}\, n^{n-z-1/2}\prod_{k=0}^{n-1}\Gamma\left(z + \frac{k}{n}\right), \quad (B.2)$$

we may write

$$E\left[\left(\Lambda_1^*\right)^h\right] = \left\{\prod_{j=2}^{p}\frac{\Gamma((n-1)/2+(j-2)/(p-1))\Gamma((n-j)/2+h)}{\Gamma((n-j)/2)\Gamma((n-1)/2+(j-2)/(p-1)+h)}\right\} \cdot \frac{\Gamma(n(p-1)/2)\Gamma((n-1)(p-1)/2+(p-1)h)}{\Gamma((n-1)(p-1)/2)\Gamma(n(p-1)/2+(p-1)h)}$$

$$= \left\{\prod_{j=2}^{p}\frac{\Gamma((n-1)/2+(j-2)/(p-1))\Gamma((n-j)/2+h)}{\Gamma((n-j)/2)\Gamma((n-1)/2+(j-2)/(p-1)+h)}\right\} \cdot \left\{\prod_{j=0}^{p-2}\frac{\Gamma(n/2+j/(p-1))\Gamma((n-1)/2+j/(p-1)+h)}{\Gamma((n-1)/2+j/(p-1))\Gamma(n/2+j/(p-1)+h)}\right\}$$

$$= \prod_{j=2}^{p}\frac{\Gamma(n/2+(j-2)/(p-1))\Gamma((n-j)/2+h)}{\Gamma((n-j)/2)\Gamma(n/2+(j-2)/(p-1)+h)} = E\left[\left(\Lambda_2^*\right)^h\right]. \quad \text{(B.3)}$$

But then, since both Λ_1^* and Λ_2^* have a delimited support, more precisely, between zero and 1, their distribution is determined by their moments and as such $\Lambda_1^* \stackrel{d}{\equiv} \Lambda_2^*$.

C. Derivation of the l.r. Statistic to Test Equality of Means, Assuming Sphericity of the Covariance Matrix

Let us suppose that $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, where $\underline{\mu} = [\mu_1, \ldots, \mu_p]'$, and that we want to test the hypotheses

$$H_{02|01}: \mu_1 = \cdots = \mu_p, \text{ assuming } \Sigma = \sigma^2 I_p \quad \text{versus } H_{12|01}: \Sigma = \sigma^2 I_p \ \left(\text{and no structure for } \underline{\mu}\right), \quad \text{(C.1)}$$

based on a sample of size n.

Under $H_{12|01}$, the m.l.e. of $\underline{\mu}$ is

$$\widehat{\underline{\mu}} = \underline{\overline{X}} = \frac{1}{n}X'E_{n1}, \quad \text{(C.2)}$$

with

$$\underline{\overline{X}} = \left[\overline{X}_1, \ldots, \overline{X}_p\right], \quad \text{(C.3)}$$

where X is the $n \times p$ sample matrix and E_{n1} is a matrix of dimensions $n \times 1$ of 1's, that is, a column vector of n 1's. Also, under this same hypothesis, the m.l.e. of Σ is

$$\widehat{\Sigma} = \widehat{\sigma}^2 I_p, \quad \text{(C.4)}$$

where

$$\widehat{\sigma}^2 = \frac{1}{p}\sum_{j=1}^{p}\frac{1}{n}\sum_{i=1}^{n}\left(X_{ij} - \overline{X}_j\right)^2 = \frac{1}{np}\operatorname{tr}(A), \quad \text{(C.5)}$$

where X_{ij} is the element in the ith row and jth column of X and A is the matrix in (27).

Since the likelihood function is

$$L = (2\pi)^{-np/2}|\Sigma|^{-n/2} e^{-(1/2)\operatorname{tr}[(X-E_{n1}\underline{\mu}')'(X-E_{n1}\underline{\mu}')\Sigma^{-1}]}, \quad \text{(C.6)}$$

its maximum under $H_{12|01}$ in (C.1) is thus

$$\max L_1 = L\left(\widehat{\underline{\mu}}, \widehat{\Sigma}\right) = (2\pi)^{-np/2}\left|\widehat{\sigma}^2 I_p\right|^{-n/2} \cdot e^{-(1/2)\operatorname{tr}[(X-E_{n1}\underline{\overline{X}}')'(X-E_{n1}\underline{\overline{X}}')(1/\widehat{\sigma}^2)I_p]} = (2\pi)^{-np/2} \cdot \left(\widehat{\sigma}^2\right)^{-np/2} e^{-(1/2\widehat{\sigma}^2)\operatorname{tr}[(X-E_{n1}\underline{\overline{X}}')'(X-E_{n1}\underline{\overline{X}}')]}$$

$$= (2\pi)^{-np/2}\left(\widehat{\sigma}^2\right)^{-np/2} e^{-(1/2\widehat{\sigma}^2)\operatorname{tr}(A)} = (2\pi)^{-np/2} \cdot \left(\frac{\operatorname{tr}(A)}{np}\right)^{-np/2} e^{-np/2}. \quad \text{(C.7)}$$

Under the null hypothesis $H_{02|01}$ in (C.1), the m.l.e. of $\underline{\mu}$ is

$$\widehat{\underline{\mu}}_0 = \underline{\overline{X}}^* = \left(\frac{1}{p}\sum_{j=1}^{p}\overline{X}_j\right)E_{p1} = \frac{1}{np}E_{pp}X'E_{n1} = \frac{1}{p}E_{pp}\underline{\overline{X}}, \quad \text{(C.8)}$$

where $(1/p)\sum_{j=1}^{p}\overline{X}_j = \overline{X}$ and the m.l.e. of Σ is

$$\widehat{\Sigma}_0 = \widehat{\sigma}_0^2 I_p, \quad \text{(C.9)}$$

with

$$\widehat{\sigma}_0^2 = \frac{1}{p}\sum_{j=1}^{p}\frac{1}{n}\sum_{i=1}^{n}\left(X_{ij} - \overline{X}\right)^2 = \frac{1}{p}\operatorname{tr}(A_0), \quad \text{(C.10)}$$

where A_0 is the matrix in (62), so that the maximum of the likelihood function under $H_{02|01}$ in (C.1) is

$$\max L_0 = L\left(\widehat{\underline{\mu}}_0, \widehat{\Sigma}_0\right) = (2\pi)^{-np/2}\left|\widehat{\sigma}_0^2 I_p\right|^{-n/2} \cdot e^{-(1/2)\operatorname{tr}[(X-E_{n1}\widehat{\underline{\mu}}_0')'(X-E_{n1}\widehat{\underline{\mu}}_0')(1/\widehat{\sigma}_0^2)I_p]} = (2\pi)^{-np/2} \cdot \left(\widehat{\sigma}_0^2\right)^{-np/2} e^{-(1/2\widehat{\sigma}^2)\operatorname{tr}(A_0)} = (2\pi)^{-np/2} \cdot \left(\frac{\operatorname{tr}(A_0)}{np}\right)^{-np/2} e^{-np/2}. \quad \text{(C.11)}$$

The l.r. statistic to test the hypotheses in (C.1) is thus

$$\Lambda_{2|1}^* = \frac{\max L_0}{\max L_1} = \frac{\left(\operatorname{tr}(A_0)/np\right)^{-np/2}}{\left(\operatorname{tr}(A)/np\right)^{-np/2}} = \left(\frac{\operatorname{tr}(A)}{\operatorname{tr}(A_0)}\right)^{np/2}, \quad \text{(C.12)}$$

so that its $(2/n)$th power is the statistic $\Lambda_{2|1}$ in (72).

D. Derivation of the l.r. Statistic to Test the Nullity of Means, Assuming Sphericity of the Covariance Matrix

Let us suppose that $\underline{X} \sim N_p(\underline{\mu}, \Sigma)$, where $\underline{\mu} = [\mu_1, \dots, \mu_p]'$, and that we want to test the hypotheses

$$\begin{aligned} &H_{02|01} : \underline{\mu} = \underline{0}, \text{ assuming } \Sigma = \sigma^2 I_p \\ &\text{versus } H_{12|01} : \Sigma = \sigma^2 I_p \ \left(\text{and no structure for } \underline{\mu}\right), \end{aligned} \quad \text{(D.1)}$$

based on a sample of size n.

Under $H_{12|01}$ in (D.1), the m.l.e.'s of $\underline{\mu}$ and Σ are the same as those in Appendix C, since this hypothesis is the same as the alternative hypothesis in (C.1) in Appendix C, and, as such, the function L_1, the maximum of the likelihood function under $H_{12|01}$ in (D.1), is the same as L_1 in (C.7) in Appendix C.

Under the null hypothesis $H_{02|01}$ in (D.1), the m.l.e. of Σ is

$$\widehat{\Sigma}_0 = \widehat{\sigma}_0^2 I_p, \quad \text{(D.2)}$$

with

$$\widehat{\sigma}_0^2 = \frac{1}{p}\sum_{j=1}^{p}\frac{1}{n}\sum_{i=1}^{n}\left(X_{ij}\right)^2 = \frac{1}{p}\operatorname{tr}\left(A_0\right), \quad \text{(D.3)}$$

where A_0 is the matrix in (92), so that the maximum of the likelihood function under $H_{02|01}$ in (D.1) is given by a similar function to that in (C.11), now with A_0 given by (92) and $\widehat{\sigma}_0^2$ given by (D.3).

The l.r. statistic to test the hypotheses in (D.1) is thus, for A in (27) and A_0 in (92),

$$\begin{aligned} \Lambda^*_{2|1} &= \frac{\max L_0}{\max L_1} = \frac{\left(\operatorname{tr}\left(A_0\right)/np\right)^{-np/2}}{\left(\operatorname{tr}\left(A\right)/np\right)^{-np/2}} \\ &= \left(\frac{\operatorname{tr}\left(A\right)}{\operatorname{tr}\left(A_0\right)}\right)^{np/2}, \end{aligned} \quad \text{(D.4)}$$

so that its $(2/n)$th power is the statistic $\Lambda_{2|1}$ in (91).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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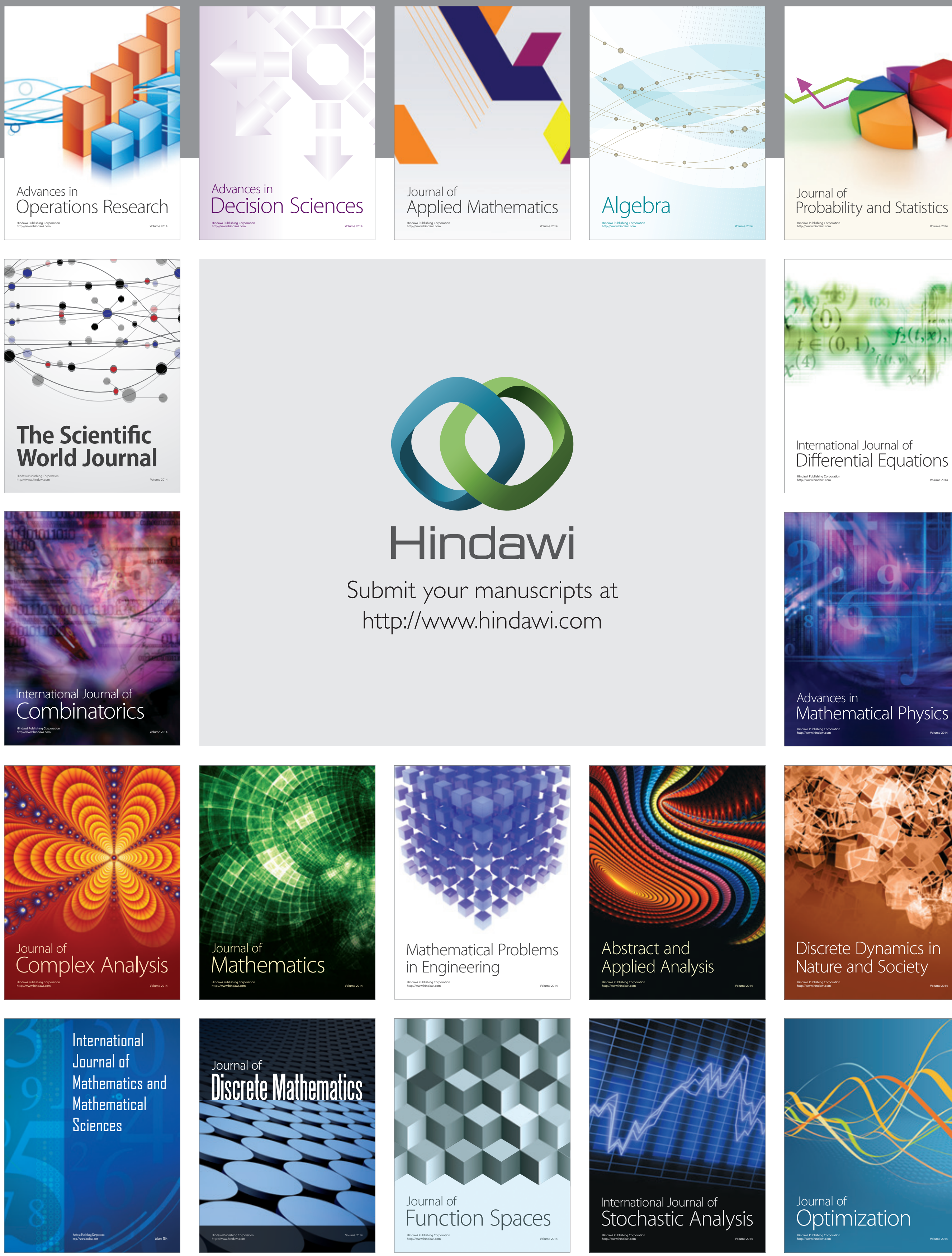
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